THE CANONICAL ENDOMORPHISM FOR INFINITE INDEX INCLUSIONS

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Abstract. Continuing the analysis in [19, 5], we give purely algebraic characterization of the canonical endomorphism in interesting infinite index cases. We apply these results when compact and discrete (but not necessarily finite dimensional) Woronowicz algebras [22] act alternately on the factors in the various levels of Jones' tower. We characterise when the acting algebra is a Kac algebra.

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1. Introduction

The purpose of this work is to extend some results of [19] beyond the finite index case, namely to find necessary and sufficient algebraic conditions on an endomorphism γ of a von Neumann algebra M which guarantee the existence of a subalgebra $N \subset M$ for which γ is the associated canonical endomorphism. We solve this problem in the cases (dual to each other) when there is a faithful normal conditional expectation either from M to N or from N' to M'. We use Pimsner-Popa basis (relative to an inclusion possessing a faithful conditional expectation) made of elements of the bigger algebra (and not merely affiliated to it), whose existence we show in case the algebras involved are properly infinite. Finally we apply our results to the context of Longo's Q-systems [19] and to give a different characterisation of (semicompact or semidiscrete) depth 2 inclusions, which have been recently proved in [2, 3] to be generated as crossed products by Woronowicz algebras [22]. Besides we characterise when the Woronowicz algebra is indeed a Kac algebra, using the formalism of Q-systems.

Our main motivation for studying, here and in [5], the canonical endomorphism γ of an inclusion $N \subset M$ of von Neumann algebras, is that the latter can be interpreted as being generated by means of a crossed product by the action, on N, of an implicitly defined "quantum object", and γ should be regarded in some sense as the "regular representation" of the "quantum object". This is to be interpreted by analogy with the case of inclusions $N \subset N \times_\alpha G$ coming from outer actions of finite groups, where the irreducible decomposition of λ is $\lambda \cong \oplus_{g\in G}\alpha_g$, while $\lambda|_N \cong \oplus_{\pi\in G} d_\pi\rho_\pi$, where ρ_π's are (irreducible) endomorphisms of N in $1-1$ correspondence with the irreducible representations π of G (see [14] for example).

This paper is organized as follows.
After a preliminary section, Section 3 is devoted to Pimsner-Popa bases; namely, given an inclusion $N \subset M$ of properly infinite von Neumann algebras with a faithful normal conditional expectation $E : M \to N$, we construct a Pimsner-Popa basis for the left N-module ${}_N M$, completely made of elements of M (not just affiliated to M, as it happens in the type II_1 case considered by Popa [26]). Using this basis, we characterise in a purely algebraic way when an endomorphism $\lambda \in End(M)$ is a canonical one. This is made in Section 4 in two cases (dual to each other), first when there exists a faithful conditional expectation $E : N \to \lambda(M)$, which we call semidiscrete, and secondly when there exists a faithful conditional expectation $E : M \to N$, which we call semicompact.
Section 5 concerns the extension of the notion of Longo's Q-system [19] to the semicompact and semidiscrete cases, and to the proof of a duality theory between them. If one implements a Q-system concretely as an inclusion of von Neumann algebras, the dual Q-system appears, using the canonical endomorphism, in a natural way in the Jones-Longo tunnel

$$\cdots \subset \gamma(N) \subset \gamma(M) \subset N \subset M.$$

We conclude with a section where we prove an extension of a Frobenius reciprocity result of [19] and apply it to (semidiscrete and semicompact) Q-systems based on a factor-subfactor inclusion of depth 2, which we prove are characterised as those for which the canonical endomorphism has the "*absorbing*" property $\lambda^2 \cong d\lambda$, already known in the case of compact groups [9], and of finite dimensional Kac algebras [19]. Therefore, using recent results in [2, 3], we can show that an irreducible semicompact Q-system based on a factor-subfactor inclusion of depth 2 will appear as the crossed product of an irreducible semidiscrete (hence automatically discrete according to the terminology in [11]) Q-system by an outer action of a discrete Woronowicz algebra, so that Jones-Longo tunnel is obtained via (alternate) crossed product procedures. The dual case, corresponding to prime actions of compact Woronowicz algebras, leads to a more complicated situation, already well known in case of compact group actions, see [23, 27]. We can characterize the case when a Kac algebra appears, improving on [3], namely a discrete Kac algebra, together with its compact dual algebra, appears *iff* the canonical endomorphism $\gamma \in Sect(M)$ decomposes as

$$\gamma = \oplus_i d(\rho_i)\rho_i,$$

where $\{\rho_i\}_{i\in I} \subset Sect(M)$ is a basis of finite index irreducible sectors for the *-semiring generated by the same $\{\rho_i\}$. The above condition is well known for compact (or discrete) groups and finite dimensional Hopf *-algebras [19].
The cases of arbitrary Woronowicz algebras or more complicated quantum symmetries such as weak and quasi-weak Hopf algebras, seem to be very difficult; we hope to return on these open problems in the future.

2. Notation and preliminaries

We consider in the following, for simplicity, only inclusions of von Neumann algebras with separable predual. For the reader's convenience we recall some notation, used throughout the paper.
Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then $s_M(e)$ is the support in M of the projection $e \in \mathcal{B}(\mathcal{H})$, and $Proj(M)$ is the set of projections in M. If N is a von Neumann subalgebra of M then $C(M,N)$ and $E(M,N)$ are the set of normal, resp. normal faithful, conditional expectations from M onto N, whereas $P(M,N)$ is the set of normal semifinite faithful N-valued weights on M, and, if $T \in P(M,N)$, $\mathfrak{N}_T := \{x \in M : T(x^*x) \in N\}$, $\mathfrak{M}_T := \mathrm{span}\{x \in M_+ : T(x) \in N\} \equiv \mathfrak{N}_T^*\mathfrak{N}_T$ is the domain of T, and $(N' \wedge M)_T := \{x \in N' \wedge M : \sigma_t^T(x) = x, t \in \mathbb{R}\}$ is the centralizer of T. $End(M)$ is the set of normal faithful unital endomorphisms of M, and for $\rho, \sigma \in End(M)$, (σ, ρ) is the vector space of intertwiners between ρ and σ,

$$(\sigma, \rho) := \{v \in M : v\sigma(x) = \rho(x)v,\ x \in M\}.$$

Finally γ is a canonical endomorphism for the inclusion $N \subset M$, $\bar{\rho}$ is a conjugate endomorphism of ρ given by $\bar{\rho} = \rho^{-1}\cdot\gamma$, and $[\rho] \in Sect(M)$ is the sector determined by ρ in $End(M)$ modulo inner automorphisms.
For the general theory of von Neumann algebras we refer to [12, 28, 27, 29].

Let us now recall the definition of H. Kosaki's index ([13]) based on A. Connes' spatial theory and U. Haagerup's operator valued weights (see [27]). If $N \subset M$ are von Neumann algebras, associated to every $E \in E(M,N)$ there is an M'-valued operator weight on N', E^{-1}, uniquely determined by

$$d(\varphi \cdot E)/d\psi = d\varphi/d(\psi \cdot E^{-1}),$$

for all normal semifinite faithful weights φ on N, ψ on M'. Observe that the index of E, $Ind(E) := E^{-1}(1) \in \overline{Z(M)}_{+}$, the extended positive part of $Z(M)$, and does not depend on the representation of M (as the same proof of [13], Theorem 2.2 works). Let φ be a faithful normal state on N, and set $\psi := \varphi \cdot E$; let $\Psi \in \mathcal{H} \equiv \mathcal{H}_{\psi}$ cyclic and separating for M and such that $\psi = (\cdot\Psi, \Psi)$, and set $e := [N\Psi] \in N'$, the Jones projection of the inclusion. The following propositions summarize standard results on index theory for inclusions.

Proposition 2.1. ([13])
(i) *$E^{-1}(e) = 1$.*
(ii) *$M_1 := \langle M, e\rangle \equiv JN'J$, where $J \equiv J^{\Psi}_{M}$. This algebra is called Jones basic construction.*
(iii) *If J is a modular conjugation for M, and $j := adJ$, then $E_1 := j \cdot E^{-1} \cdot j(\cdot) \in P(M_1, M)$. E_1 is called the dual (operator valued) weight of E.*

Proposition 2.2. ([5], *Theorem 2.1*)
Let $N \subset M \subset L$ be von Neumann algebras, $E \in E(M,N)$, $f \in Proj(L)$ s.t.
(i) *$fxf = E(x)f$, $x \in M$,*
(ii) *$L = \langle M, f\rangle$,*
(iii) *$s_{Z(L)}(f) = 1$, $s_N(f) = 1$.*
Then there is an isomorphism $\phi : L \to M_1$ s.t. $\phi|_M = id_M$ and $\phi(f) = e$.

Proposition 2.3. *Let $N \subset M$ be properly infinite von Neumann algebras. Then for any $E \in E(M,N)$, there is an isometry $v \in N$ s.t. $E(x) = v^*\gamma(x)v$, $x \in M$, $v \in (id|_N, \gamma|_N)$ and $\gamma^{-1}(vv^*)$ is Jones' projection for the inclusion $N \subset M$.*

Proof. See ([16], Proposition 5.1), or also ([5] Lemma 3.3) $\square$

For the reader's convenience we report some results in [5] the first of them being based on an argument in ([17], Theorem 4.1).

Proposition 2.4. *Let $N \subset M$ be properly infinite von Neumann algebras, $\lambda \in End(M)$, $v \in M$ an isometry s.t. $vx = \lambda(x)v$, $x \in N$, and $\lambda(M) \subset N \subset \langle N, vv^* \rangle \equiv M$, is Jones' basic construction.*
Then λ is a canonical endomorphism for $N \subset M$.

Proof. Let Ω be a cyclic and separating vector for $\lambda(M)$, N, and set $J := J_N^\Omega$, $J_0 := J_{\lambda(M)}^\Omega$. Set $v_0 := JvJ \in J\langle N, vv^* \rangle J = \lambda(M)'$. Let ξ be cyclic and separating for N s.t. $vv^* = [\lambda(M)\xi]$. The canonical implementation of $y \in \lambda(M) \to yvv^* \in \lambda(M)vv^*$ w.r.t. Ω and ξ is given by $w_0 = v_0 z$, where $z \in \lambda(M)'$ is unitary. Then by ([16], Proposition 3.1) $\Gamma := J_0 J = w_0^* J w_0 J = z^* v_0^* J v_0 J J z J$, so that, to compute the sector of $\gamma := ad\Gamma|_N$, it is sufficient to assume $w_0 = v_0$. Then, for all $x \in N$,

$$\begin{aligned}\Gamma x \Gamma^* &= v_0^* J v_0 J x J v_0^* J v_0 = J v^* J v x v^* J v J \\ &= J v^* J \lambda(x) v v^* J v J = \lambda(x) J v^* J v v^* J v J \\ &= \lambda(x) J v^* v v^* v J = \lambda(x),\end{aligned}$$

where we used $Jvv^*J = vv^*$, as vv^* is Jones' projection for $\lambda(M) \subset N$. Therefore $\lambda|_N$ is a canonical endomorphism for $\lambda(M) \subset N$, so that λ is a canonical endomorphism for $N \subset M$. $\square$

Proposition 2.5. *Let $N \subset M$ be properly infinite von Neumann algebras, $\lambda \in End(M)$, $v \in N$ an isometry s.t. $vx = \lambda(x)v$, $x \in \lambda(M)$ and $\lambda(N) \subset \lambda(M) \subset \langle \lambda(M), vv^* \rangle \equiv N$, is Jones' basic construction.*
Then λ is a canonical endomorphism for $N \subset M$.

Lemma 2.6. *Let $N \subset M$ be properly infinite von Neumann algebras, $\rho \in End(M)$, $v \in (id, \rho)$ be s.t.*
(i) $\rho(M) \subset N \subset \langle N, vv^* \rangle =: L$ is Jones' basic construction,
(ii) $s_{Z(L)}(vv^*) = 1$.
Then $M = \langle N, vv^* \rangle$.

Proof. Set $f := vv^*$, and observe that $f\langle N, f \rangle f = f\rho(M)f = \{\rho(x)f : x \in M\} = \{vxv^* : x \in M\} = fMf$, that is $\langle N, vv^* \rangle_f = M_f$. Therefore $L'_f = \langle N, f \rangle'_f = M'_f$, and, as $s_{Z(L)}(f) = 1$ the map $L' \to L'_f$ is an isomorphism which restricts to the isomorphism $M' \to M'_f \equiv L'_f$, so that $L' = M'$, that is $L = M$. $\square$

3. On the Pimsner-Popa basis

Let $N \subset M$ be an inclusion of von Neumann algebras and $T \in P(M,N)$. A Pimsner-Popa basis is a basis for the self dual completion X_T of the left N-module ${}_N(\mathfrak{M}_T)$ relative to the N-valued inner product $< x, y >_T = T(xy^*)$ as in [24].
It is a celebrated result that, if $E(M,N) \neq \emptyset$, there exists a Pimsner-Popa basis consisting of elements affiliated to M, in the sense explained in [26].
In this section we strengthen this result by showing that, if $N \subset M$ are properly infinite von Neumann algebras, a Pimsner-Popa basis consisting of elements in M can be chosen. The proof also shows how such a basis can be constructed. We need some preliminary lemmas.

Lemma 3.1. *Let $N \subset M$ be an inclusion of properly infinite von Neumann algebras, with $E(M,N) \neq \emptyset$. Then*
(i) *Jones' projection $e \in M_1$ is properly infinite,*
(ii) *if $\hat{q} \in Proj(M_1)$ is majorized by e, then there exists $q \in Proj(N)$ such that $qe = \hat{q}$.*

Proof. (i) We have isometries $v, w \in N$ with orthogonal ranges satisfying $vv^* + ww^* = 1$. We can define $\hat{v} := ve$, $\hat{w} := we$ and obtain two partial isometries with ortogonal ranges that satisfy $\hat{v}^*\hat{v} = e = \hat{w}^*\hat{w}$, $\hat{v}\hat{v}^* + \hat{w}\hat{w}^* = e$. The assertion follows by [28], 4.12.
(ii) As $eM_1e \sim Ne$, see [25], there exists $q \in N$ such that $qe = \hat{q} \equiv e\hat{q}e$; moreover $\hat{q} \in \mathfrak{M}_{E_1}$ and $q = E_1(\hat{q})$, where $E_1 \in P(M_1, M)$ is the dual weight of $E \in E(M,N)$. Finally $q \in Proj(N)$ as $q^*q = E_1(q^*qe) = E_1(eq^*qe) = E_1(\hat{q}^*\hat{q}) = E_1(\hat{q}) = q$. □

The proof of the following lemma is heavily based on ([8], Lemma 2.2).

Lemma 3.2. *Let $N \subset M$ be an inclusion of von Neumann algebras and T a normal N-valued weight on M. The following are equivalent*
(i) *T is semifinite,*
(ii) *there is $x \in (\mathfrak{M}_T)_+$ s.t. $s_M(x) = 1$,*
(iii) *there is a sequence $\{p_n\}$ of orthogonal projections in $\mathfrak{M}_T$ s.t. $\sum p_n = 1$.*

Proof. $(i) \Rightarrow (ii)$ As the unit ball of M_+ is metrizable in the strong topology ([12], 5.7.46), there is a strongly dense sequence $\{x_n\}$ in the unit ball of $(\mathfrak{M}_T)_+$. Then $\sum_{n=1}^{\infty} 2^{-n} \frac{x_n}{\|1+T(x_n)\|}$ converges in norm to $x \in M_+$. As T is normal, $x \in (\mathfrak{M}_T)_+$, in fact $T(x) = \sum_{n=1}^{\infty} 2^{-n} \frac{T(x_n)}{\|1+T(x_n)\|} \in$

N. Finally, if $p \in Proj(M)$ is s.t. $xp = 0$, then $0 = \sum_{n=1}^{\infty} 2^{-n} \frac{px_np}{\|1+T(x_n)\|}$, which implies $px_np = 0$ for all n, that is $px_n = 0$, which implies $p = 0$, due to the density of $\{x_n\}$.
$(ii) \Rightarrow (iii)$ Set $p_n := \chi_{[\frac{1}{n+1},\frac{1}{n})}(x)$. Then $\{p_n\}$ are mutually orthogonal projections in M, and $\sum_n p_n = s_M(x) = 1$. Finally $T(p_n) \leq (n+1)T(xp_n) \leq T(x) \in N_+$.
$(iii) \Rightarrow (i)$ is obvious. □

The following lemma is essentially ([11], Lemma 2.2) (which is only proved in case $N \subset M$ are factors).

Lemma 3.3. *(Push-down Lemma)*
Let $N \subset M$ be an inclusion of von Neumann algebras, $E \in E(M,N)$. Then, for all $x \in \mathfrak{N}_{E_1}$, we have $eE_1(ex) = ex$.

Proof. The same proof of ([11], Lemma 2.2) works. Just observe that $\mathcal{A} := MeM$ is still weakly dense in M_1. Indeed, as $\bar{\mathcal{A}}$ is a weakly-closed two-sided ideal in M_1, there is a central projection $z \in M_1$, s.t. $zM_1 = \bar{\mathcal{A}}$. Then $z = 1$, as $(1-z)e = 0$, so that $z \geq s_{Z(M_1)}(e) = 1$. □

We can now prove the main result of this Section, namely that a basis $\{m_i\}_{i\in I}$ for M made of elements in M can be chosen.

Remark 3.4. Recall [26] that the following equivalent properties characterize a Pimsner-Popa basis:
(i) $E(m_im_j^*) = \delta_{ij}q_j \in Proj(N) \setminus \{0\}$, and $\sum_{i\in I} m_i^*em_i = 1$
(ii) $m_i^*em_i$ are mutually ortogonal projections in M_1, and $(\sum_i m_i^*e\mathcal{H})^- = \mathcal{H}$.
In general the sequence $\{m_i\}$ is made of elements affiliated to M, as was explained in [26].

Theorem 3.5. *Let $N \subset M$ be an inclusion of properly infinite von Neumann algebras and $E \in E(M,N)$. Then there exists a Pimsner-Popa basis for the left module ${}_NM$ made of elements of M.*

Proof. Let $\{p_i\}_{i\in I}$ be a family of ortogonal projections of M_1 such that $p_i \in \mathfrak{M}_{E_1}$ and $\sum_{i\in I} p_i = 1$, as in Lemma 3.2. Thanks to $s_{Z(M_1)}(p_i) \leq s_{Z(M_1)}(e) \equiv 1$, and e being properly infinite, we get a collection $\{v_i\}_{i\in I} \subset M_1$ of partial isometries such that

$$v_i^*v_i = p_i,\ \hat{q}_i := v_iv_i^* \leq e \quad i \in I.$$

Hence $v_i \equiv ev_ip_i \in \mathfrak{M}_{E_1}$ and we take $m_i := E_1(v_i)$ as a Pimsner-Popa basis. Indeed m_i is the push-down of v_i, as $v_i = ev_i = eE_1(ev_i) = eE_1(v_i)$. Besides

$$p_i = v_i^*v_i = v_i^*ev_i = m_i^*em_i$$

and summing up we obtain $\sum_i m_i^* e m_i = 1$.
Moreover we get

$$\begin{aligned} E(m_i m_j^*) &= E_1(eE(m_i m_j^*)) = E_1(e m_i m_j^* e) \\ &= E_1(e v_i v_j^* e) = \delta_{ij} E_1(e\hat{q}_j) \\ &= \delta_{ij} E_1(e q_j) = \delta_{ij} q_j . \end{aligned}$$

□

Proposition 3.6. *If $\{m_i\}_{i\in I} \subset M$ is a Pimsner-Popa basis for $N \subset M$ then every $x \in M$ has the following expansion*

$$x = \sum_{i\in I} E(x m_i^*) m_i \tag{1}$$

where the last sum converges unconditionally relative to the topology generated by the separating family of seminorms $\{p_\varphi : \varphi \in (N_*)_+\}$, with $p_\varphi(x) := \varphi \circ E(xx^*)^{1/2}$.

Proof. Observe that $m_i = q_i m_i$. In fact $E((m_i - q_i m_i)(m_i - q_i m_i)^*) = E(m_i m_i^*) - E(q_i m_i m_i^*) - E(m_i m_i^* q_i) + E(q_i m_i m_i^* q_i) = 0$, and the thesis follows by faithfulness of E.
We now prove the convergence of (1) in the topology generated by the above seminorms.
As $p_A := \sum_{i\in A} m_i^* e m_i \nearrow 1$, in the strong operator topology when the finite subset $A \subset I$ tends to the whole index set I, for $x \in M$ we have

$$\|(1 - p_A)\Lambda_{\hat{\varphi}}(x^* e)\|^2_{\hat{\varphi}} \longrightarrow 0$$

that is

$$\hat{\varphi}(ex(1 - p_A)x^* e) \longrightarrow 0$$

for every $\hat{\varphi}$ normal semifinite faithful weight on M_1 given by $\hat{\varphi} := \varphi \circ E \circ E_1$, $\varphi \in (N_*)_+$. Finally we get

$$\begin{aligned} \hat{\varphi}(ex(1 - p_A)x^* e) &= \varphi \circ E(xx^*) - \sum_{i\in A} \varphi \circ E(E(x m_i^*) E(m_i x^*)) \\ &= \varphi \circ E(xx^*) - \sum_{i,j\in A} \varphi \circ E(E(x m_i^*) m_i m_j^* E(m_j x^*)) \\ &\equiv \varphi \circ E((x - x_A)(x^* - x_A^*)) \end{aligned}$$

where $x_A := \sum_{i\in A} E(x m_i^*) m_i$ and we have applied $m_i = q_i m_i$, $E(m_i m_j^*) = \delta_{ij} q_j$. □

Remark 3.7. Note that

$$IndE = \sum_{i\in I} m_i^* m_i \in \overline{Z(M)}_+$$

and the index is finite iff $\sum_{i\in I} m_i^* m_i \in Z(M)$.

As a direct consequence of the Theorem 3.5 we have the following

Corollary 3.8. *The self-dual completion ${}_N X$ of the left N-module ${}_N M$ is isomorphic to the ultraweak direct sum*

$$ {}_N X \cong \bigoplus_{j\in J} N q_j . $$

Proof. Follows by the above considerations and [24] Theorem 3.12. $\square$

4. The canonical endomorphism in the semidiscrete and semicompact cases

In this Section we provide the announced conditions which are equivalent to the fact that an endomorphism is canonical. We treat only the semidiscrete and semicompact cases, that is the cases when the canonical endomorphism λ is associated to an inclusion $N \subset M$ such that $E(N, \lambda(M)) \neq \emptyset$ or $E(M, N) \neq \emptyset$. These conditions allow us to extend the definition of a Q-system given in [19] to nontrivial examples of infinite index.

Theorem 4.1. *(semidiscrete case) Let M be a properly infinite von Neumann algebra with separable predual, $\lambda \in End(M)$. Then the following are equivalent*
(i) *there is $N \subset M$, with $E(N, \lambda(M)) \neq \emptyset$, s.t. λ is a canonical endomorphism for $N \subset M$*
(ii) *there are an isometry $v \in (id, \lambda)$ and $\{m_i\} \subset M$ s.t. $\sum_i m_i^* v v^* m_i = 1$, and, setting $N := \langle \lambda(M), \{m_i\}\rangle$, one has $x \in N$, $xv = 0 \Rightarrow x = 0$.*
Moreover, if $v^* m_i m_j^* v = \delta_{ij} q_i$, $q_i \in Proj(\lambda(M)) \setminus \{0\}$, then $\{m_i\}$ is a Pimsner-Popa basis for the inclusion $\lambda(M) \subset N$.

Proof. $(i) \Rightarrow (ii)$
Let $E \in E(N, \lambda(M))$, then from Proposition 2.3 it follows that there is an isometry $v \in M$ s.t. $\lambda(x)v = vx$, $x \in M$, $\lambda(v^* \cdot v) = E$, and finally vv^* is Jones' projection for the inclusion $\lambda(M) \subset N$.
By Theorem 3.5 there are $\{m_i\}_{i\in I} \subset N$, s.t. $\sum_{i\in I} m_i^* v v^* m_i = 1$, and $N = \langle \lambda(M), \{m_i\}\rangle$, which can be shown as in the proof of ([6], Proposition 6).
Finally, if $x \in N$ is s.t. $xv = 0$, then $0 = \lambda(v^* x^* x v) = E(x^* x)$, which implies $x = 0$, as E is faithful.

$(ii) \Rightarrow (i)$
Let us set $E := \lambda(v^* \cdot v) \in C(N, \lambda(M))$. Then E is faithful as $E(p) = 0$ for some $p \in Proj(N)$ implies $pv = 0$, and therefore $p = 0$.
We want to prove that $\lambda(M) \subset N \subset \langle N, vv^* \rangle$ is Jones' basic construction. Indeed, setting $f := vv^*$, we observe that $E(x)f = fxf$, $x \in N$, so that $s_{\lambda(M)}(f) = 1$, from ([5], Lemma 3.2(ii)). Finally $s_{Z(\langle N,vv^*\rangle)}(f) \geq s_{N' \wedge M}(f) = 1$, because, if $p \in Proj(N' \wedge M)$ is s.t. $pvv^* = 0$, then $p = p\sum_i m_i^* vv^* m_i = \sum_i m_i^* pvv^* m_i = 0$. From Proposition 2.2 we conclude.
Therefore, by Lemma 2.6, $\langle N, vv^* \rangle \equiv M$, and, by Lemma 2.4, λ is a canonical endomorphism for $N \subset M$. □

Theorem 4.2. *(semicompact case) Let M be a properly infinite von Neumann algebra with separable predual, $\lambda \in End(M)$. Then the following are equivalent*
(i) *there is $N \subset M$, with $E(M, N) \neq \emptyset$, s.t. λ is a canonical endomorphism for $N \subset M$*
(ii) *there are an isometry $v \in (\lambda, \lambda^2)$ and $\{m_i\} \subset \lambda(M)$ s.t. $\sum_i m_i^* vv^* m_i = 1$, $\lambda(v)v = v^2$, $\lambda(v)^* v = vv^*$, and $x \in M$, $\lambda(x)v = 0 \Rightarrow x = 0$.*
Moreover, if $v^* m_i m_j^* v = \delta_{ij} q_i$, $q_i \in Proj(N) \setminus \{0\}$, then $\{m_i\}$ is a Pimsner-Popa basis for the inclusion $N \subset M$.

Proof. $(i) \Rightarrow (ii)$
Let $E \in E(M, N)$, then from Proposition 2.3 it follows that there are an isometry $v \in N$ s.t. $\lambda(x)v = vx$, $x \in N$, $v^*\lambda(\cdot)v = E$, and finally vv^* is Jones' projection for the inclusion $\lambda(N) \subset \lambda(M)$.
Therefore $\lambda(\lambda(x))v = v\lambda(x)$, $x \in M$, that is $v \in (\lambda, \lambda^2)$; $\lambda(v)v = v^2$, and $\lambda(v)^* v = vv^*$ are immediate.
Besides, if $x \in M$, $\lambda(x)v = 0$, then $\lambda(x)vv^* = 0$, so that $\lambda(x) = 0$, and $x = 0$, as vv^* is separating for $\lambda(M)$.
Finally by Theorem 3.5 there are $\{m_i\} \subset \lambda(M)$ s.t. $\sum_i m_i^* vv^* m_i = 1$.
$(ii) \Rightarrow (i)$
Let us set $E := v^*\lambda(\cdot)v$. Then, as in ([19], Proposition 5.2), $E \in C(M, N)$, where $N := E(M)$ is a von Neumann subalgebra of M. We want to show that E is faithful, so let $x \in M$, be s.t. $E(x^*x) = 0$, then $v^*\lambda(x^*x)v = 0$, that is $\lambda(x)v = 0$, and $x = 0$, from the hypothesis.
Let us observe that $f := vv^* \in N$, as $E(vv^*) = v^*\lambda(v)\lambda(v)^*v = vv^*vv^* = vv^*$.
We want to prove that $\lambda(N) \subset \lambda(M) \subset \langle \lambda(M), f \rangle$ is Jones' basic construction, by using Proposition 2.2.
At first $s_{\lambda(M)}(f) = 1$, because, if $\lambda(p) \in Proj(\lambda(M))$ is s.t. $\lambda(p)vv^* = 0$, then $p = 0$.
Besides $s_{Z(\langle \lambda(M), f \rangle)}(f) \geq s_{\lambda(M)' \wedge N}(f) = 1$, because, if $p \in Proj(\lambda(M)' \wedge$

N) is s.t. $pvv^* = 0$, then $p = p\sum_i m_i^* vv^* m_i = \sum_i m_i^* pvv^* m_i = 0$.
Finally, observing that $F := \lambda \cdot E \cdot \lambda^{-1} \in E(\lambda(M), \lambda(N))$, is s.t. $F(x)f = fxf$, $x \in \lambda(M)$, we get the thesis.
Therefore, by Lemma 2.6, $\langle \lambda(M), f \rangle \equiv N$, and, by Lemma 2.5, λ is a canonical endomorphism for $N \subset M$. □

Remark 4.3. The above properties are also equivalent to

(iii) there exists an isometry $v \in (\lambda, \lambda^2)$ s.t. $\lambda(v)v = v^2$, $\lambda(v)^*v = vv^*$, and
$$s_{\lambda(M)' \wedge M}(vv^*) = s_{\lambda(\lambda(M)' \wedge M)}(vv^*) = 1,$$
as follows from ([5], Proposition 6.1).

In the following Sections we extend the notion of Q-system to semidiscrete and semicompact inclusions and analyse the Takesaki duality which naturally appears considering the Jones-Longo tunnel, and the canonical mirroring on it. As an application we have conditions on the canonical endomorphism which are equivalent to the fact that an inclusion arises as a crossed product by a compact or discrete Woronowicz (or Kac) algebra. In the discrete case these conditions can be stated directly in terms of Q-systems.

5. Duality for Q-systems

Q-systems in the finite index case were introduced in [19] to consider the canonical endomorphism as a relevant means to handle the problem of the actions of "quantum symmetries" on von Neumann algebras. This situation typically appears in QFT where in the physical Minkowski space an ordinary compact group acts on the algebra of fields, but, for low-dimentional theories, a braid group statistics appears, and it is expected that a quantum symmetry acts [21, 20].
We analyse Q-systems in the (semidiscrete and semicompact) infinite index case and apply the results to the depth 2 factor-subfactor inclusions, that is when compact or discrete Woronowicz algebras naturally appear [2]. This Section extends [19] Section 6.

Although some of the properties required in the following definition are unnecessary to characterize a canonical endomorphism λ we prefer to define a Q-system s.t. the sequence $\{m_i\}$ directly provides a Pimsner-Popa basis for the relevant inclusions.

Definition 5.1.

(i) A Q-system of semidiscrete type (semidiscrete Q-system for short) is a couple (M, Λ) where M is a properly infinite von Neumann algebra and $\Lambda := (\lambda, v, \{m_i\}_{i\in I})$ satisfies the properties given in Proposition 4.1, that is $v \in (id, \lambda)$ is an isometry, and $\{m_i\} \subset M$ are s.t.

$$\sum_i m_i^* vv^* m_i = 1$$
$$v^* m_i m_j^* v = \delta_{ij} q_i,$$

$q_i \in Proj(\lambda(M))\backslash\{0\}$, and, setting $N := \langle \lambda(M), \{m_i\}\rangle$, one has $x \in N$, $xv = 0 \Rightarrow x = 0$.

(ii) A Q-system of semicompact type (semicompact Q-system for short) is a couple (M, Λ) where M is a properly infinite von Neumann algebra and $\Lambda := (\lambda, v, \{m_i\}_{i\in I})$ satisfies the properties given in Proposition 4.2, that is $v \in (\lambda, \lambda^2)$ is an isometry s.t. $\lambda(v)v = v^2$, $\lambda(v)^*v = vv^*$, and $\{m_i\} \subset \lambda(M)$ are s.t.

$$\sum_i m_i^* vv^* m_i = 1$$
$$v^* m_i m_j^* v = \delta_{ij} q_i,$$

$q_i \in Proj(N) \setminus \{0\}$, and $x \in M$, $\lambda(x)v = 0 \Rightarrow x = 0$.

As we have proved, λ is the canonical endomorphism for an inclusion of von Neumann algebras $N \subset M$. Contrary to what we did in [5], but in accordance with [11] we say that a subalgebra N of a properly infinite von Neumann algebra M is semicompact if $E(M, N) \neq \emptyset$, and semidiscrete if $E(N', M') \neq \emptyset$, which is equivalent to $E(M_1, M) \neq \emptyset$ and to $E(N, \gamma(M)) \neq \emptyset$, with γ a canonical endomorphism for $N \subset M$. Therefore, if we say that an inclusion $N \subset M$ is a *concrete* Q-system, the two previous definitions coincide.

The irreducibility of an inclusion $N \subset M$ seems to be deeply related to the structure of $(id_N, \gamma|_N)$, where γ is the canonical endomorphism of the inclusion, see [16, 5]. The following result partially confirms the above considerations.

Theorem 5.2. *Let $A \subset B$ be a semicompact inclusion of properly infinite von Neumann algebras. The following are equivalent:*
(i) $A \subset B$ is irreducible, that is $A' \wedge B = Z(A)$
(ii) $E(B, A)$ is a singleton
(iii) The $Z(A)$-module $(id_A, \gamma|_A)$ is cyclic.

Proof. $(i) \iff (ii)$ is a well-known result by Combes-Delaroche [1].
$(iii) \Rightarrow (ii)$ Let v_0 be a generator of the $Z(A)$-module $(id_A, \gamma|_A)$, which must be an isometry, as follows from Proposition 2.3. Therefore, again by the same Proposition, $E(B, A)$ must be a singleton.

$(ii) \Rightarrow (iii)$ Let $s \in (id_A, \gamma|_A)$, then $s^*s \in (id_A, id_A) \equiv Z(A)$ and we can restrict ourselves to the case when $s = v = vp$, where $v \in (id_A, \gamma|_A)$ is a partial isometry with $p \in Z(A)$ as domain projection, and a subprojection of p as range projection. As by Proposition 2.3 there is an isometry $v_0 \in (id_A, \gamma|_A)$, we can construct $w := vp + v_0 p^\perp$, which is an isometry. By the following Lemma 5.3, there is $z \in Z(A)$ s.t. $w = zv_0$, so that $s = v = vp = zpv_0$ that is v_0 is a generator. □

Lemma 5.3. *Let $A \subset B$ be properly infinite von Neumann algebras, and assume that $E(B, A)$ is a singleton. If $v, w \in (id_A, \gamma|_A)$ are isometries, then there is a unitary $z \in Z(A)$ s.t. $v = zw$.*

Proof. As $v^*\gamma(\cdot)v$, $w^*\gamma(\cdot)w \in C(B, A) \equiv E(B, A)$, because of the hypothesis, we get $E := v^*\gamma(\cdot)v = w^*\gamma(\cdot)w$. Let us prove that $e := vv^*$, $f := ww^*$ are Jones' projections for the inclusion $\gamma(A) \subset \gamma(B)$. Indeed, setting $F := \gamma \circ E \circ \gamma^{-1} \in E(\gamma(B), \gamma(A))$, we have $F(x)e = exe$, $x \in \gamma(B)$, so that $s_{\gamma(A)}(e) = 1$, and $s_{Z(\langle\gamma(B),e\rangle)}(e) \geq s_{\gamma(B)' \wedge A}(e) = s_{Z(A)}(e) = 1$, because, if $p \in Proj(Z(A))$ is s.t. $pe = 0$, then $p = 0$, therefore from Proposition 2.2 we conclude. Then $\langle\gamma(B), e\rangle = A$ as follows from Lemma 2.6. Analogously f is Jones' projections for the inclusion $\gamma(A) \subset \gamma(B)$ and $\langle\gamma(B), f\rangle = A$. Therefore, by ([14], Appendix A), $f = e$. Let us now consider $vw^* = evw^*e \in eAe = \gamma(A)e$, that is, there exists $z \in A$ s.t. $vw^* = \gamma(z)e = \gamma(z)ww^*$, which implies $v = \gamma(z)w = wz$, and $w^*v = z$. But we have also $w^*v \in Z(A)$, so that $1 = v^*wz = z^*z$, and z is a unitary operator in $Z(A)$. □

The above Theorem suggests the following

Definition 5.4.
(i) A Q-system of semidiscrete type (M, Λ) is called irreducible if there is a unique $v \in (id, \lambda)$, up to multiplication by a unitary operator in $Z(M)$.
(ii) A Q-system of semicompact type (M, Λ) is called irreducible if there is a unique $v \in (id_N, \lambda|_N)$, up to multiplication by a unitary operator in $Z(N)$.

Hence in both cases the irreducibility condition on the Q-system means that a suitable inclusion in the Jones-Longo tower

$$\cdots \subset \gamma(M) \subset N \subset M \subset M_1 \subset M_2 \subset \cdots$$

is irreducible.

There is a natural notion of isomorphism, namely (M_1, Λ_1) and (M_2, Λ_2) are isomorphic if there is an isomorphism $\varphi : M_1 \to M_2$, s.t. $\lambda_2 = \varphi \circ \lambda_1 \circ \varphi^{-1}$, $v_2 = \varphi(v_1)$, and $m_{2i} = \varphi(m_{1i})$. It is easy to see that a Q-system isomorphic to a semidiscrete one, is itself semidiscrete. Analogously for the semicompact case.

Definition 5.5.
(i) Q-systems of discrete type (M,Λ_1) and (M,Λ_2) are inner conjugate if there is a unitary operator $u\in M$, s.t. $\lambda_2=u\lambda_1(\cdot)u^*$, $v_2=uv_1$, $m_{2i}=um_{1i}$.
(ii) Q-systems of compact type (M,Λ_1) and (M,Λ_2) are inner conjugate if there is a unitary operator $u\in M$, s.t. $\lambda_2=u\lambda_1(\cdot)u^*$, $v_2=u\lambda_1(u)v_1u^*$, $m_{2i}=u\lambda_1(u)m_{1i}u^*$.

Finally, Q-systems are cocycle equivalent if the first is isomorphic to an inner conjugate copy of the second.

Theorem 5.6. (*cfr.* [19]*, Theorem 6.1*)
Let M be a properly infinite von Neumann algebra. Then there is a bijective correspondence between (irreducible) semidiscrete subalgebras N of M, and (irreducible) semidiscrete Q systems based on M. Conjugate inclusions correspond to cocycle equivalent Q-systems.
The same holds if we replace semidiscrete with semicompact everywhere.

Proof. Bijective correspondence follows from Propositions 4.1 e 4.2. Irreducibility assumption for a Q-system corresponds to irreducibility of the subalgebra by Theorem 5.2. Finally cocycle equivalence follows from the Radon-Nikodym property of the canonical endomorphism [15]. □

We now deal with Takesaki duality in the context of Q-systems.

Let $(M,\lambda,v,\{m_i\})$ be a semidiscrete Q-system, $N\subset M$ the corresponding subalgebra. Then λ is a canonical endomorphism for $N\subset M$, that is $\lambda=ad\Gamma$, with $\Gamma:=J_NJ_M$. Then

Definition 5.7. $(\tilde M,\tilde\lambda,\tilde v,\{\tilde m_i\})$ is called the dual Q-system, where $\tilde M:=\Gamma^*N\Gamma$ is the *crossed product* of M by Λ, $\tilde\lambda:=ad\Gamma|_{\tilde M}$, $\tilde v:=v$, $\tilde m_i:=m_i$.

Let $(M,\lambda,v,\{m_i\})$ be a semicompact Q-system, $N\subset M$ the corresponding subalgebra. Then λ is a canonical endomorphism for $N\subset M$, that is $\lambda=ad\Gamma$, with $\Gamma:=J_NJ_M$. Then

Definition 5.8. $(\tilde M,\tilde\lambda,\tilde v,\{\tilde m_i\})$ is called the dual Q-system, where $\tilde M:=\Gamma^*N\Gamma$ is the *crossed product* of M by Λ, $\tilde\lambda:=ad\Gamma|_{\tilde M}$, $\tilde v:=\Gamma^*v\Gamma$, $\tilde m_i:=\Gamma^*m_i\Gamma$.

It is easy to see that the dual of a semidiscrete Q-system is semicompact and viceversa.

In this context Takesaki duality holds, too. The bidual Q-system, that is the double crossed product, is obtained by shifting all the structure two steps upwards in Jones' tower.

6. Actions of Woronowicz and Kac algebras

In this Section we analyse more deeply what happens in case of depth 2 irreducible inclusions of infinite factors.
The following Proposition, while being the crucial step in proving Theorem 6.2, could be considered as a version of Frobenius reciprocity in its own right.

Proposition 6.1. *Let $\rho(M) \subset M$ be an irreducible endomorphic inclusion of infinite factors and $\sigma \in End(M)$.*
(i) *Suppose that $\sigma\rho \succ id$. Then $\sigma \succ \bar{\rho}$.*
(ii) *Suppose that $\rho\sigma \succ id$. Then $\sigma \succ \bar{\rho}$.*

Proof. (i) Let $v \in (id, \sigma\rho)$, and let $u \in M$ be an isometry s.t. $uu^* = p := s_{\sigma(M)'\wedge M}(vv^*) \leq 1$ (as p is an infinite projection), so that $uu^*v = v$, and set $\tau := u^*\sigma(\cdot)u$. Then $\tau \in End(M)$, as it is easily shown. Set $z := u^*v$, which is an isometry, and satisfies $\tau\rho(x)z = u^*\sigma\rho(x)uu^*v = u^*\sigma\rho(x)v = u^*vx = zx$, so that $z \in (id, \tau\rho)$, that is $\tau\rho \succ id$.
Besides $s_{\tau\rho(M)'\wedge\tau(M)}(zz^*) = 1$, as $\rho(M)' \wedge M = \mathbb{C}$.
If we could prove that $s_{\tau(M)'\wedge M}(zz^*) = 1$, by Proposition 7.1, we would have $\sigma \succ \tau \cong \bar{\rho}$. So all that is left to prove is $s_{\tau(M)'\wedge M}(zz^*) = 1$.
First let us prove that $(\tau, \tau) = u^*(\sigma, \sigma)u$. Indeed $a \in (\tau, \tau)$ is equivalent to $au^*\sigma(x)u = u^*\sigma(x)ua$, $x \in M$, which implies $uau^*\sigma(x)uu^* = uu^*\sigma(x)uau^*$, that is $uau^*\sigma(x) = \sigma(x)uau^*$, as $uu^* \in (\sigma, \sigma)$. Setting $b := uau^*$, we get $a = u^*bu$, with $b \in (\sigma, \sigma)$. Conversely, if $b \in (\sigma, \sigma)$, we get

$$\begin{aligned} u^*bu\tau(x) &= u^*buu^*\sigma(x)u = u^*\sigma(x)bu \\ &= u^*\sigma(x)uu^*bu = \tau(x)u^*bu. \end{aligned}$$

Besides, if $q \in (\tau, \tau)$ is a projection s.t. $qzz^* = 0$, then $q = u^*fu$, with $f \in (\sigma, \sigma)$. As $q = q^*q$ implies $u^*fu = u^*f^*uu^*fu$, we can, substituting f^*uu^*f for f, consider f positive. Then $qzz^* = 0 \iff qz = 0$, that is

$u^*fuu^*v = 0$, which implies $v^*uu^*fuu^*v = v^*fv = 0$, which is equivalent to $fvv^* = 0$, that is $s(f)vv^* = 0$, which, recalling $uu^* = s_{(\sigma,\sigma)}(vv^*)$, implies $s(f)uu^* = 0$, that is $fuu^* = 0$, and finally $q = 0$.

(ii) The proof is the same as above if one looks at $v \in (id, \rho\sigma)$ and takes $p := \rho^{-1}(s_{\rho(\sigma(M)' \wedge M)}(vv^*))$. p is an infinite projection hence there exists an isometry u such that $uu^* = p$. In this case $\tau := u^*\sigma(\cdot)u$ gives rise to a conjugate of ρ with the isometry $\rho(u^*)v$ which intertwiners id and $\rho\tau$. □

We are now ready to extend Frobenius reciprocity, in Longo's setting, to the semidiscrete and semicompact cases.

Theorem 6.2. *Let M be an infinite factor and $\rho, \eta \in Sect(M)$ be irreducible sectors.*

(i) *(semidiscrete case) Suppose that $\rho\bar{\rho} \succ id$, $\eta\bar{\eta} \succ id$ and $\alpha \in Sect(M)$ is a sum of finite index sectors. Then, for every $\beta \in Sect(M)$ we have*

$$\alpha\rho\beta \succ \eta \iff \bar{\alpha}\eta\bar{\beta} \succ \rho$$

with equal multiplicities.

(ii) *(semicompact case) Suppose that $\bar{\rho}\rho \succ id$, $\bar{\eta}\eta \succ id$ and $\beta \in Sect(M)$ is a sum of finite index sectors. Then, for every $\alpha \in Sect(M)$ we have*

$$\alpha\rho\beta \succ \eta \iff \bar{\alpha}\eta\bar{\beta} \succ \rho$$

with equal multiplicities.

Proof. Same as [19], making a repeated use of the above Proposition. □

Now we can apply the previous results to the duality for semidiscrete (or equally well semicompact) factor-subfactor inclusions of depth 2, that is when discrete and compact dual Woronowicz algebras (see [22]) alternately act on Jones' tower [2, 3].
Moreover we provide a condition on the canonical endomorphism for the above inclusion to be generated by the crossed product by a (discrete or compact) Kac algebra.

Theorem 6.3. *Let $N \subset M$ be an irreducible inclusion of infinite factors.*
(a) *Suppose that γ contains the identity sector. Then the following are equivalent.*
(i) *$N \subset M$ is depth 2*
(ii) *$\gamma^2 \cong d \cdot \gamma$ for some $d \in \mathbb{N} \cup \{\infty\}$*

(iii) M is the crossed product of N by an outer action of a compact Woronowicz algebra.

(b) Suppose that $\gamma|_N$ contains the identity sector. Then the following are equivalent.
(i) $N \subset M$ is depth 2
(ii) $\gamma^2 \cong d \cdot \gamma$ for some $d \in \mathbb{N} \cup \{\infty\}$
(iii) M is the crossed product of N by an outer action of a discrete Woronowicz algebra.

Proof. It is a consequence of the following Proposition and [2, 3]. □

Proposition 6.4. *Let $N \subset M$ be an inclusion of infinite factors and $\gamma : M \to N$ the canonical endomorphism. Suppose that γ contains the identity sector and consider the following statements*
(i) $N \subset M$ is depth 2
(ii) $\gamma^2 \cong d \cdot \gamma$ for some $d \in \mathbb{N} \cup \{\infty\}$
(iii) there exists a sequence of finite index irreducible sectors $\{\rho_i\}_{i \in I} \subset Sect(M)$ which is a basis for the *-semiring generated by the same $\{\rho_i\}$, such that

$$\gamma = \oplus_i d(\rho_i)\rho_i.$$

Then $(i) \iff (ii) \Leftarrow (iii)$.

Proof. Up to tensoring with an absorbing factor [18], we may assume that $N = \rho(M)$ for some irreducible endomorphism $\rho \in End(M)$. We have $v \in (id, \rho\bar{\rho}) \subset (\rho, \rho\bar{\rho}\rho)$, $\bar{\rho}(v) \in (\bar{\rho}, \bar{\rho}\rho\bar{\rho})$ hence the inclusion is regular according to [2], sec.5.

By [2], Proposition 6.3, condition (i) means that $\rho\bar{\rho}\rho \cong d \cdot \rho$, whereas condition (ii) translates into $\rho\bar{\rho}\rho\bar{\rho} \cong d \cdot \rho\bar{\rho}$.
$(i) \Rightarrow (ii)$ Follows multiplying on the right by $\bar{\rho}$.
$(ii) \Rightarrow (i)$
Note that if $d \in \mathbb{N}$ the proof is contained in ([19], Lemma 6.3), in the general case we may proceed as follows.
Let us set $\sigma := \rho\bar{\rho}\rho$. Then $\sigma\bar{\rho} \cong d\rho\bar{\rho} \succ d \cdot id \succ id$. Therefore, by a repeated application of Proposition 6.1, we obtain $\sigma \cong k\rho \oplus \tau$, where $k \in \mathbb{N} \cup \{\infty\}$, $\tau \not\succ \rho$. We want to prove that $k = d$ and $\tau = 0$.
Suppose that $\tau \neq 0$. Then, as $\rho\bar{\rho}\rho \cong k\rho \oplus \tau$, we have $d\rho\bar{\rho} \cong \rho\bar{\rho}\rho\bar{\rho} \cong k\rho\bar{\rho} \oplus \tau\bar{\rho}$. Therefore $\tau\bar{\rho} \succ \rho\bar{\rho}$, then $\tau\bar{\rho} \succ id$, and, by Proposition 6.1 applied to $\bar{\rho}$, $\tau \succ \rho$ which is absurd. Then $\tau = 0$, so that $k = d$.

$(iii) \Rightarrow (ii)$ Due to completeness we have

$$\rho_i\rho_j = \oplus_k N^k_{ij}\rho_k$$

with

$$\sum_k N^k_{ij} d_k = d_i d_j$$

with finite sum, because of finite index condition. It follows from Frobenius reciprocity that

$$N^k_{ij}\rho_k \prec \rho_i\rho_j \iff N^k_{ij}\rho_i \prec \rho_k\bar{\rho}_j.$$

Again by completeness we get

$$\rho_k\bar{\rho}_j = \oplus_k N^i_{k,l(j)}\rho_i$$

where $j \to l(j)$ is the permutation relative to the conjugation, hence we have $N^k_{i,j} = N^i_{k,l(j)}$. Finally we have

$$\begin{aligned}\gamma^2 &= \bigoplus_{ijk} d_i d_j N^k_{i,j}\rho_k \\ &= \bigoplus_{ijk} d_i d_j N^i_{k,l(j)}\rho_k \\ &= \bigoplus_k \sum_{ij} d_i d_j N^i_{k,l(j)}\rho_k \\ &= \bigoplus_k \sum_j d_j d_k d_{l(j)}\rho_k \\ &= \bigoplus_k d_k (\sum_j d_j^2)\rho_k \\ &= d\cdot\gamma,\end{aligned}$$

where $d \equiv \sum_j d(\rho_j)^2$. □

We apply this result to Q-systems of semidiscrete type.

Corollary 6.5. *Let (M, Λ) be an irreducible semidiscrete Q-system s.t. M is an infinite factor and $\lambda^2 \cong d\cdot\lambda$. Then the dual Q-system $(\tilde{M}, \tilde{\Lambda})$ is the crossed product of (M, Λ) by an outer action of a discrete Woronowicz algebra.*

Proof. The irreducibility condition means that $N' \wedge M = Z(M) = \mathbb{C}$. The assertion now follows by the above considerations. □

We cannot give the result involving semicompact Q-systems. Even for actions of compact groups on factors we need additional conditions to assure that a crossed product of a factor by a prime action of a compact group is itself a factor, see [27], Section 21, and [23], Section IV.3.

This problem seems to be directly related to the fact that the irreducibility condition for both semidiscrete and semicompact Q-systems is equivalent to the irreducibility of the same kind of inclusion in Jones tower, namely the one with the conditional expectation, (see 5.2), hence we have the inapplicability of the result by Combes-Delaroche ([1]) for the dual statement in Theorem 5.6.

As we said previously, we now show that property (iii) in Proposition 6.4 characterises when a depth 2 factor-subfactor inclusion arises as the crossed product by a Kac algebra.

Theorem 6.6. *Let $N \subset M$ be an irreducible inclusion of infinite factors.*
(a) *The following are equivalent.*
(i) *M is the crossed product of N by an outer action of a compact Kac algebra.*
(ii) *there exists a sequence of finite index irreducible sectors $\{\rho_i\}_{i\in I} \subset Sect(M)$ which is a basis for the *-semiring generated by the same $\{\rho_i\}$, such that*

$$\gamma = \oplus_i d(\rho_i)\rho_i.$$

(b) *The following are equivalent.*
(i) *M is the crossed product of N by an outer action of a discrete Kac algebra.*
(ii) *there exists a sequence of finite index irreducible sectors $\{\rho_i\}_{i\in I} \subset Sect(N)$ which is a basis for the *-semiring generated by the same $\{\rho_i\}$, such that*

$$\gamma|_N = \oplus_i d(\rho_i)\rho_i.$$

Proof. We only prove part (a), part (b) being analogous.
$(i) \Rightarrow (ii)$ Assume M is the crossed product of N by an outer action of a compact Kac algebra, which is isomorphic to $\gamma(N)' \wedge N$ ([3]). Then $\gamma(M)' \wedge M$ is a discrete Kac algebra isomorphic to $\oplus_i \mathcal{B}(\mathcal{H}_i)$, and $d_i := \dim(\mathcal{H}_i) < \infty$ ([4]). Finally, if $E \in E(N, \gamma(M))$ is given by $E := \gamma(v^* \cdot v)$, and $E_1 := j_N \circ E^{-1} \circ j_N \in P(M,N)$, with $j_N := adJ_N$, is the dual weight, it follows from [4], that $E_1|_{\gamma(M)'\wedge M} = \sum_i d_i Tr_i$, where Tr_i is the canonical (unnormalized) trace on $\mathcal{B}(\mathcal{H}_i)$. Set $T := E \circ E_1 \in P(M, \gamma(M))$, let p_i be the minimal central projections in $\gamma(M)' \wedge M$, and $v_i \in M$ be isometries s.t. $p_i = v_i v_i^*$, so that, with $\sigma_i := v_i^* \gamma(\cdot) v_i \in End(M)$, we have $\gamma \cong \oplus \sigma_i$ and $\sigma_i(M)' \wedge M \cong \mathcal{B}(\mathcal{H}_i)$. Then, by [7] Theorem 6.6 and Corollary 6.10, $p_i \in (\gamma(M)' \wedge M)_T \cap \mathfrak{M}_T$, so that, by the following Lemma 6.7, we have $Ind(T_{p_i}) = (T_{p_i})^{-1}(p_i) = T^{-1}(T(p_i)p_i)p_i$. As $T(p_i) = d_i^2$, and $j_M \circ T^{-1} \circ j_M = \gamma^{-1} \circ T \circ \gamma$, so that $T^{-1}(p_i) = d_i^2$, we get $Ind(\sigma_i) = Ind(T_{p_i}) = d_i^4$. Decomposing each σ_i in irreducible equivalent sectors $\sigma_i \cong d_i \rho_i$, we get $d_i^2 = d(\sigma_i) = d_i d(\rho_i)$,

so that $d(\rho_i) = d_i$, and therefore $\gamma \cong \oplus_i d(\rho_i)\rho_i$.
Finally, as Proposition 6.4 (ii) is true, we obtain $\oplus_{ij} d_i d_j \rho_i\rho_j \cong \gamma^2 \cong d\gamma \cong \oplus d d_i \rho_i$, that is the irreducibles contained in $\rho_i\rho_j$ are a subset of $\{\rho_k\}$. Moreover $\oplus d_i\bar{\rho}_i \cong \bar{\gamma} \cong \gamma \cong \oplus d_i\rho_i$, so that $\bar{\rho}_i \cong \rho_{j(i)}$ for some $j(i)$. All this shows that $\{\rho_k\}$ is a basis for the *-semiring generated by $\{\rho_k\}$.
$(ii) \Rightarrow (i)$ By Theorem 6.3, $M = N \times_\alpha \mathcal{A}$, where $\mathcal{A}$ is a compact Woronowicz algebra, which appears as $\mathcal{A} = \gamma(N)' \wedge N$. The dual algebra $\hat{\mathcal{A}}$, which is a discrete Woronowicz algebra, is given by $\gamma(M)' \wedge M$ and is isomorphic to $\oplus_i \mathcal{B}(\mathcal{H}_i)$, $\dim \mathcal{H}_i = d(\rho_i)$. Let $E \in E(N, \gamma(M))$ be the unique expectation, $E_1 \in P(M, N)$ the dual weight, and consider $T := E \circ E_1 \in P(M, N)$. It is enough to prove that $\sigma_t^T = id$, see [3]. Let $\{p_i\}$ be the minimal central projections of $\gamma(M)' \wedge M$, then, using ([11], Proposition 2.8), we obtain that $E_1|_{\gamma(M)'\wedge M} \in P(\gamma(M)' \wedge M)$ and $p_i \in \mathfrak{M}_{E_1}$. Therefore, using next Lemma 6.7, we can define $T_{p_i} \in E(M, \sigma_i(M))$, where $\sigma_i := d(\rho_i)\rho_i$, and obtain $Ind(T_{p_i}) = d(\rho_i)^4$, which means that T_{p_i} is the minimal expectation. Therefore $T_{p_i}|_{\sigma_i(M)'\wedge M}$ is a trace. As $T|_{\gamma(M)'\wedge M} = \sum_i T(p_i) T_{p_i}|_{\sigma_i(M)'\wedge M}$, it is a trace. The thesis follows from ([7], Corollary 6.10). □

Lemma 6.7. *Let $A \subset B$ be an inclusion of von Neumann algebras, $G \in P(B, A)$, $p \in (A' \wedge B)_G \cap \mathfrak{M}_G$ a non-zero projection in B, and set $G_p := G(x)G(p)^{-1}p$, $x \in (B_p)_+$. Then $G_p \in E(B_p, A_p)$ and $(G_p)^{-1}(x) = G^{-1}(G(p)x)p$, $x \in (A'_p)_+$.*

Proof. The proof is the same as [10], Proposition 1.4, with obvious modifications. □

We remark that Corollary 6.5 can be stated also in the case of Kac algebras, with obvious minor modification.

7. Appendix

Although we have already given a characterization of the conjugate endomorphism (in the semidiscrete and semicompact cases) in ([5]), we give here a different one, based on the methods of this paper.

Proposition 7.1. *Let M be a properly infinite von Neumann algebra with separable predual, $\rho, \sigma \in End(M)$. Then the following are equivalent*

(i) *$E(M, \rho(M)) \neq \emptyset$, $\sigma = \bar{\rho}$,*
(ii) *$\sigma\rho \succ id$ and there is an isometry $v \in (id, \sigma\rho)$ s.t. $x \in M$, $xv = 0$ $\Rightarrow x = 0$,*

(iii) $\sigma\rho \succ id$*, with* $v \in (id, \sigma\rho)$ *an isometry s.t.*

$$s_{\sigma(M)'\wedge M}(vv^*) = 1$$
$$s_{\sigma\rho(M)'\wedge\sigma(M)}(vv^*) = 1.$$

Proof. $(i) \iff (iii)$ is ([5], Theorem 3.4)
$(i) \Rightarrow (ii)$
Let $E \in E(M, \rho(M))$ and consider the inclusion $\bar{\rho}\rho(M) \subset \bar{\rho}(M)$, and set $F := \bar{\rho} \cdot E \cdot \bar{\rho}^{-1} \in E(\bar{\rho}(M), \bar{\rho}\rho(M))$. From Proposition 2.3 it follows that there are an isometry $V \in \bar{\rho}\rho(M)$ and a choice of a canonical endomorphism γ' for the inclusion $\bar{\rho}\rho(M) \subset \bar{\rho}(M)$, s.t. $\gamma'(x)V = Vx$, $x \in \bar{\rho}\rho(M)$, and $V^*\gamma'(\cdot)V = F$.
Set $V = \bar{\rho}\rho(v)$, $v \in M$. Then $\bar{\rho}\rho(x)v = vx$, $x \in M$. Indeed, as $\gamma' = ad\Gamma|_{\bar{\rho}(M)} = \bar{\rho}\rho|_{\bar{\rho}(M)}$ ([16], Proposition 2.4), we get, for all $x \in M$, $\bar{\rho}\rho(\bar{\rho}\rho(x)v) = \bar{\rho}\rho(\bar{\rho}\rho(x))V = V\bar{\rho}\rho(x) = \bar{\rho}\rho(vx)$, and the thesis follows from the injectivity of $\bar{\rho}\rho$.
Besides $E(x) = \rho(v^*\bar{\rho}(x)v)$, $x \in M$. Indeed $\bar{\rho}\rho(v^*\bar{\rho}(x)v) = V\bar{\rho}\rho(\bar{\rho}(x))V = F(\bar{\rho}(x)) = \bar{\rho}(E(x))$, and the thesis follows from the injectivity of $\bar{\rho}$.
Therefore, from Proposition 2.3, it follows that $\Gamma^*VV^*\Gamma = \Gamma^*\bar{\rho}\rho(vv^*)\Gamma = vv^*$ is Jones' projection for the inclusion $\bar{\rho}\rho(M) \subset \bar{\rho}(M)$.
Hence, if $x \in M$ is s.t. $xv = 0$, then $xvv^* = 0$, so that $x = 0$, as vv^* is separating for M.
$(ii) \Rightarrow (i)$
Let us set $F(x) := \sigma\rho(v^*xv)$, $x \in \sigma(M)$. Then $F \in E(\sigma(M), \sigma\rho(M))$, as, if $x \in \sigma(M)$ is s.t. $F(x^*x) = 0$, then $\sigma\rho(v^*x^*xv) = 0$, that is $xv = 0$, so that $x = 0$.
Setting now, $f := vv^*$ and $L := \langle\sigma(M), f\rangle$, we want to show that $\sigma\rho(M) \subset \sigma(M) \subset L$ is Jones' basic construction for the inclusion $\sigma\rho(M) \subset \sigma(M)$, by using Proposition 2.2. So we must prove $s_{\sigma\rho(M)}(f) = 1$, which follows from ([5], Lemma 3.2), and $s_{Z(L)}(f) = 1$. But we have $s_{Z(L)}(f) \geq s_{\sigma(M)'\wedge M}(f) = 1$, as follows from ([5], Lemma 3.1 (i) and (ii)).
Therefore, by lemma 2.6, $\langle\sigma(M), f\rangle = M$.
Finally to show that σ is conjugate to ρ, set $\lambda := \sigma\rho$ and observe that Lemma 2.4 applied to λ gives that λ is a canonical endomorphism for $N \subset M$, hence ρ and σ are conjugate. $\square$

References

[1] Combes, F., Delaroche, C. *Groupe modulaire d' une espérance conditionnelle dans une algèbre de von Neumann*, Bull. Soc. Math. France, **103** (1975), 385-426.

[2] Enock, M., Nest, R., *Irreducible inclusion of factors, multiplicative unitaries and Kac algebras*, J. Funct. Anal., **137** (1996), 466-543.

[3] Enock, M., *Inclusions irreductibles des facteurs et unitaires multiplicatifs, II*, Preprint (1996).

[4] Enock, M., Schwartz, J. M., *Kac algebras and duality of locally compact groups*, Springer (1992).

[5] Fidaleo, F., Isola, T. *On the conjugate endomorphism in the infinite index case*, Math. Scand. **77** (1995), 289-300.

[6] Fidaleo, F., Isola, T. *Minimal expectations for inclusions with atomic centres*, International J. Math. **7** (1996), 307-327.

[7] Haagerup, U. *Operator valued weights in von Neumann algebras, II*, J. Funct. Anal. **33** (1979), 339-361.

[8] Halpern, H, Kaftal, V., Zsido, L. *Finite weight projections in von Neumann algebras*, Pacific J. Math. **147** (1991), 81-121.

[9] Hewitt, E., Ross, K. A. *Abstract harmonic analysis II.* Springer, New York, 1970.

[10] Hiai, F. *Minimum index for subfactors and entropy*, J. Op. Th. **24** (1990), 301–336.

[11] Izumi, M., Longo, R., Popa, S., *A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras*, Preprint (1996).

[12] Kadison, R. V., Ringrose, J. R. *Fundamentals of the theory of operator algebras. I, II*, Academic Press, 1983,1986.

[13] Kosaki, H. *Extension of Jones' theory of index to arbitrary factors*, J. Funct. Anal. **66** (1986), 123–140.

[14] Kosaki, H. *Characterization of crossed product (properly infinite case)*, Pacific J. Math. **137** (1989), 159–167.

[15] Longo, R. *Simple injective subfactors*, Adv. Math., **63**, (1987), 152–171.

[16] Longo, R., *Index of subfactors and statistics of quantum fields. I*, Commun. Math. Phys. **126** (1989), 217–247.

[17] Longo, R., *Index of subfactors and statistics of quantum fields. II*, Commun. Math. Phys. **130** (1990), 285–309.

[18] Longo, R., *Minimal index and braided subfactors*, J. Funct. Anal. **109** (1992), 98–112.

[19] Longo, R. *A duality for Hopf algebras and for subfactors. I*, Commun. Math. Phys., **159**, (1994), 133–150.

[20] Longo, R., Rehren, K. H. *Nets of subfactors*, Rev. Math. Phys., **7** (1995), 567-597.

[21] Mack, G., Schomerus, V. *Quasi Hopf quantum symmetry in quantum theory*, Nucl. Phys., **B370**, (1992), 185–

[22] Masuda, T., Nakagami, Y. *A von Neumann algebra framework for the duality of the quantum groups*, Publ. R.I.M.S. Kyoto, **30** (1994), 799–850.

[23] Nakagami, Y., Takesaki, M. *Duality for crossed products of von Neumann algebras*, Lecture Notes in Math., **731**, Springer-Verlag, New York (1979).

[24] Paschke, W. L., *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc., **182** (1973), 443–468.
[25] Pimsner, M., Popa, S., *Entropy and index for subfactors*, Ann. Scient. Ec. Norm. Sup. (4), **19** (1986), 57–106.
[26] Popa, S., *Classification of subfactors and of their endomorphisms*, CBMS Lecture Notes Series n. 86, American Math. Soc., Providence, Rhode Island, 1995.
[27] Stratila, S., *Modular theory in operator algebras*, Abacus Press, Tunbridge Wells, Kent, England (1981).
[28] Stratila, S., Zsido, L. *Lectures on von Neumann algebras.* Abacus Press, Turnbridge Wells, 1979.
[29] Takesaki, M., *Theory of operator algebras, I*, Springer Verlag, Berlin, Heidelberg, New York, 1979.

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