ON THE CONJUGATE ENDOMORPHISM IN THE INFINITE INDEX CASE

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Abstract

We give an algebraic characterization for the conjugate endomorphism $\bar{\rho}$ of an endomorphism ρ of infinite index of a properly infinite von Neumann algebra M such that the set of normal faithful conditional expectations $E(M, \rho(M))$ is not empty. In the particular case of irreducible endomorphisms we obtain the same result holding in finite index case and in the representation theory of compact groups, that is if ρ is an irreducible endomorphism of an infinite factor, with $E(M, \rho(M)) \neq \emptyset$, then an irreducible endomorphism σ is conjugate to ρ iff $\sigma\rho \succ id$; moreover the identity is contained only once in $\sigma\rho$. Some applications of the above results are also given.

1. Introduction

In the category of representations of a compact group, the notion of conjugate representation exists naturally via a spatial definition [10]. For irreducible representations, one can easily characterize the conjugate by a simple algebraic property: if σ, π are irreducible, σ is conjugate to π iff the tensor product $\pi \otimes \sigma$ contains the identity representation.

Motivated by early works in local quantum theory [2], [3], where a (compact) gauge group was strongly related to the particle-antiparticle symmetry, S. Doplicher and J.E. Roberts have recently shown an isomorphism between a monoidal C*-category of endomorphisms with conjugates and permutation symmetry and the category of representations of a compact group, see [4], [5], [6].

The superselection structure, naturally appearing in local quantum theory, was a point of departure also for R. Longo, in connecting [15] the statistical dimension of a sector [2] with Jones' index theory for inclusions [12], and for giving [16] a notion of a conjugate map for the semigroup $Sect(M)$ of (normal faithful unital) endomorphisms, of a properly infinite von Neumann algebra M, modulo inner automorphisms. This conjugation assumes the following simple form

$$\bar{\rho} = \rho^{-1}\gamma$$

where γ is Longo's canonical endomorphism relative to the inclusion $\rho(M) \subset M$, [14]. This formula seems to cover all interesting cases (locally compact groups, particle-antiparticle symmetry for infinite statistics in local quantum theory, quantum groups). In this context also there is a characterization, analogous to that holding in the representation category of a compact group, for the conjugate of an irreducible endomorphism ρ of finite index: σ is a conjugate to ρ iff the product endomorphism $\sigma\rho$ contains the identity automorphism [16].

In this paper we want to give a purely algebraic characterization of the conjugate sector (*i.e.* endomorphisms modulo inner automorphisms) for inclusions of von Neumann algebras arising from some relevant classes of endomorphisms (see below). This covers some interesting cases of non-factor inclusions of arbitrary (*i.e.* non necessarily finite) index.

In more detail, we start with an endomorphism $\rho \in End(M)$ of a properly infinite von Neumann algebra M for which there is a normal faithful conditional expectation from M onto $\rho(M)$, and determine, using algebraic conditions on Jones' projection, the conjugate endomorphism of ρ. As an immediate corollary we have an algebraic characterization of an endomorphism ρ for which there exists a normal faithful conditional expectation from $\rho(M)'$ onto M'.

In case ρ is irreducible (and $E(M, \rho(M)) \neq \emptyset$) one obtains the same condition holding in finite index inclusions, that is an irreducible endomorphism σ is conjugate to ρ iff $\sigma\rho \succ id$; besides we prove that the identity is contained only once (as in the finite index case).

For an endomorphic inclusion of von Neumann algebras with arbitrary but finite index, we give necessary and sufficient conditions for the conjugate which extend the result in [11]. This characterization coincides with the definition of the conjugate map appearing in [2],[3] for particle-antiparticle symmetry in local quantum theory, and in the more general context of monoidal C*-categories [5], [6].

An interesting problem is to give an abstract characterization of the canonical endomorphism, as in [18] for the finite index case. Unfortunately we are presently able to answer this question only in a particular case, but we hope to return to this problem in the near future.

For convenience of the reader we recall the following notations, used throughout the paper.

We consider in the following, for simplicity, only inclusions of von Neumann algebras with separable predual. Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then $s_M(e)$ is the support in M of the projection $e \in \mathcal{B}(\mathcal{H})$. If N is a von Neumann subalgebra of M then $C(M,N)$ and $E(M,N)$ are the set of normal, resp. normal faithful, conditional expectations from M onto N. $End(M)$ is the set of normal faithful unital endomorphisms of M, and for $\rho, \sigma \in End(M)$, (σ, ρ) is the vector space of intertwiners between ρ and σ,

$$(\sigma, \rho) := \{v \in M : v\sigma(x) = \rho(x)v, \ x \in M\}.$$

Finally γ is a canonical endomorphism for the inclusion $N \subset M$, $\bar{\rho}$ is a conjugate endomorphism of ρ given by $\bar{\rho} = \rho^{-1} \cdot \gamma$, and $[\rho] \in Sect(M)$ is the sector determined by ρ in $End(M)$ modulo inner automorphisms.
For the general theory of von Neumann algebras we refer to [23], [22], [24]

2. An algebraic characterization of the basic construction

In this section we provide a purely algebraic characterization of Jones' basic construction, based on spatial theory, for an inclusion $N \subset M$ of von Neumann algebras [12]. This is a generalization to the case of non factor inclusions, possibly with non finite index, of theorem 1.2 of [21], and is crucial for all that follows.

Theorem 2.1. *Let $N \subset M \subset L$ be von Neumann algebras, $E \in E(M,N)$, $f \in Proj(L)$ s.t.*
(i) $fxf = E(x)f, \ x \in M$,
(ii) $L = \langle M, f \rangle$,
(iii) $s_{Z(L)}(f) = 1$, $s_N(f) = 1$.
Then there exists an isomorphism $\phi : L \to M_1$, where $M_1 := \langle M, e \rangle$ is Jones' basic construction of $N \subset M$, s.t. $\phi|_M = id_M$ and $\phi(f) = e$.
Proof.
Let us first suppose that $N \subset M \subset L \subset \mathcal{B}(\mathcal{H})$ be properly infinite and act standardly on $\mathcal{H}$, and f be infinite relative to N'.
Observe that $f \in N' \wedge L$ as $fxf = E(x)f = xf, \ x \in N$. Besides N_f, which is isomorphic to N by (iii), acts standardly on $f\mathcal{H}$, therefore $\exists \xi \in f\mathcal{H}$, cyclic and separating for N_f. It follows that ξ is separating for both N, M. Indeed, if $x \in N, \ x\xi = 0$ then $fxf\xi = x\xi = 0$, which implies, as ξ is separating for N_f, $x_f = 0$ that is $x = 0$. Now, if $x \in M, \ x\xi = 0$ then $0 = fx^*xf\xi = E(x^*x)f\xi = E(x^*x)\xi$, which implies, as ξ is separating for N, $E(x^*x) = 0$, that is $x = 0$.
Note that $\omega_\xi \cdot E = \omega_\xi$, where $\omega_\xi := (\xi, \cdot\xi)$, and $[N\xi] = [N_f\xi] = f$.
Introduce now $p := [M\xi] \in M'$ and observe that $p \geq f$ so that $pf = fp$ and, by (ii), $p \in L' \equiv \{x \in M' : xf = fx\}$.

Therefore we can define an homomorphism $\phi : x \in L \to x_p \in L_p$ which is in fact an isomorphism as $s_{Z(L)}(p) \geq s_{Z(L)}(f) = 1$.
Let us now observe that $L_p = \langle M_p, f_p \rangle$ and that $N_p \subset M_p \subset \langle M_p, f_p \rangle$ is Jones' basic construction. Indeed ξ is cyclic and separating for M_p, and, setting $\phi_0 := \phi|_M$, $E_p := \phi_0 \cdot E \cdot \phi_0^{-1} \in E(M_p, N_p)$, and $\omega := \omega_\xi \cdot \phi_0^{-1} \in E(M_p)$, we get $\omega \cdot E_p = \omega$, $\omega = (\xi, \cdot \xi)$ and $f_p = [N_p \xi]$ is Jones' projection.
In the general case let us take the tensor product of L with a type III factor F and apply the above reasoning to $N \otimes F \subset M \otimes F \subset L \otimes F$ with $E \otimes id$ and $f \otimes 1$ in place of E and f respectively. Therefore we obtain the existence of an isomorphism $\tilde{\phi} : L \otimes F \to M_1 \otimes F$, where $M_1 \otimes F$ is Jones basic construction for the inclusion $N \otimes F \subset M \otimes F$, and $\tilde{\phi}|_{M \otimes F} = id_{M \otimes F}$. Then $\tilde{\phi}(L \otimes 1) = M_1 \otimes 1$, as $L = \langle M, f \rangle$ and $\tilde{\phi}(f \otimes 1) = e \otimes 1$.

□

3. The conjugate endomorphism for some infinite index inclusions

In this section we characterize the conjugate sector in a purely algebraic fashion in the spirit of [16], [18]. This result is applied in the following section to some interesting cases.
We begin with some technical lemmas, which are used repeatedly in the sequel.

Lemma 3.1. *Let $N \subset M \subset \mathcal{B}(\mathcal{H})$, $E \in C(M, N)$, $f \in Proj(\mathcal{B}(\mathcal{H}))$ s.t. $E(x)f = fxf$, $x \in M$. Then*
(i) $s_{N' \wedge M}(f) = s_M(f)$;
(ii) E is faithful and $s_N(f) = 1 \iff s_M(f) = 1$.
Proof.
(i) It follows from $s_{N' \wedge M}(f) = [(N \vee M')f\mathcal{H}] = [\sum n_i m_i' f \mathcal{H}] = [\sum m_i' f n_i \mathcal{H}] = [M' f \mathcal{H}] = s_M(f)$.
(ii) $(\Rightarrow)$ Let $p \in Proj(M)$ be s.t. $pf = 0$. Then $0 = fpf = E(p)f \Rightarrow E(p) = 0 \Rightarrow p = 0$.
$(\Leftarrow)$ Let $p \in Proj(M)$ be s.t. $E(p) = 0$. Then $0 = E(p)f = fpf \Rightarrow pf = 0 \Rightarrow p = 0$.

□

We want to show two cases where the support of a projection is very easily calculated.

Lemma 3.2.
(i) Let $N \subset M \subset M_1 = \langle M, e \rangle$ be Jones' basic construction. Then $s_{M' \wedge M_1}(e) = s_{N' \wedge M}(e) = 1$.
(ii) Let M be properly infinite, $\rho \in End(M)$, $v \in (id, \rho)$ isometry. Then $s_{\rho(M)}(vv^*) = 1$.

Proof.
(i) It follows immediately from the fact that the range of e contains, by construction, a cyclic and separating vector for M.
(ii) Let $p = \rho(q) \in Proj(\rho(M))$ s.t. $pvv^* = 0$. Then $0 = pv = \rho(q)v = vq$ so that $q = 0$ that is $p = 0$.

$\square$

For the reader's convenience we report the following

Lemma 3.3. [15, Prop. 5.1] *Let $N \subset M$ be properly infinite von Neumann algebras. Then $\forall E \in E(M,N)$, there is an isometry $v \in N$ s.t. $E(x) = v^*\gamma(x)v$, $x \in M$, $v \in (id|_N, \gamma|_N)$ and $\gamma^{-1}(vv^*)$ is Jones' projection for the inclusion $N \subset M$.*
Proof.
Let $\Omega \in \mathcal{H}$ be a cyclic and separating vector for $N \subset M$, $\omega := (\Omega, \cdot\Omega)$ and set $\varphi := \omega \cdot E$. Let $\xi \in L^2(M,\Omega)_+$ be a vector representative of φ. The map $V : x\Omega \in N\Omega \to x\xi \in N\xi$ determines an isometry $V \in N'$. Set $e := VV^* \equiv [N\xi] \in N'$, so that $e = J^\Omega_M e J^\Omega_M$, and $v := J^\Omega_N V J^\Omega_N \in N$. Then, with $\Gamma := J^\Omega_N J^\Omega_M$, we get $\Gamma^* v = \Gamma^* J^\Omega_N V J^\Omega_N = J^\Omega_M V J^\Omega_N = J^\Omega_M J^\xi_N V = V$, where $J^\xi_N = J^\Omega_M|_{e\mathcal{H}}$, so that $VJ^\Omega_N V^* = J^\xi_N$. There follows that

$$v^*\gamma(x)v = v^*\Gamma x\Gamma^* v = V^* x V = V^* exe V = V^* E(x) e V = V^* E(x) V = E(x), \ x \in M,$$

and $\gamma(x)v = \Gamma x \Gamma^* v = \Gamma x V = \Gamma V x = vx$, $x \in N$. Finally $vv^* = J^\Omega_N VV^* J^\Omega_N = J^\Omega_N e J^\Omega_N = J^\Omega_N J^\Omega_M e J^\Omega_M J^\Omega_N = \Gamma e \Gamma^* = \gamma(e)$, hence $\gamma^{-1}(vv^*) = e$.

$\square$

We can now prove the main result of this section, concerning the algebraic characterization of the conjugate sector.

Theorem 3.4. *Let M be a properly infinite von Neumann algebra and $\rho, \sigma \in End(M)$. Then the following are equivalent:*
(i) $E(M, \rho(M)) \neq \emptyset$ and $\sigma = \bar{\rho}$,
(ii) $\sigma\rho \succ id$, with $v \in (id, \sigma\rho)$ an isometry s.t.

$$s_{\sigma(M)' \wedge M}(vv^*) = 1$$
$$s_{\sigma\rho(M)' \wedge \sigma(M)}(vv^*) = 1.$$

Remark. Observe that $\sigma(M)' \wedge M \equiv (\sigma, \sigma)$ and $\sigma\rho(M)' \wedge \sigma(M) \equiv \sigma((\rho, \rho))$.

Proof.

$(i) \Rightarrow (ii)$ Let $E \in E(M, \rho(M))$ and consider the inclusions $M \supset \bar\rho(M) \supset \bar\rho\rho(M)$. Then by lemma 3.3 there exists an isometry $V = \bar\rho\rho(v) \in \bar\rho\rho(M)$ s.t. $\gamma'(x)V = Vx,\ x \in \bar\rho\rho(M)$ and $V^*\gamma'(x)V = F(x),\ x \in \bar\rho(M)$, where $\gamma' : \bar\rho(M) \to \bar\rho\rho(M)$ is the canonical endomorphism and $F := \bar\rho \cdot E \cdot \bar\rho^{-1} \in E(\bar\rho(M), \bar\rho\rho(M))$.

Then $\bar\rho\rho(x)v = vx,\ x \in M$ and $E(x) = \rho(v^*\bar\rho(x)v),\ x \in M$. Indeed, as $\gamma' = ad\Gamma|_{\bar\rho(M)} = \bar\rho\rho|_{\bar\rho(M)}$, [15, prop. 2.4], we have $\bar\rho\rho(\bar\rho\rho(x)v) = \bar\rho\rho(\bar\rho\rho(x))V = V\bar\rho\rho(x) = \bar\rho\rho(vx),\ x \in M$, and $\bar\rho\rho(v^*\bar\rho(x)v) = V^*\bar\rho\rho(\bar\rho(x))V = F(\bar\rho(x)) = \bar\rho \cdot E(x),\ x \in M$.

Finally, from lemma 3.3 it follows that $\Gamma^*VV^*\Gamma = \Gamma^*\bar\rho\rho(vv^*)\Gamma = vv^*$ is Jones' projection for the inclusion $\bar\rho\rho(M) \subset \bar\rho(M)$, so that by lemma 3.2 (i) we have

$$s_{\bar\rho(M)'\wedge M}(vv^*) = s_{\bar\rho\rho(M)'\wedge\bar\rho(M)}(vv^*) = 1.$$

$(ii) \Rightarrow (i)$ From lemma 3.1 (i) it follows that $s_{\sigma(M)}(f) = s_{\sigma\rho(M)'\wedge\sigma(M)}(f) = 1$, where $f := vv^*$ is the range projection of v, so that by lemma 3.1 (ii) the expectation $F := \sigma\rho(v^* \cdot v) \in C(\sigma(M), \sigma\rho(M))$ is faithful, as $F(x)f = \sigma\rho(v^*xv)vv^* = v(v^*xv)v^* = fxf,\ x \in \sigma(M)$. Besides $s_{\sigma\rho(M)}(f) \geq s_{\sigma(M)}(f) = 1$. Finally, with $L := \langle\sigma(M), f\rangle$, we have $s_{Z(L)}(f) \geq s_{\sigma(M)'\wedge M}(f) = 1$. Therefore, from theorem 2.1, $\sigma\rho(M) \subset \sigma(M) \subset L$ is Jones' basic construction for the inclusion $\sigma\rho(M) \subset \sigma(M)$.

We now want to prove that $\langle\sigma(M), f\rangle = M$. Let us first observe that $f\langle\sigma(M), f\rangle f = f\sigma\rho(M)f = \{\sigma\rho(x)f : x \in M\} = \{vxv^* : x \in M\} = fMf$ so that $\langle\sigma(M), f\rangle_f = M_f$. Therefore $L'_f = \langle\sigma(M), f\rangle'_f = M'_f$, and, as $s_{Z(L)}(f) = 1$ the map $L' \to L'_f$ is an isomorphism which restricts to the isomorphism $M' \to M'_f \equiv L'_f$ so that $L' = M'$ that is $L = M$.

We want to show that σ is conjugate to ρ; to do this we follow the same computations of [16, page 296], which we report here for the reader's convenience.

Choose $\Omega \in \mathcal{H}$ cyclic and separating for $M, \rho(M), \sigma(M)$ and set $J := J^\Omega_M,\ J_\rho := J^\Omega_{\rho(M)}$. Let U be the canonical unitary implementation of σ with respect to Ω, and let $\sigma^{-1} := adU^*$. From the hypotheses it follows that $\sigma^{-1}(M) = \langle M, \sigma^{-1}(e)\rangle = M_1 = J\rho(M)'J$. Therefore $\sigma^{-1}(v), \sigma^{-1}(e) \in M_1 = J\rho(M)'J$, so that $v_0 := J\sigma^{-1}(v)J \in \rho(M)'$. From the proof of theorem 2.1 it follows that there exists $\xi \in \sigma^{-1}(e)\mathcal{H}$ cyclic and separating for M s.t. $\sigma^{-1}(e) = [\rho(M)\xi]$. The canonical unitary implementation of the isomorphism $y \in \rho(M) \to y\sigma^{-1}(e) \in \rho(M)\sigma^{-1}(e)$ with respect to Ω and ξ, is given by the isometry $w_0 = v_0z$, where $z \in \rho(M)'$ is unitary. Then, from [15, prop. 3.1], we get $\Gamma_\rho := J_\rho J = w_0^*Jw_0J = z^*v_0^*Jv_0zJ = z^*v_0^*Jv_0JJzJ$, thus, to compute the class of $\gamma_\rho := ad\Gamma_\rho$, the canonical endomorphism of M into $\rho(M)$, we may assume $w_0 = v_0$.

Then we have, $\forall x \in M$,

$$\begin{aligned}
\Gamma_\rho x \Gamma_\rho^* &= v_0^* J v_0 J x J v_0^* J v_0 \\
&= J\sigma^{-1}(v)^* J\sigma^{-1}(v) x \sigma^{-1}(v)^* J\sigma^{-1}(v) J \\
&= J\sigma^{-1}(v)^* J\sigma^{-1}(v\sigma(x)v^*) J\sigma^{-1}(v) J \\
&= J\sigma^{-1}(v)^* J\sigma^{-1}(vv^*\sigma\rho\sigma(x)) J\sigma^{-1}(v) J \\
&= J\sigma^{-1}(v)^* J\sigma^{-1}(e)\rho\sigma(x) J\sigma^{-1}(v) J \\
&= J\sigma^{-1}(v)^* J\sigma^{-1}(e) J\sigma^{-1}(v) J\rho\sigma(x) \\
&= J\sigma^{-1}(v^*ev) J\rho\sigma(x) \\
&= \rho\sigma(x),
\end{aligned}$$

because $J\sigma^{-1}(v)J \in \rho(M)'$ and $J\sigma^{-1}(e)J = \sigma^{-1}(e)$. Hence we get $[\rho\sigma] = [\gamma_\rho]$, that is ρ and σ are conjugate.

□

As an immediate consequence of theorem 3.4 we have:

Corollary 3.5. *Let M be a properly infinite von Neumann algebra and $\rho, \sigma \in End(M)$. Then the following are equivalent:*
(i) $E(\rho(M)', M') \neq \emptyset$ and $\sigma = \bar{\rho}$,
(ii) $\rho\sigma \succ id$, with $w \in (id, \rho\sigma)$ an isometry s.t.

$$s_{\rho(M)' \wedge M}(ww^*) = 1$$
$$s_{\rho\sigma(M)' \wedge \rho(M)}(ww^*) = 1.$$

□

Remark. Observe that in [9] Herman and Ocneanu adopt the following terminology. Let $N \subset M$ be an inclusion of von Neumann algebras. They say
(i) $N \subset M$ is *discrete* if $E(M, N) \neq \emptyset$,
(ii) $N \subset M$ is *compact* if $E(N', M') \neq \emptyset$.
According to this definition, one could say an endomorphism ρ of M is discrete (compact) if $\rho(M) \subset M$ is discrete (compact).
As Herman and Ocneanu observed in [9], if $M = N \times_\alpha G$ where α is an action of an abelian locally compact group G, the inclusion $N \subset M$ is discrete (resp. compact) iff G is discrete (resp. compact).

4. An interesting case: irreducible inclusions

In the category of unitary representations of a compact group G we have a very simple characterization for the conjugate class of an irreducible representation, as pointed out in the introduction, namely $\pi, \sigma \in Irr(G)$ are conjugate to each other iff $\pi \otimes \sigma \succ id$, besides the identity representation is contained with multiplicity one.
These results are also true for irreducible endomorphisms of a factor M, that is $\rho,\ \sigma \in End(M)$, irreducible and of finite index, are conjugate to each other iff $\rho\sigma \succ id$, moreover the identity automorphism appears only once [16, th 4.1].
In the compact group case, due to Peter-Weyl theorem [10], the roles of π and σ are symmetric. The same is true in the case of endomorphisms of finite index of a factor, in fact the inclusion $\rho(M) \subset M$ has finite index iff it is both discrete and compact, in the terminology of [9], so that one can exchange the roles of ρ and σ.
In the general case of possibly infinite index, one cannot exchange ρ and σ but we prove that the above characterization also holds for the discrete case. Besides we show that id appears only once in the product $\bar{\rho}\rho$, also in this case.

Theorem 4.1. *Let M be an infinite factor, $\sigma, \rho \in End(M)$ and irreducible; then the following are equivalent:*
(i) $E(M, \rho(M)) \neq \emptyset$ and $\sigma = \bar{\rho}$;
(ii) $\sigma\rho \succ id$.
Moreover $(id, \sigma\rho)$ is one dimensional.
Proof.
The equivalence of (i) and (ii) follows easily from theorem 3.4. Let $e, f \in Proj(M)$ be two Jones' projections for the inclusion $\sigma\rho(M) \subset \sigma(M)$ s.t. $\langle \sigma(M), e\rangle = \langle \sigma(M), f\rangle = M$ and let $v, w \in (id, \sigma\rho)$ isometries s.t. $vv^* = e,\ ww^* = f$. Then, by theorem 2.1, there exists $\alpha \in Aut(M)$ s.t. $\alpha(e) = f$ and $\alpha|_{\sigma(M)} = id$. Therefore $\alpha(v) \in (\sigma\rho, \alpha)$, indeed $\sigma\rho(x)\alpha(v) = \alpha(\sigma\rho(x)v) = \alpha(vx) = \alpha(v)\alpha(x)\ x \in M$, so that $\sigma\rho \succ \alpha$. As $\alpha(v)^*\alpha(v) = f = ww^*$, $u := \alpha(v)^*w \in M$ is unitary and $xu^* = xw^*\alpha(v) = w^*\sigma\rho(x)\alpha(v) = w^*\alpha(v)\alpha(x) = u^*\alpha(x),\ x \in M$, so that $\alpha(x) = uxu^*,\ x \in M$; as $\alpha|_{\sigma(M)} = id$, we get $u \in \sigma(M)' \wedge M \equiv \mathbf{C}$, because σ is irreducible. Therefore $\alpha = id$ and $e = f$.
Let us now consider $vw^* = evw^*e \in eMe = \sigma\rho(M)e$, that is, there exists $\lambda \in M$ s.t. $vw^* = \sigma\rho(\lambda)e = \sigma\rho(\lambda)ww^*$, which implies $v = \sigma\rho(\lambda)w = w\lambda$, and $w^*v = \lambda$. But we have also $w^*v \in Z(M) \equiv \mathbf{C}$, so that $v = \lambda w$, with $\lambda \in \mathbf{C},\ |\lambda| = 1$.

□

The last part of the previous theorem holds also in the general case of irreducible inclusions of infinite factors.

Corollary 4.2. *Let $N \subset M$ be an irreducible inclusion of infinite factors. Then $(id|_N, \gamma|_N)$ is at most one dimensional.*

Proof.
If $E(M,N) = \emptyset$, supposing that there were $v \in (id|_N, \gamma|_N)$ then $E := v^*\gamma(\cdot)v$ would belong to $C(M,N)$, and by ([22], 10.17) it would be (unique) and faithful, contradicting the hypothesis.
If $E(M,N) \neq \emptyset$, upon tensoring with and absorbing factor ([17], lemma 2.3) we may assume that N is isomorphic with M, that is $N = \rho(M)$ with $\rho \in End(M)$, and from theorem 3.4 it follows that there is $v \in (id|_N, \gamma|_N)$, which is unique by theorem 4.1.

□

5. Finite index case

For finite index inclusions one can characterize the conjugate endomorphism via conditions on some intertwiner operators between the products $\sigma\rho$, $\rho\sigma$ and the identity. This is known in the factor case [16] and, at least as a sufficient condition, for scalar index inclusions of general von Neumann algebras [11].
Analogous conditions can be used to provide a purely algebraic definition of a conjugation in braided categories of endomorphisms [5], [1].
In the following theorem we adopt the definition of index given by Kosaki in [13].

Theorem 5.1. *Let M be a properly infinite von Neumann algebra, $\rho, \sigma \in End(M)$, then the following are equivalent:*
(i) σ is conjugate to ρ and $Ind(\rho) < \infty$;
(ii) there exist $v \in (id, \sigma\rho)$, $w \in (id, \rho\sigma)$ isometries s.t. $w^*\rho(v) = z$, $v^*\sigma(w) = \sigma(z')$, with $z, z' \in Z(M)_+$ and invertible.
Moreover, setting $E := \rho(v^*\sigma(\cdot)v) \in C(M, \rho(M))$, E is faithful and $Ind_E(\rho) = z^{-2}$;
(iii) there exist $w' \in (id, \rho\sigma)$, $v' \in (id, \sigma\rho)$ isometries s.t. $v'^*\sigma(w') = c$, $w'^*\rho(v) = \rho(c')$, with $c, c' \in Z(M)_+$ and invertible.
Moreover, setting $\mathcal{E} := \sigma(w'^*\rho(\cdot)w') \in C(M, \sigma(M))$, $\mathcal{E}$ is faithful and $Ind_{\mathcal{E}}(\sigma) = c^{-2}$.
Proof.
$(i) \Rightarrow (ii)$ Let $E \in E(M, \rho(M))$ be s.t. $Ind_E(\rho) < \infty$. Then by lemma 3.3 there is $v \in (id, \sigma\rho)$ isometry s.t. $E(x) = \rho(v^*\sigma(x)v)$, $x \in M$. Besides $e := vv^*$ is Jones' projection for the inclusion $\sigma\rho(M) \subset \sigma(M)$. Let F be the dual expectation of $\sigma \cdot E \cdot \sigma^{-1}$, then we get $\sigma\rho\sigma(y)F(v) = F(\sigma\rho\sigma(y)v) = F(v\sigma(y)) = F(v)\sigma(y)$, $y \in M$, and applying σ^{-1} we get $\rho\sigma(y)\phi(v) = \phi(v)y$, $y \in M$, where $\phi := \sigma^{-1} \cdot F$ is the left inverse of σ relative to F. Then $\sigma(Ind(E)) \equiv Ind(\sigma \cdot E \cdot \sigma^{-1}) = F(e)^{-1} \in \sigma(Z(M)) \equiv Z(\sigma(M))$ and, by the push-down lemma [20], we get $v = ev = e\sigma(m)$ with $m \in M$ uniquely determined by $\sigma(m) = F(e)^{-1}F(v)$, so that $F(v)^*F(v) = \sigma(m)^*F(e)F(e)\sigma(m) = F(e)\sigma(m)^*F(e)\sigma(m) = F(e)F(\sigma(m)^*e\sigma(m)) = F(e)F(v^*v) = F(e)$. So, applying σ^{-1} we obtain $\phi(v)^*\phi(v) = \phi(e) = \sigma^{-1} \cdot \sigma(Ind(E)^{-1}) = Ind(E)^{-1} \in Z(M)$. Let now

$\phi(v) = wz$ be the polar decomposition of $\phi(v)$, so that $z^2 \equiv \phi(v)^*\phi(v) = Ind(E)^{-1}$, therefore $z = Ind(E)^{-1/2} \in Z(M)_+$ and invertible, which implies that w is an isometry, and from $\rho\sigma(y)wz = wzy,\ y \in M$ we obtain $w \in (id, \rho\sigma)$. Finally we compute:

$$\begin{aligned} w^*\rho(v) &= z^{-1}\phi(v)^*\rho(v) = z^{-1}\phi(v^*\sigma\rho(v)) \\ &= z^{-1}\phi(vv^*) = z^{-1}\phi(e) \\ &= z^{-1}Ind(E)^{-1} = z \end{aligned}$$

$$\begin{aligned} v^*\sigma(w) &= v^*\sigma(\phi(v)z^{-1}) = v^*F(v)\sigma(z)^{-1} \\ &= v^*F(e\sigma(m))\sigma(z)^{-1} = v^*F(e)\sigma(m)\sigma(z)^{-1} \\ &= v^*\sigma(z^2)\sigma(m)\sigma(z)^{-1} = v^*\sigma(m)\sigma(z) \\ &= v^*e\sigma(m)\sigma(z) = v^*v\sigma(z) = \sigma(z). \end{aligned}$$

$(ii) \Rightarrow (i)$ We divide the proof in two lemmas.

Lemma 5.2. $E := \rho(v^*\sigma(\cdot)v) \in E(M, \rho(M))$.
Proof.
We have only to prove faithfulness. Let us set $G(y) := \sigma \cdot E \cdot \sigma^{-1}(y)$ and prove that $G \in E(\sigma(M), \sigma\rho(M))$ from which it will follow immediately that $E \in E(M, \rho(M))$.
Set $e := vv^* \in Proj(M)$; then $\sigma\rho(x)e = \sigma\rho(x)vv^* = vxv^* = vv^*\sigma\rho(x) = e\sigma\rho(x)$, $\forall x \in M$, that is $e \in \sigma\rho(M)' \wedge M$. Besides

$$e\sigma(x)e = vv^*\sigma(x)vv^* = \sigma\rho(v^*\sigma(x)v)vv^* = \sigma(E(x))e, \forall x \in M,$$

that is $G(y)e = eye$, $\forall y \in \sigma(M)$.
Now, if $y = \sigma(x) \in \sigma(M)$ and $G(y^*y) = 0$, we have $0 = G(y^*y)e = ey^*ye$, that is $0 = ye = \sigma(x)e$, that is $\sigma(x)v = 0$, that is $\rho\sigma(x)\rho(v) = 0$, which implies $0 = w^*\rho\sigma(x)\rho(v) = xw^*\rho(v) = xz$, that is $x = 0$, which implies $y = 0$ and the faithfulness of G.

□

Lemma 5.3. $M = \langle\sigma(M), e\rangle$, *that is* $\sigma\rho(M) \subset \sigma(M) \subset M = \langle\sigma(M), e\rangle$ *is Jones' basic construction.*
Proof.
Let us observe that $\langle\sigma(M), e\rangle_e = \sigma\rho(M)_e = M_e$ and $\langle\sigma(M), e\rangle'_e = M'_e$.
Since

$$\begin{aligned} s_{Z(\langle\sigma(M),e\rangle)}(e) &\geq s_{\sigma(M)'\wedge\langle\sigma(M),e\rangle}(e) \geq s_{\sigma(M)'}(e) \\ &= [\sigma(M)vv^*\mathcal{H}] \geq [\sigma(w)^*vv^*\mathcal{H}] = [\sigma(z')\mathcal{H}] = 1 \end{aligned}$$

we get $M' = \langle\sigma(M), e\rangle'$ that is $M = \langle\sigma(M), e\rangle$.

Now from lemma 3.2 (ii) it follows that $s_{\sigma\rho(M)}(e) = 1$ so that, by theorem 2.1 we get the thesis.

$\square$

End of proof of theorem 5.1.
From lemma 5.3 and theorem 3.4 we get the conclusion.

$\square$

6. The canonical endomorphism in the 'discrete' case

Theorem 3.4 immediately allows us to characterize the canonical endomorphism of non necessarily finite index inclusions of von Neumann algebras, at least in the discrete case.

Proposition 6.1. *Let M be a properly infinite von Neumann algebra, $\lambda \in End(M)$; then the following are equivalent:*
(i) there exist $N \subset M$, $E \in E(M, N)$ s.t. $\lambda : M \to N$ is the canonical endomorphism;
(ii) there exists an isometry $v \in (\lambda, \lambda^2)$ s.t.

$$\lambda(v)v = v^2, \qquad \lambda(v)^*v = vv^*,$$

and,

$$s_{\lambda(M)' \wedge M}(vv^*) = s_{\lambda(\lambda(M)' \wedge M)}(vv^*) = 1.$$

Proof.
$(i) \Rightarrow (ii)$ From lemma 3.3 there exists $v \in N$ isometry s.t. $v^*\lambda(x)v = E(x),\ x \in M$, and $\lambda(x)v = vx,\ x \in N$, and finally vv^* is Jones' projection for the inclusion $\lambda(N) \subset \lambda(M)$. Therefore $\lambda(\lambda(x))v = v\lambda(x),\ x \in M$, that is $v \in (\lambda, \lambda^2)$; $\lambda(v)v = v^2$, and $\lambda(v)^*v = vv^*$ are immediate and $s_{\lambda(M)' \wedge M}(vv^*) = s_{\lambda(\lambda(M)' \wedge M)}(vv^*) = 1$ follow from lemma 3.2.
$(ii) \Rightarrow (i)$ Let us set $E := v^*\lambda(\cdot)v$. Then, as in [18, lemma 5.2], $E \in C(M, N)$, where $N := E(M)$ is a von Neumann subalgebra of M. We want to show that E is faithful. Let us set $F := \lambda \cdot E \cdot \lambda^{-1} \in C(\lambda(M), \lambda(N))$ and $e := vv^*$ and observe that $F(x)e = exe,\ x \in \lambda(M)$, and $s_{\lambda(M)}(e) = s_{\lambda(N' \wedge M)}(e) \geq s_{\lambda(\lambda(M)' \wedge M)}(e) = 1$, so that, by lemma 2.3 (ii), F is faithful, and so is E. Let us observe that $s_{\lambda(N)}(e) \geq s_{\lambda(M)}(e) = 1$ and $s_{Z(\langle\lambda(M), e\rangle)}(e) \geq s_{\lambda(M)' \wedge M}(e) = 1$ so that, by theorem 2.1, $\langle\lambda(M), e\rangle$ is Jones' basic construction for the inclusion $\lambda(N) \subset \lambda(M)$. Besides we have $N_e = \{exe : x \in N\} = \{vv^*xvv^* : x \in N\} = \{\lambda(v^*xv)vv^* : x \in N\} =$ by [18, lemma 5.3] $= \lambda(N)_e = \langle\lambda(M), e\rangle_e$ so that we can conclude, analogously to theorem 3.4, that $N = \langle\lambda(M), e\rangle$. Now calculations similar to those in the proof of theorem 3.4 can be done to show that λ is a canonical endomorphism for the inclusion $N \subset M$.

□

Remarks. We cannot give conditions under which the canonical endomorphism is associated to an irreducible inclusion.
Another interesting open problem would be to characterize the canonical endomorphism associated to a 'compact' inclusion. This has applications in the characterization of crossed products of von Neumann algebras by actions of discrete infinite dimensional Kac algebras, as in [18], [9], or by more general objects as 'quantum groups' naturally appearing in low dimensional quantum field theories [19], [7], [8], [16].

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