MINIMAL EXPECTATIONS FOR INCLUSIONS WITH ATOMIC CENTRES

FRANCESCO FIDALEO AND TOMMASO ISOLA

Abstract. We study the set of minimal conditional expectations for inclusions of von Neumann algebras with atomic centres. Contrary to the case of finite dimensional centres, the set of minimal conditional expectations with scalar index consists, in general, of more than one element. Some calculations relative to the connection between minimal expectations and entropy are also done. The last section is devoted to the existence of conditional expectations preserving a Markov trace. Simple examples show that all possibilities can occur.

Mathematics Subject Classification Numbers: Primary 46L37, Secondary 46L35.

1. Introduction

The notion of index was introduced by V. F .R. Jones as an invariant for inclusions of II_1 factors [14] although this concept appeared in preceding works [6], and, independently, in the theory of statistical dimension in quantum field theory [3]. The index was generalized to an arbitrary factor-subfactor inclusion $N \subset M$ by H. Kosaki [17] as a scalar associated to a faithful normal conditional expectation $E : M \to N$. F. Hiai [9] and, independently, R. Longo [19] introduced the notions of minimal index and minimal conditional expectation showing that for a factor-subfactor inclusion with finite index there exists a unique conditional expectation whose index is minimal. Moreover Hiai also established a remarkable connection between minimal index and entropy:

$$K_E(M,N) = \log Ind(E) \tag{1}$$

if and only if E is the minimal expectation, though M. Pimsner and S. Popa ([23]) were the first to study the relation between entropy and index in the II_1 context. Longo ([19]) pointed out the connection between the minimal index and the statistical dimension of a sector in QFT. Remarkable additivity and multiplicativity properties of the square root of the minimal index are also obtained in [18], [20], [21].

Jones' work greatly stimulated the study of the classification of inclusions (see [24] and references therein). In many of these works remarkable relations with topological invariants of knots and links, as well as representations of Artin braid group were established (see, for example, [4], [30]).

Successively J. F. Havet [8] and T. Teruya [28] studied the notion of minimal expectation for inclusions of von Neumann algebras with finite dimensional centres. In this case also, the above authors prove that there exists a unique minimal conditional expectation and its index, a function in $Z(M)$, is automatically a scalar, if the connectedness assumption $Z(N) \wedge Z(M) = \mathbb{C}$ is made. Moreover the same connection (1) with the entropy is established by Teruya, whereas Havet shows the existence of a unique Markov trace (strictly related to invariants of links, see [15], [16]).

In this paper we continue this program studying the set of minimal conditional expectations in the case of atomic centres.

We start with a preliminary section where we show why we can study inclusions with atomic centres, and inclusions with diffuse centres separately. In section 3 we set the stage for our problem and, in section 4 we prove, assuming the connectedness hypothesis $Z(N) \wedge Z(M) = \mathbb{C}$, the following result. There exists a minimal conditional expectation $E : M \to N$ with scalar index and such that $E_f : M_f \to N_f$ is minimal for the factor-subfactor inclusion $N_f \subset M_f$, f minimal projection of $Z(N) \vee Z(M)$; in general, the set of minimal conditional expectations is a compact convex set containing more than one element, but, if it consists of only one element, the two properties above are automatically satisfied. In section 5 we describe simple examples which illustrate these facts. In section 6 we develop some considerations about the connection between minimal index and entropy, some calculations are also made. Finally, section 7 is devoted to the existence of Markov traces for an inclusion $N \subset M$ with atomic (infinite dimensional) centres and connectedness assumption $Z(N) \wedge Z(M) = \mathbb{C}$. Contrary to the case of finite dimensional centres, all possibilities can occur.

Before ending this introduction we want to point out a remarkable connection established between the square root of the minimal index and the theory of dimension for tensor C^*-categories ([22]) where the most important results are established assuming the irreducibility hypothesis $(i, i) = \mathbb{C}$ for the unity object. The detailed knowledge of the set of conditional expectations for an inclusion of von Neumann algebras with arbitrary centres, and principally the minimal ones or those which preserve a Markov trace, could play a twofold role in the context of (braided) tensor C^*-categories without the irreducibility assumption, that is in the theory of dimension, and in establishing new connections with links and knots invariants.

2. Preliminaries

In this section we recall some known results on the theory of index, initiated by V. Jones [14], both for ease of reference and for fixing notations. Throughout the paper we assume that all von Neumann algebras have separable predual, and use the following notation: if $N \subset M$ are von Neumann algebras, $E(M, N)$ is the set of all normal faithful conditional expectations from M to N, and $E(M)$ is the set of all normal faithful states on M.

Let us now recall the definition of H. Kosaki's index ([17]) based on A. Connes' spatial theory and U. Haagerup's operator valued weights (see [26]). If $N \subset M$ are von Neumann algebras, associated to every $E \in E(M, N)$ there is an M'-valued operator weight on N', E^{-1}, uniquely determined by

$$d(\varphi \cdot E)/d\psi = d\varphi/d(\psi \cdot E^{-1}),$$

for all normal faithful weights φ on N, ψ on M'. Observe that $E^{-1}(1) \in \overline{Z(M)}_+$, the extended positive part of $Z(M)$, and does not depend on the representation of M (as the same proof of [17], Theorem 2.2 works).

Definition 1. If $N \subset M$ are von Neumann algebras and $E \in E(M, N)$, we say that E has finite index if $E^{-1}(1) \in Z(M)$, and that $Ind(E) := E^{-1}(1)$ is the index of E.
$N \subset M$ has finite index if there exists $E \in E(M, N)$ with finite index.

Let now $N \subset M$ be an inclusion of von Neumann algebras with finite index and let $E \in E(M, N)$. Let $\varphi \in E(N)$, and set $\psi := \varphi \cdot E \in E(M)$; let $\Psi \in \mathcal{H} \equiv \mathcal{H}_\psi$ cyclic and separating for M and such that $\psi = (\Psi, \cdot\Psi)$, and set $e := [N\Psi] \in N'$, the Jones projection of the inclusion. Then one has

Proposition 1. ([17])
(i) $E^{-1}(e) = 1$. *In particular* $Ind(E) \geq 1$ *and* $Ind(E) = 1 \iff N = M$.
(ii) $\langle M, e\rangle \equiv span\{aeb : a, b \in M\} \equiv JN'J$, *where* $J \equiv J_M^\Psi$. *This algebra is called Jones basic construction.*

(iii) If J is a modular conjugation for M, and $j := adJ$, then $E^1 := Ind(E)^{-1}j \cdot E^{-1} \cdot j(\cdot) \in E(\langle M, e\rangle, M)$ and $Ind(E^1) = j(E(Ind(E)))$; E^1 is the expectation dual of E.

Definition 2. We say $N \subset M$ is connected, if $Z(N) \wedge Z(M) = \mathbb{C}$.

The property of connectedness for an inclusion $N \subset M$ is related to some type of connectedness for the spectrum of the commutative von Neumann algebra $Z(N) \vee Z(M)$, viewed as a measure space on the cartesian product of the spectra of $Z(N)$ and $Z(M)$. In the case of atomic centres, this property directly translates in the connectedness of a graph naturally associated to the inclusion (see section 3).
The following results show that we can reduce our study of general inclusions to the cases of connected inclusions with atomic or diffuse centres separately.

Proposition 2. ([11]) *Let $N \subset M$ be von Neumann algebras, $E \in E(M, N)$. If $N = \int^{\oplus} N_\omega d\mu(\omega)$, $M = \int^{\oplus} M_\omega d\mu(\omega)$ are their decompositions with respect to $L^\infty(\Omega, \mu) \cong Z \subset Z(N) \wedge Z(M)$, then for almost all ω, there exists $E_\omega \in E(M_\omega, N_\omega)$ such that $E(x) = \int^{\oplus} E_\omega(x_\omega) d\mu(\omega)$, $\forall x = \int^{\oplus} x_\omega d\mu(\omega) \in M$. Besides $Ind(E) = \int^{\oplus} Ind(E_\omega) d\mu(\omega)$.*

Proposition 3. ([10], *Proposition 3.5 and Corollary 4.4) Let $N \subset M$ be von Neumann algebras with finite index. Then the following are equivalent*
(i) $Z(M)$ is atomic,
(ii) $Z(N)$ is atomic,
(iii) $N' \wedge M$ is atomic,
(iv) $Z(N' \wedge M)$ is atomic.

Theorem 1. *Let $N \subset M$ be a finite index inclusion of von Neumann algebras. Then there exists a projection $z \in Z(N) \wedge Z(M)$ s.t. $N_z \subset M_z$ has a common decomposition with almost all fibres having atomic centres, whereas $N_{1-z} \subset M_{1-z}$ has a common decomposition with almost all fibres having diffuse centres.*

Proof. Let $N = \int^{\oplus} N_\omega d\mu(\omega)$, $M = \int^{\oplus} M_\omega d\mu(\omega)$ be their decompositions with respect to $L^\infty(\Omega, \mu) \cong Z(N) \wedge Z(M)$ and set $\Omega_z := \{\omega \in \Omega : Z(M_\omega)$ is atomic $\}$, which is a measurable set, as follows from [27].

Then $Z(N_\omega)$ is atomic, for almost all $\omega \in \Omega_z$, as follows from the previous propositions. From next proposition $Z(M_\omega)$ is diffuse, for almost all $\omega \in \Omega \setminus \Omega_z$, so that $Z(N_\omega)$ is diffuse too, because of Proposition 3. So the projection associated to Ω_z is the one we were looking for. □

Proposition 4. *Let $N \subset M$ be a finite index inclusion of von Neumann algebras s.t. $Z(N) \wedge Z(M) = \mathbb{C}$. Then $Z(N)$ and $Z(M)$ are either both atomic or both diffuse.*

Proof. Let $p \in Z(M)$, resp. $q \in Z(N)$, be the projections corresponding to the atomic part of $Z(M)$, resp. $Z(N)$, and let $c \in Z(N)$ be the support of p in $Z(N)$. As $N_c \cong N_p$, $Z(N_c)$ is atomic, because of Proposition 3, so that $p \le c \le q$. Suppose now that $q - p \neq 0$, then, as $N_{q-p} \subset M_{q-p}$ has finite index, we obtain that $Z(N_{q-p})$ is diffuse, which is absurd. □

In the sequel the atomic case will be considered, but we hope to discuss the diffuse case in the future.

3. Inclusions with atomic centres

Let $N \subset M$ be a finite index inclusion of von Neumann algebras with atomic centres, and let $\{p_i\}_{i\in I}$, $\{q_j\}_{j\in J}$ be the minimal central projections of M, N respectively.
Let us set $\mathcal{A} := \{(i,j) \in I \times J : p_i q_j \neq 0\}$. Introduce the shorthand notation $i \sim j \iff j \sim i \iff (i,j) \in \mathcal{A}$, and observe that for all $i \in I$ there are only finitely many $j \in J$ s.t. $j \sim i$, and for all $j \in J$ there are only finitely many $i \in I$ s.t. $i \sim j$. This follows easily from the fact that $E_{p_i}(x) := E(x)E(p_i)^{-1}p_i$ has finite index ([10], Proposition 1.4) so that N_{p_i} has finite dimensional centre that is $p_i q_j \neq 0$ for only finitely many j's; passing to commutants we get the second half of the statement.
Let us set $M_i := M_{p_i}$, $N_j := N_{q_j}$, $M_{ij} := M_{p_i q_j}$, $N_{ij} := N_{p_i q_j}$ and let $\sigma_{ij} : N_{ij} \to N_j$ be the inverse of the induction isomorphism $N_j \to N_{ij}$. Then

Proposition 5.
(i) *There exists a unique matrix of positive real numbers $\gamma^E := [\gamma_{ij}]$ s.t. $E(p_i) = \sum_j \gamma_{ij} q_j$. This matrix is column-Markovian i.e. $\sum_i \gamma_{ij} = 1$, and $\gamma_{ij} \neq 0 \iff i \sim j$.*

(ii) For all $(i,j) \in \mathcal{A}$ *one can define* $E_{ij} \in E(M_{ij}, N_{ij})$ *by* $E_{ij}(x) := \gamma_{ij}^{-1} E(x) p_i$, $x \in M_{ij}$.
(iii) One has $E(x) = \sum_{(i,j)\in\mathcal{A}} \gamma_{ij}\sigma_{ij}(E_{ij}(p_i q_j x q_j))$, $x \in M$, *where the sum converges in the ultraweak topology.*
(iv) Conversely, given γ *a column-Markovian matrix s.t.* $\gamma_{ij} \neq 0 \iff (i,j) \in \mathcal{A}$, *and* $F_{ij} \in E(M_{ij}, N_{ij})$ *for all* $(i,j) \in \mathcal{A}$, *there exists a unique* $E \in E(M,N)$, *given by* $E(x) = \sum_{(i,j)\in\mathcal{A}} \gamma_{ij}\sigma_{ij}(F_{ij}(p_i q_j x q_j))$, $x \in M$, *s.t.* $\gamma^E = \gamma$, *and* $E_{ij} = F_{ij}$, $(i,j) \in \mathcal{A}$.

Proof. It is analogous to ([8], Propositions 2.2, 2.3). □

Definition 3. We call index-matrix the matrix α given by

$$\alpha_{ij} = \begin{cases} 0 & \text{if } (i,j) \notin \mathcal{A}, \\ Ind(E_{ij}) & \text{if } (i,j) \in \mathcal{A}. \end{cases}$$

Proposition 6. *Let* $E \in E(M,N)$ *be of finite index. Then* α *has finite entries, and*

$$Ind(E) = \sum_i \Big(\sum_j \frac{\alpha_{ij}}{\gamma_{ij}}\Big) p_i \tag{2}$$

Proof. The proof of [8], Theorem 2.5, also works in this case. One has only to observe that

$$(\oplus_{ijl}\Lambda_E(p_i q_l M q_j))^{-uw} = \Lambda_E(M)$$

where we use Λ_E to identify the algebras with their images in X_E, and

$$(\oplus_{ijl}\Lambda_E(p_i q_l M q_j))^{-\sigma} = \Lambda_E(M)^{-\sigma} = X_E$$

as the σ-topology is weaker then the ultraweak topology (see [2], par. 1.2). Hence, a (possibly infinite) Pimsner-Popa basis for X_E can be choosen collecting the bases of all the orthogonal summands $\Lambda_E(p_i q_l M q_j)$. □

Propositions 5 and 6 can be also shown analogously to [28].

Let us denote by $f_i(\alpha,\gamma) := \sum_j \frac{\alpha_{ij}}{\gamma_{ij}}$ and by $f(\alpha,\gamma) := \sup_i f_i(\alpha,\gamma)$ so that $\|Ind(E)\| \equiv f(\alpha,\gamma)$. We denote simply by $f_i(\gamma)$ and $f(\gamma)$ these functions, if the matrix α is understood from the context.

Remark 1. For the dual conditional expectation $E^1 \in E(M^1, M)$ one easily obtains $\alpha^1_{ji} = \alpha_{ij}$ and $\gamma^1_{ji} = \frac{\frac{\alpha_{ij}}{\gamma_{ij}}}{\sum_h \frac{\alpha_{ih}}{\gamma_{ih}}}$.

Let us recall ([8]) that one can associate to the inclusion $N \subset M$ a bipartite graph $\mathcal{G}$ (see [29] for graph theoretical notions) where the upper vertices correspond to the $\{p_i\}$'s and the lower ones to the $\{q_j\}$'s, and there is an edge between i and j iff $p_i q_j \neq 0$. It is easy to see that the above graph is connected iff $Z(N) \wedge Z(M) = \mathbb{C}$, that it is locally finite if there is a conditional expectation E of finite index, and the valency of every vertex is uniformly bounded by $\|Ind(E)\|$. Due to the important role played in this paper by one particular graph we introduce for it a special notation, that is we denote by $\mathcal{G}_\infty$ the bipartite graph specified by the connection matrix $[a_{ij}]_{i,j\in\mathbb{N}}$, where

$$a_{ij} := \begin{cases} 1 & \text{if } j = i, i-1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Proposition 7. *Let $N \subset M$ be a connected inclusion with infinite dimensional centres. Then for all $E \in E(M,N)$ one has $\|Ind(E)\| \geq 4$.*

Proof. Due to the connectedness of $\mathcal{G}$ we can choose a subgraph of $\mathcal{G}$ isomorphic to $\mathcal{G}_\infty$ and this will select $I' \subset I$ and $J' \subset J$. Then one has

$$\|Ind(E)\| = \sup_{i\in I} \sum_{j\in J} \frac{\alpha_{ij}}{\gamma_{ij}} \geq \sup_{i\in I'} \sum_{j\in J'} \frac{\alpha_{ij}}{\gamma_{ij}} \geq \sup_{i\in I'} \sum_{j\in J'} \frac{1}{\hat{\gamma}_{ij}} \geq 4$$

where we have chosen $\hat{\gamma}_{ij} \geq \gamma_{ij}$ s.t. $\sum_{i\in I'} \hat{\gamma}_{ij} = 1$, for all $j \in J'$, and the last inequality follows from a result on $\mathcal{G}_\infty$ proved in Section 5. □

Proposition 8. *If $E \in E(M,N)$ is of finite index then*

$$1 \leq \alpha_{ij} \leq a, \qquad \gamma_{ij} \in [\varepsilon, 1-\varepsilon], \qquad (i,j) \in I \times J,$$

for suitable $a, \varepsilon \geq 0$.

Proof. Follows by direct computation. □

4. The set of Minimal expectations

Let us introduce a partition of $E(M,N)$. For every matrix of expectations $\mathbb{G} := \{G_{ij}\}$, $G_{ij} \in E(M_{ij}, N_{ij})$, let $\mathcal{E}(\mathbb{G}) := \{E \in E(M,N) : E_{ij} = G_{ij}\}$, so that every expectation in $\mathcal{E}(\mathbb{G})$ is uniquely determined by a matrix $\gamma \in \Gamma := \{\gamma \in [0,1]^{\mathcal{A}} : \sum_i \gamma_{ij} = 1,\ \forall j\}$, and we sometimes identify $E \in \mathcal{E}(\mathbb{G})$ with its associated $\gamma \in \Gamma$. Conversely $\gamma \in \Gamma$ determines an expectation which is faithful iff $\gamma_{ij} > 0$, $(i,j) \in \mathcal{A}$. Introduce the topology $\mathcal{T}$ on $\mathcal{E}(\mathbb{G})$ defined as $E^n \to E$ in the $\mathcal{T}$-topology $\iff$ $E^n(x)q_j \to E(x)q_j$ in norm, for all $x \in M$, $j \in J$. Observe that convergence in this topology is equivalent to $\gamma^n_{ij} \to \gamma_{ij}$, for all $i \in I$, $j \in J$, for the associated γ's.

Recall from [8], Lemme 2.8, the following

Proposition 9. *Let $N \subset M$ be a finite-index inclusion of von Neumann algebras with finite dimensional centres, and $Z(N) \wedge Z(M) = \mathbb{C}$, then for every matrix of expectations $\mathbb{G}$, one has*
(i) *there is a unique $E \in \mathcal{E}(\mathbb{G})$ s.t. $\|Ind(E)\| = \inf\{\|Ind(F)\| : F \in \mathcal{E}(\mathbb{G})\}$,*
(ii) *the above E has scalar index.*

In the atomic infinite dimensional case one does not have the uniqueness (see the examples in section 5); indeed

Proposition 10. *Let $N \subset M$ be a finite-index inclusion of von Neumann algebras with atomic centres, and $Z(N) \wedge Z(M) = \mathbb{C}$, then for every matrix of expectations $\mathbb{G}$, one has*
(i) *there is $E \in \mathcal{E}(\mathbb{G})$ s.t. $\|Ind(E)\| = \inf\{\|Ind(F)\| : F \in \mathcal{E}(\mathbb{G})\}$,*
(ii) *the above E has scalar index.*

Proof. Let us introduce the following subsets of the vertices of $\mathcal{G}$ defined inductively by: $I_1 := \{1\}$, $J_1 := \{j \in J : j \sim 1\}$, $I_n := \{i \in I : \exists\ j \in J_{n-1}, j \sim i\}$, $J_n := \{j \in J : \exists\ i \in I_n, i \sim j\}$, and observe that $\{I_n\}$, $\{J_n\}$ are all finite because of the finite-index condition and $\cup I_n = I$, $\cup J_n = J$. Observe that $\Gamma := \{\gamma \in [0,1]^{\mathcal{A}} : \sum_i \gamma_{ij} = 1,\ \forall j\}$ is compact in the product topology and $f(\gamma)$ is lower semicontinuous, so that there is $\gamma^0 \in \Gamma$ s.t. $f(\gamma^0) = \min f(\gamma)$. Let $\gamma^n \in \Gamma$ be s.t. $\sup_{k\in I_n} f_k(\gamma^n) = \min_\Gamma \sup_{k\in I_n} f_k(\gamma)$, which, by the previous proposition, exists and satisfies $f_k(\gamma^n) = const,\ k \in I_n$, and

$$f(\gamma^0) = \sup_{k\in I} f_k(\gamma^0) \geq \sup_{k\in I_n} f_k(\gamma^0) \geq \sup_{k\in I_n} f_k(\gamma^n).$$

In fact one can choose γ^n as the unique minimal solution for $(i,j) \in I_{n+1} \times J_n$ and s.t. $\gamma^n_{ij} > 0$ otherwise. Observe that $\forall i \in I, \exists \nu_i \in \mathbb{N}$ s.t. $\{1, 2, \dots, i\} \subset I_n$ and $f_i(\gamma^n) = f_{i+1}(\gamma^n)$, for all $n \geq \nu_i$. Let $\tilde\gamma$ be a limit point of $\{\gamma^n\}$ for the product topology, and $\gamma^{n_h} \to \tilde\gamma$, then, by lower semicontinuity of $\sup_{k\leq i} f_k$, one gets

$$\sup_{k\leq i} f_k(\tilde\gamma) \leq \liminf_{h\to\infty}\ \sup_{k\leq i} f_k(\gamma^{n_h}) \leq f(\gamma^0)$$

so that $f(\tilde{\gamma}) \leq f(\gamma^0)$ and, by minimality of γ^0, we get equality.
Finally we obtain

$$f_i(\tilde{\gamma}) = \lim_h f_i(\gamma^{n_h}) = \lim_h f_{i+1}(\gamma^{n_h}) = f_{i+1}(\tilde{\gamma})$$

from which we conclude that $f_i(\tilde{\gamma}) = const.$ □

Choosing the expectations G_{ij} as the minimal ones, we obtain

Theorem 2. *Let $N \subset M$ be a finite-index inclusion of von Neumann algebras with atomic infinite dimensional centres, and $Z(N) \wedge Z(M) = \mathbb{C}$. Then there exists an expectation E which satisfies:*
(i) $\|Ind(E)\| = \inf\{\|Ind(F)\| : F \in E(M, N)\}$,
(ii) *E has scalar index,*
(iii) $Ind(E_f) = \min\{Ind(F) : F \in E(M_f, N_f)\}$, *for all minimal projections $f \in Z(M) \vee Z(N)$.*

Proof. Let E_{ij} be the minimal expectation for the inclusion $N_{ij} \subset M_{ij}$, and let E be the expectation obtained in the previous proposition for this choice of the matrix of expectations; we want to prove that E satisfies (i).
So, let $F \in E(M, N)$ be s.t. $\|Ind(F)\| \leq \|Ind(E)\|$, and let $G \in E(M, N)$ be obtained from γ^F and $\{E_{ij}\}$, as in Proposition 5 (iv). Then $Ind(G) \leq Ind(F)$ so that $\|Ind(G)\| \leq \|Ind(F)\| \leq \|Ind(E)\|$, but, from the construction of E we have $\|Ind(G)\| \geq \|Ind(E)\|$, therefore $\|Ind(F)\| = \|Ind(E)\|$ and the thesis follows.

□

Definition 4. Let us call minimal an expectation which satisfies (i) of the above theorem, and special an expectation which satisfies $(i)-(iii)$ of the above theorem.

Remark 2. In [13], Theorem 1.8, Jolissaint proves the existence of minimal expectations in full generality, but he doesn't show that this expectation also satisfies other properties like those of Theorem 2.

Proposition 11. *Let $N \subset M$ be as above, then special expectations E are characterized by:*
(i) *E has scalar index,*
(ii) $Ind(E) = \inf\{Ind(F) : F \in E(M, N), Ind(F) \in \mathbb{R}\}$,
(iii) $Ind(E_f) = \min\{Ind(F) : F \in E(M_f, N_f)\}$, *for all minimal projections $f \in Z(M) \vee Z(N)$.*

Proof. Let E^0 be a special expectation, then the thesis follows from $Ind(E) \leq Ind(E^0) = \|Ind(E^0)\| \leq \|Ind(E)\| = Ind(E)$ which implies E is special. $\square$

In the following, if not otherwise mentioned, we deal only with connected inclusions with finite index and atomic infinite dimensional centres. In this context we now study the set of minimal expectations more closely.

Proposition 12. *Let $\mathbb{G}$ be a fixed matrix of expectations, and set $\mathcal{M} \equiv \mathcal{M}(\mathbb{G}) := \{E \in \mathcal{E}(\mathbb{G}) : \|Ind(E)\| = \inf\{\|Ind(F)\| : F \in \mathcal{E}(\mathbb{G})\}\}$. Then $\mathcal{M}$ is a $\mathcal{T}$-compact convex set.*

Proof. Let $E^1, E^2 \in \mathcal{M}$, $t \in (0,1)$, and set $E := (1-t)E^1 + tE^2$; then $f_k(\gamma) \leq (1-t)f_k(\gamma^1) + tf_k(\gamma^2)$, $\forall k$, so that $\|Ind(E)\| \leq (1-t)\|Ind(E_1)\| + t\|Ind(E_2)\|$ and $E \in \mathcal{M}$.
$\mathcal{M}$ is compact as it is closed in Γ: let $\gamma^n \to \gamma^0$, $\gamma^n \in \mathcal{M}$; then $f(\gamma^0) \leq \liminf_n f(\gamma^n) = \min\{\|Ind(F)\| : F \in \mathcal{E}(\mathbb{G})\}$, by lower semicontinuity of f; so that $E \in \mathcal{M}$.
$\square$

Definition 5. Let us say that $i', i'' \in I$ are at distance k if the shortest path in $\mathcal{G}$ from i' to i'' has length $2k$. Analogously for $j', j'' \in J$.

Proposition 13. *The set of extremal points of $\mathcal{M}(\mathbb{G})$ consists precisely of the expectations $E = E(\gamma)$ that satisfy the following property: set $d := \|Ind(F)\|$, $F \in \mathcal{M}$, then the distance between two indices $i_1, i_2 \in I$ s.t. $f_{i_1}(\gamma) < d$, $f_{i_2}(\gamma) < d$ is at least two.*

Proof. Let $E \in \mathcal{M}$ satisfy the above property and set $S := \{i \in I : f_i(\gamma) < d\}$, and suppose that there are $E^1, E^2 \in \mathcal{M}$ s.t. $E = (1-t)E^1 + tE^2$, with $t \in (0,1)$. Then, for all $i \notin S$, $d = f_i(\gamma) \leq (1-t)f_i(\gamma^1) + tf_i(\gamma^2) \leq d$, so that $\frac{1}{\gamma_{ij}} = (1-t)\frac{1}{\gamma^1_{ij}} + t\frac{1}{\gamma^2_{ij}}$, for all $i \notin S$, $j \sim i$. Then $\gamma^1_{ij} = \gamma^2_{ij} = \gamma_{ij}$, for all $i \notin S$, $j \sim i$. Finally, as for $i \in S$ $\gamma_{ij} = 1 - \sum_{k \notin S} \gamma_{kj}$, for all $j \sim i$, and analogously for γ^1_{ij} and γ^2_{ij}, we get $\gamma^1_{ij} = \gamma^2_{ij} = \gamma_{ij}$, for all i, j. That is $E^1 = E^2 = E$, and E is extremal.
Conversely, let us suppose that there are $i_1, i_2 \in I$, at distance one, s.t. $f_{i_1}(\gamma) < d$, $f_{i_2}(\gamma) < d$, and let j_0 be connected to i_1 and i_2. Let us

choose $\varepsilon > 0$ s.t., with

$$\gamma^1_{ij} = \begin{cases} \gamma_{i_1 j_0} + \varepsilon & \text{if } i = i_1,\ j = j_0, \\ \gamma_{i_2 j_0} - \varepsilon & \text{if } i = i_2,\ j = j_0, \\ \gamma_{ij} & \text{otherwise,} \end{cases}$$

$$\gamma^2_{ij} = \begin{cases} \gamma_{i_1 j_0} - \varepsilon & \text{if } i = i_1,\ j = j_0, \\ \gamma_{i_2 j_0} + \varepsilon & \text{if } i = i_2,\ j = j_0, \\ \gamma_{ij} & \text{otherwise,} \end{cases}$$

we have $f_{i_1}(\gamma^1), f_{i_2}(\gamma^1), f_{i_1}(\gamma^2), f_{i_2}(\gamma^2) < d$. Then $E = \frac{1}{2}(E^1 + E^2)$, with $E^1, E^2 \in \mathcal{M}$, so that E is not extremal. □

Remark 3. As $\mathcal{M}$ is $\mathcal{T}$-metrizable, Choquet's theory ([1]) asserts that the extreme boundary $\partial\mathcal{M}$ is a G_δ set and every $E \in \mathcal{M}$ is represented as the barycentre of a Borel probability measure supported on $\partial\mathcal{M}$. In general $\mathcal{M}$ is not a simplex (see Example 1), hence the above measure is not unique.

Remark 4. If the matrix $\mathbb{G}$ consists of minimal expectations for the inclusions $N_{ij} \subset M_{ij}$, and $\mathcal{M}(\mathbb{G})$ consists of only one element, then the set of all minimal expectations for the inclusion $N \subset M$ consists also of one element.

5. Some illustrative examples

The following examples, which are chosen to point out some aspects of the theory, are given in terms of graphs and matrices α's and γ's over them, but they can be easily translated in terms of connected inclusions of von Neumann algebras with atomic centres, and expectations between them via Proposition 26 in the Appendix (fixing an arbitrary trace-matrix).

Example 1. Let us consider the graph $\mathcal{G}_\infty$, with connection matrix $[a_{ij}]$ given by (3). Let $a, \lambda > 0$ be s.t. $\lambda, a\lambda$ belong to the Jones set $\{4\cos^2 \frac{\pi}{n} : n \in \mathbb{N}, n \geq 3\} \cup [4, \infty)$, and set

$$\alpha_{ij} := \begin{cases} a\lambda & \text{if } i = j = 1, \\ \lambda & \text{otherwise.} \end{cases}$$

We want to find the unique special expectation given by Theorem 2 corresponding to an inclusion with the index-matrix α as above; one has to find the minimum d which satisfies the following system

$$\begin{cases} \frac{a\lambda}{\gamma_{11}} = d \\ \frac{\lambda}{1-\gamma_{k-1,k-1}} + \frac{\lambda}{\gamma_{kk}} = d, \ k > 1. \end{cases}$$

By means of direct calculations one finds

$$d = \begin{cases} 4\lambda & \text{if } 0 < a \leq 2, \\ \frac{a^2\lambda}{a-1} & \text{if } a \geq 2. \end{cases}$$

Indeed, the sequence

$$\begin{cases} \gamma_{kk} = \frac{\lambda(1-\gamma_{k-1,k-1})}{d(1-\gamma_{k-1,k-1})-\lambda} \\ \gamma_{11} = \frac{a\lambda}{d} \end{cases}$$

converges iff $d = 4\lambda$ and $0 < a \leq 2$ or $d > 4\lambda$ and $a \leq \frac{d+\sqrt{d^2-4d\lambda}}{2\lambda}$. Besides, if $0 < a \leq 2$, $d = 4\lambda$ is the minimum, whereas, if $a > 2$ the minimum d is found by solving

$$\frac{d+\sqrt{d^2-4d\lambda}}{2d} = \frac{a\lambda}{d}.$$

Using the above results, we obtain examples of:
1) inclusions with only one minimal expectation, just take $a \geq 2$, and think of α_{ij} as the minimal indices;
2) inclusions with lots of minimal expectations, which satisfy (iii) but not (ii) of Theorem 2, just take $0 < a < 2$, λ large enough and $\gamma_{11} \geq \frac{a}{4}$ and think of α_{ij} as the minimal indices;
3) inclusions with minimal expectations with scalar index which don't satisfy (iii), just take $a = 2$ and $\lambda \geq 4$, but think of λ as the minimal index for all the factor-subfactor inclusions $N_{ij} \subset M_{ij}$ (see [9], Theorem 1).

Remark 5. Simple calculations applied to the case $a = \lambda = 1$ show that $\mathcal{M}$ is not a simplex. Indeed denoting with $\mathcal{P} := \{\gamma \in \Gamma : \gamma_{kk} = \frac{1}{2}, k \geq 3\}$, the set $\mathcal{M} \cap \mathcal{P}$, which is a face of $\mathcal{M}$, is given by $\mathcal{M} \cap \mathcal{P} = \{\gamma \in \mathcal{M} : \gamma_{kk} = \frac{1}{2}, k \geq 3, \ \gamma_{11} \in [\frac{1}{4}, \frac{1}{2}], \ \frac{1-\gamma_{11}}{3-4\gamma_{11}} \leq \gamma_{22} \leq \frac{1}{2}\}$, and the subset of its boundary given by $\{\gamma \in \mathcal{M} \cap \mathcal{P} : \gamma_{22} = \frac{1-\gamma_{11}}{3-4\gamma_{11}}\}$ consists of expectations with $f_1(\gamma), f_3(\gamma) \leq 4, \ f_2(\gamma) = f_k(\gamma) = 4, \ k \geq 4$, which are extremal also for all $\mathcal{M}$ as follows from Proposition 13. A simple application of [1] Proposition II.3.3, shows that $\mathcal{M}$ cannot be a simplex.

Example 2. Let us consider the graph $\mathcal{G}_\infty$ with arbitrary index-matrix α_{ij} interpreted as giving the minimal index of the inclusion $N_{ij} \subset M_{ij}$, let γ^0 correspond to the special expectation of $N \subset M$ and d be the minimal index. Then γ^0 is determined recursively by:

$$\begin{cases} \gamma^0_{11} = \frac{\alpha_{11}}{d} \\ \gamma^0_{k,k-1} = 1 - \gamma^0_{k-1,k-1} \\ \gamma^0_{kk} = g_k(\gamma^0_{k-1,k-1}) := \frac{\alpha_{kk}(1-\gamma^0_{k-1,k-1})}{d(1-\gamma^0_{k-1,k-1})-\alpha_{k,k-1}}. \end{cases}$$

We want to discuss the possibility of having minimal expectations whose index is a finite support perturbation of the minimal index. First we observe that minimal expectations, whose index differs only in the first component from the minimal index d, are obtained if we choose $\gamma_{11} > \frac{\alpha_{11}}{d}$ and if the iterative scheme

$$\begin{cases} \gamma_{k,k-1} = 1 - \gamma_{k-1,k-1} \\ \gamma_{kk} = g_k(\gamma_{k-1,k-1}) \end{cases}$$

does not abort, that is if $\gamma_{k,k-1}, \gamma_{kk} \in (0,1)$, for all $k \in \mathbb{N}$. So we get the following

Proposition 14. *The set of values γ_{11} for which the above iterative scheme does not abort is a closed interval $[\gamma^0_{11}, \bar{\gamma}_{11}]$.*

Proof. Define $K_1 \supset K_2 \supset \dots$ by $K_1 = [\gamma^0_{11}, 1]$, $K_k = \{x \in K_{k-1} : g_k \circ \dots \circ g_2(x) \in [g_k(0), 1]\}$. Looking the graphs of the g_k's, we inductively see that all K_k are closed intervals. The set of values γ_{11} for which the iterative scheme does not abort is $\bigcap_{k=1}^\infty K_k$ and the proof follows. □

Now the general case is contained in the following

Proposition 15. *There exist minimal expectations whose index is a suitable perturbation with finite support of the minimal index iff $(\gamma^0_{11}, \bar{\gamma}_{11}) \neq \emptyset$.*

Proof. Let $\delta = \{\delta_i\}$ be a finite support positive sequence, let $k \in \mathbb{N}$ be s.t. $\delta_i = 0$ for all $i > k$, and set $\tilde{g}_i(x) := \frac{\alpha_{ii}(1-x)}{(d-\delta_i)(1-x)-\alpha_{i,i-1}}$. Then it is easy to see that there exist minimal expectations whose index is $\{d - \delta_i\}$ iff $\gamma_{11} := g_2^{-1} \circ \dots \circ g_k^{-1} \circ \tilde{g}_k \circ \dots \circ \tilde{g}_2(\frac{\alpha_{11}}{d-\delta_1}) \in (\gamma^0_{11}, \bar{\gamma}_{11}]$. □

Remark 6. In the case $\alpha_{ij} = 1$ a simple example of a minimal expectation F whose index is a perturbation of the minimal index is obtained by $F := \frac{1}{2}(E^0 + E)$, where E^0 is the unique special expectation, and E corresponds to $\gamma_{ij} = \frac{1}{2}$. Indeed $f_i(\gamma^F) < 4$ and $f_i(\gamma^F) \nearrow 4$.

Example 3. Let us consider the graph specified by the connection matrix $[a_{ij}]$, where

$$a_{ij} := \begin{cases} 1 & \text{if } j = i, i+1 \text{ or } (i,j) = (2,1), \\ 0 & \text{otherwise.} \end{cases}$$

Let the index-matrix α be given by

$$\alpha_{ij} := \begin{cases} 1 & \text{if } i = 1, 2, \\ 4 & \text{otherwise.} \end{cases}$$

Then the minimal index is 16 and this system has lots of special expectations. Indeed, setting $\gamma_{22} = b$ and imposing scalar index d, one can calculate $\gamma_{11},\ \gamma_{12},\ \gamma_{21},\ \gamma_{23}$ as functions of b and d. If $\gamma_{23} \leq \frac{1}{2}$ the subgraph $\mathcal{G}_\infty$ has index 16 and one has an interval $[\frac{49-\sqrt{1722}}{97}, \frac{49+\sqrt{1722}}{97}]$ of admissible values for b, that is a one-parameter family of special conditional expectations.

Example 4. This last example furnishes another method for calculating the minimal index in some special cases.
Let us consider the graph specified by the connection matrix

$$[a_{ij}] = \begin{pmatrix} \mathbb{I}_1 & 0 & 0 & 0 & 0 & \cdot \\ \mathbb{J}_1 & \mathbb{I}_2 & 0 & 0 & 0 & \cdot \\ 0 & \mathbb{J}_2 & \mathbb{I}_3 & 0 & 0 & \cdot \\ 0 & 0 & \mathbb{J}_3 & \mathbb{I}_4 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

where $\mathbb{I}_k := \mathrm{diag}(1 \dots 1)$, 2^{k-1} times,

$$\mathbb{J}_k := \begin{pmatrix} \mathbb{I}_k \\ \mathbb{I}_k \end{pmatrix},$$

and k is the index of the block. Let the index-matrix α be given by

$$\alpha_{ij} := \begin{cases} c & (i,j) = (1,1), \\ 1 & \text{otherwise.} \end{cases}$$

Let us consider the connected sub-graph consisting of the first $2^n - 1$ lower vertices, and compute the γ^n_{ij}'s of an expectation which has scalar index d_n. The scalar index condition forces the off-diagonal γ^n_{ij}'s belonging to the k-th block to be equal to each other and we denote this common value by $\frac{N^n_k}{D^n_k}$; also the diagonal elements belonging to the same block are equal to each other and we denote it by $\frac{N^n_{k-1}}{D^n_k}$. Exploiting column-markovianity of γ^n and the scalar index condition from bottom to top with $N_n^n = 1$, we get the following recursive system

$$\begin{cases} N_{k-1}^n = D_k^n - 2N_k^n \\ D_{k-1}^n = (d_n - 1)D_k^n - 2d_n N_k^n, \;\; k = 2, \dots, n \\ N_n^n = 1 \\ D_n^n = d_n \end{cases}$$

and N_1^n, D_1^n must satisfy the equation

$$(d_n - c)D_1^n - 2d_n N_1^n = 0.$$

That is to say, if we introduce the polynomial $Q_n(d) := (d-c)D_1^n(d) - 2dN_1^n(d)$, d_n will be a value for a scalar-index expectation of the subsystem $\iff$ d_n is a root of Q_n. Observe that, performing some tedious calculations and making use of the fact that $N_{k+1}^{n+1}(d) = N_k^n(d)$ and $D_{k+1}^{n+1}(d) = D_k^n(d)$, for $k = 1, \dots, n$, we obtain a recursive relation

$$\begin{cases} Q_{n+1}(d) = (d-3)Q_n(d) - 2Q_{n-1}(d), \;\; n \geq 1 \\ Q_1(d) = d - (c+2) \\ Q_0(d) := 1. \end{cases}$$

As the Q_n's are orthogonal polynomials ([5]), to obtain the value of the minimal index of the inclusion specified by the above data, one has to calculate the limit of the sequence of the greatest roots of the Q_n's. To do this, observe that if we introduce the matrix

$$A = \begin{pmatrix} 2+c & \sqrt{2} & 0 & 0 & 0 & \cdot \\ \sqrt{2} & 3 & \sqrt{2} & 0 & 0 & \cdot \\ 0 & \sqrt{2} & 3 & \sqrt{2} & 0 & \cdot \\ 0 & 0 & \sqrt{2} & 3 & \sqrt{2} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

then Q_n is the characteristic polynomial of the matrix A_n consisting of the first n rows and columns of A, so that the greatest root of Q_n, that is the spectral radius of A_n, converges to $\|A\|$. As an example, if $c = 1$ then the minimal index is $3 + 2\sqrt{2}$.

6. On the minimal index and the entropy

In this section we develop some connections between the minimal index and the entropy for an inclusion $N \subset M$ of von Neumann algebras with atomic centres. Recall that in case of finite dimensional centres Teruya proved the following

Proposition 16. ([28]) *Let $N \subset M$ be a connected inclusion with finite index and finite dimensional centres and $E \in E(M, N)$. Then the following are equivalent*
(i) $E \in E(M, N)$ is the minimal expectation,
(ii) E_{ij} is minimal, $Ind(E) \in \mathbb{R}$ and exist two finite sequences of positive numbers $\{\mu_i\}$, $\{\nu_j\}$ s.t. $\frac{\alpha_{ij}}{\gamma_{ij}^2} = \mu_i \nu_j$,
(iii) $K_E(M, N) = \log Ind(E)$.

As we have seen in Section 5, Example 3, there exists in general more than one minimal expectation with scalar index; we fix our attention on those which are limit of expectations, minimal when restricted to suitable finite dimensional systems, as in the proof of Proposition 10.

Proposition 17. *Let γ correspond to a limit expectation E as above. Then there exist two sequences of positive numbers $\{\mu_i\}$, $\{\nu_j\}$ s.t.*

$$\frac{\alpha_{ij}}{\gamma_{ij}^2} = \mu_i \nu_j. \tag{4}$$

Proof. Let $\{\gamma^n\}$ be a sequence converging component-wise to γ, as in the proof of Proposition 10. Then for all i and j we define two sequences $\{\mu_i^n\}$ and $\{\nu_j^n\}$, defined for all n large enough as ([28], page 446) and set $\mu_i := \lim_{n\to\infty} \mu_i^n$ and $\nu_j := \lim_{n\to\infty} \nu_j^n$, for these limits exist because μ_i^n, ν_j^n are continuous functions of γ. □

Recall that the entropy of an expectation $E \in E(M, N)$ is given by

$$K_E(M, N) = \sup\{\sum_{i,j} \gamma_{ij}\nu_j \log \frac{(\sum_k \gamma_{ik}\nu_k)\beta_{ij}}{\gamma_{ij}^2 \nu_j}\}$$

where $\beta_{ij} := \exp K_{E_{ij}}(M_{ij}, N_{ij})$, and the supremum is taken in $\{\nu \in \ell^1 : \nu_j > 0,\ \sum_j \nu_j = 1\}$ (see [28]).
It is easy to see that the supremum may be taken in $\{\nu \in \ell^1 : \nu_j \geq 0,\ \sum_j \nu_j = 1\}$ as well.

Proposition 18. *Let $E \in E(M, N)$ be a special expectation s.t. there exist two sequences of positive real numbers $\{\mu_i\}$, $\{\nu_j\}$ satisfying $\frac{\alpha_{ij}}{\gamma_{ij}^2} = \mu_i \nu_j$.*

Suppose that there exists an increasing sequence $\{S_n\}$ of finite subsets of J s.t.

$$\lim_{n\to\infty} \frac{\sum_{j\in S'_n} \nu_j}{\sum_{j\in S_n} \nu_j} \to 0,$$

where $S' := \{j \in S : dist(j, J \setminus S) = 1\}$, for all $S \subset J$.
Then

$$K_E(M, N) = \log Ind(E).$$

Proof. From the above remark and the fact that the E_{ij}'s are minimal we get

$$\begin{aligned}
K_E \geq & \frac{1}{\sum_{j\in S_n} \nu_j} \sum_{j\in S_n} \sum_{i\sim j} \gamma_{ij}\nu_j \log \frac{(\sum_{k\in S_n} \gamma_{ik}\nu_k)\beta_{ij}}{\gamma_{ij}^2 \nu_j} \\
&= \frac{1}{\sum_{j\in S_n} \nu_j} \sum_{j\in S_n} \sum_{i\sim j} \gamma_{ij}\nu_j \log(\sum_{k\in S_n} \frac{\alpha_{ik}}{\gamma_{ik}}) \\
&= \log Ind(E) - \frac{1}{\sum_{j\in S_n} \nu_j} \sum_{j\in S'_n} \sum_{i\sim j} \gamma_{ij}\nu_j \log(\frac{d}{d_i}) \\
&\geq \log Ind(E) - k \frac{\sum_{j\in S'_n} \nu_j}{\sum_{j\in S_n} \nu_j}
\end{aligned}$$

where we set $d := Ind(E)$, $d_i := \sum_{j\in S_n, j\sim i} \frac{\alpha_{ij}}{\gamma_{ij}}$ and $k > 0$ is a uniform constant, as follows easily from Proposition 8. As $K_E \leq \log Ind(E)$ ([10]), we obtain the thesis. □

In the next two propositions we apply the above results to the graph $\mathcal{G}_\infty$ of Section 3. Similar calculations could be made also for a little more complicated graphs.

Proposition 19. *Let $[\alpha_{ij}]$ be an index-matrix interpreted as giving the minimal index of the inclusion $N_{ij} \subset M_{ij}$. Let γ correspond to the special expectation E of $N \subset M$. Then we get*

$$K_E(M, N) = \log Ind(E).$$

Proof. The thesis will follow from the previous proposition as soon as we show that there exists a subsequence ν_{n_k} such that $\frac{\nu_{n_k}}{\sum_{j=1}^{n_k} \nu_j} \to 0$. So suppose on the contrary that $\frac{\sum_{j=1}^{n} \nu_j}{\nu_n} \leq M$, for all $n \in \mathbb{N}$. Then we can apply the inverse function theorem to the function $F \equiv (f_i) : \ell^\infty \to \ell^\infty$. Indeed the Frechet derivative of F in γ is given by a matrix L in $B(\ell^\infty)$

and the norm of L^{-1} in $B(\ell^\infty)$ is bounded by M. Therefore we can apply the inverse function theorem and conclude that E is not minimal, contradicting the hypothesis. $\square$

Under the condition that $\alpha_{ij} = 1$ one can also prove the following result.

Proposition 20. *Let the minimal expectation E provide a finite support perturbation of the minimal index, which exists by Proposition 15. Then*

$$K_E(M, N) = \log 4.$$

Proof. Suppose that $f_i(\gamma) = 4$ if $i > n_0$. We have

$$K_E(M, N) \geq \log 4 - k\frac{\nu_{n_0} + \nu_n}{\sum_{j=n_0}^{n} \nu_j}.$$

In this special case we can choose $\nu_j \geq 1$. Then

$$\frac{\nu_{n_0}}{\sum_{j=n_0}^{n} \nu_j} \leq \frac{\nu_{n_0}}{n - n_0} \rightarrow 0.$$

Moreover also

$$\frac{\nu_n}{\sum_{j=n_0}^{n} \nu_j} \rightarrow 0$$

otherwise we can apply the inverse function theorem and contradict the minimality of E. $\square$

Remark 7. We cannot generalize to our context Proposition 16

(i) as the previous result shows that there are minimal expectations with nonscalar index which satisfy (4) and $K_E = \log Ind(E)$,

(ii) as the above proof also works for the inclusion with graph given in Example 3, Section 5, showing that there are examples of special expectations E that do not satisfy (4) but $K_E(M, N) = \log Ind(E)$.

Actually we cannot extend the result of Proposition 19 to more general graphs, as well as the converse connection between entropy and minimal index.

However it might be reasonable to conjecture that, for the special expectations E satisfying (4), we always have

$$K_E(M, N) = \log Ind(E).$$

Indeed, in this case, one could contradict Proposition 18 and hence assume that

$$\frac{\sum_{j\in S}\nu_j}{\sum_{j\in S'}\nu_j} \le M$$

uniformly for all finite subsets S. This condition seems to be intimately related to the applicability of the inverse function theorem for constructing an expectation whose index is less than that of E. Unfortunately, at present, this relation is not well understood.

7. Markov traces

Let $N \subset M$ be finite von Neumann algebras, τ a faithful normal trace state on M, $E \in E(M,N)$ preserving τ. We say that τ is a Markov trace with modulus $\beta > 0$ if τ extends to a f.n. trace state τ^1 on M^1 s.t. $\beta\tau^1(xe) = \tau(x)$, $x \in M^1$, where $N \subset M \subset M^1$ is the Jones tower and $e \in M^1$ is the Jones projection.

For the reader's convenience, we recall some definitions and results (see [8], Section 3):

Proposition 21. *The following are equivalent:*
(i) τ is a Markov trace with modulus β,
(ii) E has index β.
In this case $\tau^1 = \tau E^1$.

Let now $N \subset M$ have atomic centres, and let tr_i be the trace state on M_i. Define the trace matrix $T := [c_{ij}]$, where $c_{ij} := tr_i(p_i q_j)$. A trace state on M is determined by a row-vector $s = (s_i) \in \ell^1$ s.t. $s_i = \tau(p_i)$, as $\tau(x) = \sum_{ij} s_i tr_i(p_i q_j x q_j)$. Set $t = (t_j) := sT$, then $t_j = \sum_k s_k c_{kj} = \tau(q_j)$.
Finally set $\tilde{T} := [\tilde{c}_{ji}]$, where

$$\tilde{c}_{ij} := \begin{cases} \frac{\alpha_{ij}}{c_{ij}} & \text{if } (i,j) \in \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases}$$

Then E preserves τ iff E_{ij} preserves tr_{ij} (the trace state on M_{ij}) and $t_j \gamma_{ij} = s_i c_{ij}$.

Proposition 22. *Let $E \in E(M,N)$ preserve a f.n. trace state τ on M. Set $\delta = (\delta_i) := sT\tilde{T}$. Then*
(i)
$$Ind(E) = \sum_i \delta_i s_i^{-1} p_i.$$
(ii) $Ind(E)$ is a scalar iff s is an ℓ^1-eigenvector of $T\tilde{T}$ corresponding to the eigenvalue $Ind(E)$.

Proposition 23. ([8], *Theorem 3.9*) *Let $N \subset M$ be a connected, finite index inclusion of finite von Neumann algebras with finite dimensional centres. Then there is a unique Markov trace τ with module β which is the Perron-Frobenius eigenvalue of $T\tilde{T}$.*

In order to ensure the existence and uniqueness of a Markov trace for the inclusion $N \subset M$, the theory of Perron-Frobenius doesn't work in this generality ([25]), because $T\tilde{T}$ is non-compact or even unbounded as an operator on l^1 in some cases (see below).
This problem is considered also by Jolissaint ([12]), but he provides only examples of existence or non-existence, whereas we have examples of inclusions with atomic (infinite dimensional) centres for which all possibilities can occur.

For a linear operator A acting on the space of all complex-valued sequences we indicate with $\sigma_M(A) := \{\lambda > 0 : \exists\, \xi \in \ell^1_+ \text{ s.t. } \xi A = \lambda\xi\}$ the set of its positive eigenvalues corresponding to ℓ^1 positive eigenvectors. Therefore, if T is the trace-matrix associated to the inclusion $N \subset M$, $\sigma_M(T\tilde{T})$ gives rise directly to all the Markov traces for $N \subset M$.

Let us consider the graph specified by the following connection matrix $[a_{ij}]$
$$a_{ij} := \begin{cases} 1 & j = i, i+1, \\ 0 & \text{otherwise}, \end{cases}$$
with $\alpha_{ij} = 1$, and the trace matrix $T := [c_{ij}]$.
In this case
$$T\tilde{T} = \begin{pmatrix} 2 & b_1^{-1} & 0 & 0 & 0 & \cdot \\ b_1 & 2 & b_2^{-1} & 0 & 0 & \cdot \\ 0 & b_2 & 2 & b_3^{-1} & 0 & \cdot \\ 0 & 0 & b_3 & 2 & b_4^{-1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

where $b_k := \frac{c_{k+1,k+1}}{c_{k,k+1}}$.
Let us observe that $T\tilde{T} - I = V\Theta V^{-1}$, where $V = \mathrm{diag}(v_0, v_1, v_2, \cdots)$, $v_0 := 1$, $v_n := b_n v_{n-1}$, and

$$\Theta = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdot \\ 1 & 1 & 1 & 0 & 0 & \cdot \\ 0 & 1 & 1 & 1 & 0 & \cdot \\ 0 & 0 & 1 & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

So that, in the space of all complex-valued sequences the equation $sT\tilde{T} = \lambda s$ can be rewritten as $sV\Theta = (\lambda - 1)sV$ and gives $s_{n+1} = v_n^{-1} Q_n(\frac{\lambda-2}{2})s_1$, where the polynomials Q_n are given by

$$\begin{cases} Q_0(x) = 1 \\ Q_1(x) = 2x \\ Q_n(x) = 2xQ_{n-1}(x) - Q_{n-2}(x), \quad n > 1. \end{cases}$$

Let us observe that the previous relations define Gegenbauer polynomials with parameter 1 ([5]).
We start with the following technical lemmas.

Lemma 1.
(i) $s_n > 0,\ n \in \mathbb{N} \iff \lambda \geq 4$,
(ii) $s \in \ell^1 \iff \sum_{n=1}^{\infty} v_n^{-1} Q_n(\frac{\lambda-2}{2}) < \infty$.

Proof. *(i)* It is equivalent to showing $Q_n(x) > 0, n \in \mathbb{N} \iff x \geq 1$. Let us observe that the Q_n are orthogonal polynomials (see [5]) so that Q_n has only real simple roots which we denote $x_{1n}, \ldots, x_{nn}$ and which satisfy the following separation property

$$x_{m-1,n} < x_{m,n+1} < x_{m,n}.$$

($\Longleftarrow$) As $2x_{nn} + 1$ is the norm of Θ_n the restriction of Θ to the first n components, $2x_{nn} + 1 \nearrow \|\Theta\| = 3$, that is $x_{nn} \nearrow 1$, and we are done.
($\Longrightarrow$) Follows from the fact that if $x \in [x_{n-1,n}, x_{nn}]$ then $Q_n(x) \leq 0$, and $\cup[x_{n-1,n}, x_{nn}] \equiv (-\infty, 1)$ by the separation property.
(ii) is immediate. □

Lemma 2.
(i) $Q_n(1) = n + 1, n \in \mathbb{N}$,
(ii) $Q_n'(1) = \frac{1}{3}n(n + 1)(n + 2), n \in \mathbb{N}$,
(iii) $Q_n^{(k)}(1) \geq 0, k, n \in \mathbb{N}$.

Proof. (i), (ii) follow by induction.
(iii) As $Q_n^{(k)}(x) = 2kQ_{n-1}^{(k-1)}(x) + 2xQ_{n-1}^{(k)}(x) - Q_{n-2}^{(k)}(x)$, we get, setting for simplicity $a_{kn} := Q_n^{(k)}(1)$, $a_{kn} = 2ka_{k-1,n-1} + 2a_{k,n-1} - a_{k,n-2}$. Now by (i) and (ii) we get $a_{0n}, a_{1n} \geq 0, n \in \mathbb{N}$. Supposing that $a_{k-1,n} \geq 0, n \in \mathbb{N}$, we show that $a_{kn} \geq 0, n \in \mathbb{N}$. Indeed, as $a_{k2} - a_{k1} \geq 0$, and assuming $a_{k,n-1} - a_{k,n-2} \geq 0$ we get $a_{kn} - a_{k,n-1} \geq 0$, we obtain that a_{kn} is increasing in n, and so it is positive for all n, as $a_{k0} = 0$. This concludes the proof. □

The following propositions show that, for this simple example, there exist choices of trace-matrices for which the corresponding inclusions have only one Markov trace or Markov traces with modulus in a semi-closed bounded or unbounded interval; an example with no Markov trace is easily obtained by considering an inclusion as above with trace-matrix given by $c_{ii} = c_{i,i+1} = 1/2$.

Proposition 24.
(i) *If* $c \in (2, 4]$, *and* $b_k := e^{\frac{c}{k}}$, *then* $\sum_{k=1}^{\infty} v_k^{-1} Q_k(x)$ *converges for* $x = 1$, *and diverges for* $x > 1$.
(ii) *For all* $x_0 > 1$ *there exists a choice of* $\{b_k\}$, $b_k > 1$, *s.t.* $\sum_{k=1}^{\infty} v_k^{-1} Q_k(x)$ *converges for* $x \in [1, x_0)$, *and diverges for* $x > x_0$.
(iii) *There exists a choice of* $b_k > 1$ *s.t.* $\sum_{n=1}^{\infty} v_n^{-1} Q_n(x)$ *converges for all* $x \geq 1$.

Proof. (i) As $v_n = \prod_{k=1}^{n} b_k \sim e^{c\gamma} n^c$, where γ is Euler's constant, and $Q_n(x) = n + 1 + \frac{1}{3}n(n+1)(n+2)(x-1) + r_n(x)$, where $r_n(x) \geq 0, x \geq 1$ as follows from (iii) of the previous Lemma, we get:
1. $\sum_{n=1}^{\infty} v_n^{-1} Q_n(1)$ converges as $\sum_{n=1}^{\infty} e^{-c\gamma} n^{-c}(n+1)$ does;
2. $\sum_{n=1}^{\infty} v_n^{-1} Q_n(x)$ diverges, if $x > 1$, as it majorizes $\sum_{n=1}^{\infty} e^{-c\gamma} n^{-c} \frac{1}{3} n(n+1)(n+2)(x-1)$, which diverges.
(ii) From ([5], page 177) we have $\sum_{n=0}^{\infty} Q_n(x) z^n = \frac{1}{1-2xz+z^2}$, if $z \in \mathbb{C}$ is s.t. $|z| < z(x) := x - \sqrt{x^2 - 1}$. Therefore if we choose $b_k := z(x_0)^{-1}$, with $x_0 > 1$, we have that $\sum_{n=1}^{\infty} v_n^{-1} Q_n(x)$ converges if $1 \leq x < x_0$, as $z(x) > z(x_0)$, and diverges if $x \geq x_0$, as, if $x > x_0$, $\sum_{n=1}^{\infty} v_n^{-1} Q_n(x) > \sum_{n=1}^{\infty} v_n^{-1} Q_n(x_0)$.
(iii) Let us observe that for all decreasing $\{q_n\}$, s.t. $q_n > 0$ and $\sum_{n=1}^{\infty} q_n < \infty$, there exists $\{b_k\}$ s.t. $(\prod_{k=1}^{n} b_k^{-1}) Q_n(n+1) = q_n$: indeed one has $b_n = \frac{Q_n(n+1)}{q_n} \frac{q_{n-1}}{Q_{n-1}(n)}$, where $q_0 := 1$, for all $n \in \mathbb{N}$.
As for all $x > 1$ there is $n_0 \in \mathbb{N}$ s.t. for all $n > n_0$ one has $Q_n(x) \leq Q_n(n+1)$, the thesis follows. □

Proposition 25. *Let* $\{b_k\}_{k\in\mathbb{N}}$*,* $\{c_{kk}\}_{k\in\mathbb{N}}$*, and* $\{v_k\}_{k\geq 0}$ *be as above. If* $b_k > 1$*, and* $\sum_{n=0}^{\infty} v_n^{-1}$ *converges, then*

(i)

$$\sum_{n=0}^{\infty}(-1)^n v_n^{-1} \in (0,1).$$

(ii) *Choosing*

$$c_{11} := \sum_{n=0}^{\infty}(-1)^n v_n^{-1}$$

we get $0 < c_{nn} < 1$*,* $n \in \mathbb{N}$*.*

Proof. Observe that $\{v_k\}$ is decreasing and $0 < \sum_{k=0}^{n}(-1)^k v_k^{-1} < 1$, $n \in \mathbb{N}$, so we get (i).

Owing to the relation $b_k = \frac{c_{k+1,k+1}}{1-c_{kk}}$, to obtain $0 < c_{kk} < 1$, $k = 1, \cdots, n$, one has to choose c_{11} satisfying the following inequalities

$$\sum_{k=0}^{n-2}(-1)^k v_k^{-1} < c_{11} < \sum_{k=0}^{n-1}(-1)^k v_k^{-1}, \quad \text{if } n \text{ is odd},$$

$$\sum_{k=0}^{n-1}(-1)^k v_k^{-1} < c_{11} < \sum_{k=0}^{n-2}(-1)^k v_k^{-1}, \quad \text{if } n \text{ is even},$$

and (ii) follows easily. □

Summing up the above results, we can state the following

Theorem 3.

(i) *There is a trace-matrix* T *s.t.* $\sigma_M(T\tilde{T}) = \emptyset$*, i.e. there is an inclusion of von Neumann algebras possessing no Markov traces.*

(ii) *There is a trace-matrix* T *s.t.* $\sigma_M(T\tilde{T}) = \{4\}$*, i.e. there is an inclusion of von Neumann algebras possessing only one Markov trace, with modulus* 4*.*

(iii) *For all* $\lambda_0 > 4$ *there is a trace-matrix* T *s.t.* $\sigma_M(T\tilde{T}) = [4, \lambda_0)$*, i.e. there is an inclusion of von Neumann algebras possessing Markov traces whose modulus is any value in* $[4, \lambda_0)$*.*

(iv) *There is a trace-matrix* T *s.t.* $\sigma_M(T\tilde{T}) = [4, \infty)$*, i.e. there is an inclusion of von Neumann algebras possessing Markov traces whose modulus is any value in* $[4, \infty)$*.*

Proof. The thesis follows easily from previous propositions, lemma 1 (ii) and proposition 26. □

8. Appendix

For the reader's convenience we give here a construction, analogous to that in [7], which shows how to associate to a bipartite graph and matrices α and T, an inclusion of hyperfinite type II_1 von Neumann algebras with atomic centres.

Proposition 26. *Given a bipartite graph, an index-matrix α and a trace-matrix T, there is an inclusion of hyperfinite type II_1 von Neumann algebras with atomic centres whose associated graph, index-matrix of minimal indices, and trace-matrix are the given ones.*

Proof. Let us choose an inclusion of hyperfinite type II_1 factors $A_{ij} \subset B_{ij}$ whose minimal index is α_{ij}, and let $T = [c_{ij}]$ be the given trace-matrix. We want to construct an inclusion of hyperfinite type II_1 von Neumann algebras $N \subset M$ with atomic centres, s.t. $N_{ij} \subset M_{ij}$ is conjugate to $A_{ij} \subset B_{ij}$, and T is the trace matrix associated to the inclusion.
Let $\vartheta_{ij} \in End(R)$ be s.t. $\vartheta_{ij}(R) \subset R$ is conjugate to $A_{ij} \subset B_{ij}$, $e_{ij} \in Proj(R)$ be s.t. $e_{ij} \perp e_{ik}$, $j \neq k$ and $\tau(e_{ij}) = c_{ij}$, $\beta_{ij} : R \to R_{e_{ij}}$ an isomorphism between hyperfinite type II_1 factors, $\pi_{ij} := \beta_{ij} \cdot \vartheta_{ij}$ so that, with $R_{ij} := \pi_{ij}(R)$, $R_{ij} \subset R_{e_{ij}}$ is conjugate to $A_{ij} \subset B_{ij}$. Let $\pi_j := \oplus_{i,i\sim j}\pi_{ij}$ so that $\pi_j(R) \subset \oplus_{i,i\sim j}R_{e_{ij}} \equiv (\oplus_{i\in I}R_i)_{q_j}$, where $R_i \equiv R$ and $q_j := \sum_{i,i\sim j} e_{ij}$ and we have identified $e_{ij} \in R$ with $e_{ij} \in R_i \subset \oplus_{k\in I}R_k$.
Finally set $N := \sum_j \pi_j(R)$ and $M := \oplus_{k\in I}R_k$ so that $N \subset M$. Therefore $p_i = \sum_{j,j\sim i} e_{ij}$ so that $p_iq_j = e_{ij}$, $\tau_i(p_iq_j) \equiv \frac{\tau(p_iq_j)}{\tau(p_i)} = c_{ij}$, $N_{p_iq_j} = \pi_{ij}(R)$, $M_{p_iq_j} = R_{e_{ij}}$. □

Acknowledgement

We thank prof. L. Zsido for some useful conversations.
We are also grateful to the referee for a careful reading of the manuscript.

References

[1] Alfsen, E. M. *Compact convex sets and boundary integrals*, Springer Verlag, Berlin, Heidelberg, New York (1971).

[2] Baillet, M., Denizeau, Y., Havet, J. F. *Indice d'une espérance conditionnelle*, Comp. Math. **66** (1988), 199–236.

[3] Doplicher, S., Haag, R., Roberts, J. E. *Local observables and particle statistics. I*, Commun. Math. Phys. **23** (1971), 199–230.

[4] de la Harpe, P., Kervaire, M., Weber, C. *On the Jones polynomial*, Enseign. Math., **32** (1986), 271–335.

[5] Erdélyi, A. (ed.). *Higher transcendental functions*, vol. II, R. E. Krieger Publ. Co., Malabar, Florida (1985).

[6] Goldman, M. *On subfactors of factors of type II_1*, Mich. Math. J. **7** (1960), 167–172.

[7] Goodman, F., de la Harpe P., Jones, V. F. R. *Coxeter graphs and towers of algebras*, MSRI Publ. **14**, Springer Verlag (1989).

[8] Havet, J. F. *Espérance conditionnelle minimale*, J. Op. Th. **24** (1990), 33–55.

[9] Hiai, F. *Minimizing indices of conditional expectations onto a subfactor*, Publ. R.I.M.S., **24** (1988), 673–678.

[10] Hiai, F. *Minimum index for subfactors and entropy*, J. Op. Th. **24** (1990), 301–336.

[11] Isola, T. *Modular structure of the crossed product by a compact group dual*, J. Op. Th. **33** (1995), 3–31.

[12] Jolissaint, P. *Index for pairs of finite von Neumann algebras*, Pacific J. Math. **146** (1990), 43–70.

[13] Jolissaint, P. *Indice d' esperances conditionnelles et algebres de von Neumann finies*, Math. Scand. **68** (1991), 221–246.

[14] Jones, V. F. R. *Index for subfactors*, Invent. math. **72** (1983), 1–25.

[15] Jones, V. F. R. *Braid groups, Hecke algebras and type II_1 factors*, in: Geometric methods in operator algebras (Kyoto, 1983), Pitman Res. Notes in Math. **123** (1986), 242–273.

[16] Jones, V. F. R. *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. (2) **126** (1987), 335–388.

[17] Kosaki, H. *Extension of Jones' theory of index to arbitrary factors*, J. Funct. Anal. **66** (1986), 123–140.

[18] Kosaki, H., Longo R., *A remark on the minimal index of subfactors*, J. Funct. Anal. **107** (1992), 458–470.

[19] Longo, R., *Index of subfactors and statistics of quantum fields. I, II*, Commun. Math. Phys. **126** (1989), 217–247; *ibidem* **130** (1990), 285–309.

[20] Longo, R., *Minimal index and unimodular sectors*, in: Proceedings of the International Conference on Non-commutative Analysis. Araki, H., Ito, K. R., Kishimoto, A., Ojima, I. (eds.). Kluwer Ac. Publ. (1993).

[21] Longo, R., *Minimal index and braided subfactors*, J. Funct. Anal. **109** (1992), 98–112.

[22] Longo, R., Roberts, J. E. *A theory of dimension*, Preprint (1995).

[23] Pimsner, M., Popa, S., *Entropy and index for subfactors*, Ann. Scient. Ec. Norm. Sup. (4), **19** (1986), 57–106.

[24] Popa, S., *Classification of subfactors and of their endomorphisms*, CBMS Lecture Notes Series (1994).

[25] Schaefer, H. H., *Banach lattices and positive operators*, Springer Verlag, Berlin, Heidelberg, New York (1974).

[26] Stratila, S., *Modular theory in operator algebras*, Abacus Press, Tunbridge Wells, Kent, England (1981).

[27] Takesaki, M., *Theory of operator algebras, I*, Springer Verlag, Berlin, Heidelberg, New York (1979).

[28] Teruya, T. *Index for von Neumann algebras with finite dimensional centers*, Publ. R.I.M.S., **28** (1992), 437–453.

[29] Tutte, W. T., *Graph theory*, Encyclopedia of Mathematics and its Applications, vol. 21, Addison-Wesley Publ. Co., Menlo Park, California (1992).

[30] Wenzl, H. *Representations of braid groups and the quantum Yang–Baxter equation*, Pacific J. Math., **145** (1990), 153–180.

Francesco Fidaleo, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy
E-mail address: `Fidaleo@mat.utovrm.it`

Tommaso Isola, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy
E-mail address: `Isola@mat.utovrm.it`