A C*-ALGEBRA OF GEOMETRIC OPERATORS ON SELF-SIMILAR CW-COMPLEXES. NOVIKOV-SHUBIN AND L²-BETTI NUMBERS

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Abstract. A class of CW-complexes, called self-similar complexes, is introduced, together with C*-algebras $\mathcal{A}_j$ of operators, endowed with a finite trace, acting on square-summable cellular j-chains. Since the Laplacian Δ_j belongs to $\mathcal{A}_j$, L^2-Betti numbers and Novikov-Shubin numbers are defined for such complexes in terms of the trace. In particular a relation involving the Euler-Poincaré characteristic is proved. L^2-Betti and Novikov-Shubin numbers are computed for some self-similar complexes arising from self-similar fractals.

1. Introduction.

In this paper we address the question of the possibility of extending the definition of some L^2-invariants, like the L^2-Betti numbers and Novikov-Shubin numbers, to geometric structures which are not coverings of compact spaces.

The first attempt in this sense is due to John Roe [29], who defined a trace on finite propagation operators on amenable manifolds, allowing the definition of L^2-Betti numbers on these spaces. However such trace is defined in terms of a suitable generalised limit, hence the corresponding L^2-Betti and Novikov-Shubin numbers also depend on this generalised limit procedure.

Here we show that, on spaces possessing a suitable self-similarity, it is possible to select a natural C*-algebra of operators, generated by operators with finite propagation and locally commuting with the transformations giving the self-similar structure, on which a Roe-type trace is well defined.

The theory of L^2-invariants was started by Atiyah, who, in a celebrated paper [1], observed that on covering manifolds $\Gamma \to M \to X$, a trace on Γ-periodic operators may be defined, called Γ-trace, with respect to which the Laplace operator has compact resolvent. Replacing the usual trace with the Γ-trace, he defined the L^2-Betti numbers and proved an index theorem for covering manifolds.

Based on this paper, Novikov and Shubin [27] observed that, since for noncompact manifolds the spectrum of the Laplacian is not discrete, new global spectral invariants can be defined, which necessarily involve the density near zero of the spectrum.

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L^2-Betti numbers were proved to be Γ-homotopy invariants by Dodziuk [7], whereas Novikov-Shubin numbers were proved to be Γ-homotopy invariants by Gromov-Shubin [12]. L^2-Betti numbers (depending on a generalised limit procedure) were subsequently defined for open manifolds by Roe, and were proved to be invariant under quasi-isometries [30]. The invariance of Novikov-Shubin numbers was proved in [13].

The basic idea of the present analysis is the notion of self-similar CW-complex, which is defined as a complex endowed with a natural exhaustion $\{K_n\}$ in such a way that K_{n+1} is a union (with small intersections) of a finite number of copies of K_n. The identification of the different copies of K_n in K_{n+1} gives rise to many local isomorphisms on such complexes. Then we consider finite propagation operators commuting with these local isomorphisms up to boundary terms, and call them geometric operators. For any j from zero to the dimension of the complex, geometric operators on the space of ℓ^2-chains of j-cells generate a C*-algebra $\mathcal{A}_j$, containing the j-Laplace operator. For any operator T in this C*-algebra, we consider the sequence of the traces of TE_n, renormalised with the volume of the j-cells of K_n, where E_n denotes the projection onto the space generated by the j-cells of K_n. Such a sequence is convergent, and the corresponding functional is indeed a finite trace on $\mathcal{A}_j$. By means of these traces, L^2-Betti numbers and Novikov-Shubin numbers are defined. For the sake of completeness, we mention that notions related to that of geometric operators have been considered in the literature, see *e.g.* [24] where they are called tight binding operators, and [10] where they are called pattern invariant operators.

In the Γ-covering case, L^2-Betti numbers are defined as Γ-dimensions of the kernels of Laplace operators, namely as Γ-traces of the corresponding projections. This is not allowed in our framework. Indeed, our traces being finite, and the C*-algebras being weakly dense in the algebra of all bounded linear operators, our traces cannot extend to the generated von Neumann algebras. In particular they are not defined on the spectral projections of the Laplace operators. Therefore we define L^2-Betti numbers as the infimum of the traces of all continuous functional calculi of the Laplacian, with functions taking value 1 at 0, namely L^2-Betti numbers are defined as the "external measure" of the spectral projections of the Laplace operators.

Since we are in an infinite setting, the Euler-Poincaré characteristic is naturally defined as a renormalised limit of the Euler-Poincaré characteristic of the truncations K_n of the complex. We prove that such characteristic coincides with the alternating sum of the L^2-Betti numbers. An analogous result, though obtained with a different proof, for amenable simplicial complexes is contained [9].

Here we do not prove directly invariance results for L^2-Betti or Novikov-Shubin numbers, however when 1-dimensional CW-complexes are considered, and in particular prefractal graphs determined by nested fractals, a result by Hambly and Kumagai applies [20], implying that Novikov-Shubin numbers are invariant under rough isometries. Further results on invariance will be proved elsewhere [6].

We then show that in some cases L^2-Betti and Novikov-Shubin numbers can be computed, relying on results of several authors concerning random walks on graphs. In particular, it turns out that the Novikov-Shubin numbers of some prefractal complexes coincide with the spectral dimensions of the corresponding fractals, thus strenghtening the interpretation of such numbers as (asymptotic) spectral dimensions given in [13].

Our framework was strongly influenced by the approach of Lott and Lück [25], in particular we also consider invariants relative to the boundary, however we are not able to prove the Poincaré duality shown in [25].

The paper is organised as follows. In Section 2 we recall some notions from the theory of CW-complexes and introduce the basic operators. Section 3 introduces the notion of local isomorphisms of CW-complexes and the algebra of geometric operators. The notion of self-similar CW-complex is given in Section 4, and a finite trace on geometric operators is constructed.

In Section 5 we introduce L^2-Betti and Novikov-Shubin numbers for the above setting, and prove the mentioned result on the Euler-Poincaré characteristic. Section 6 focuses on the subclass of self-similar CW-complexes given by prefractal complexes, and on some properties of the associated Laplacians. Computations of the Novikov-Shubin numbers for fractal graphs in terms of transition probabilities, together with an invariance result under rough isometries are discussed in Sections 7 and 8, and the top-dimensional relative Novikov-Shubin number is computed for two examples of 2-dimensional CW-complexes.

In closing this introduction, we note that the C*-algebra and the trace for self-similar graphs constructed in this paper, are used in [19] to study the Ihara zeta function for fractal graphs.

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2. CW-complexes and basic operators.

In this paper we shall consider a particular class of infinite CW-complexes, therefore we start by recalling some notions from algebraic topology, general references being [26, 28]. A *CW-complex* M of dimension $p \in \mathbb{N}$ is a Hausdorff space consisting of a disjoint union of (open) cells of dimension $j \in \{0, 1, \dots, p\}$ such that: (i) for each j-cell σ^j_α, there is a continuous map $f^j_\alpha : \{x \in \mathbb{R}^j : \|x\| \leq 1\} \to X$ that is a homeomorphism of $\{x \in \mathbb{R}^j : \|x\| < 1\}$ onto σ^j_α, and maps $\{x \in \mathbb{R}^j : \|x\| = 1\}$ into a finite union of cells of dimension $< j$; (ii) a set $A \subset X$ is closed in X iff $A \cap \overline{\sigma}^j_\alpha$ is closed in $\overline{\sigma}^j_\alpha$, for all j, α, where $\overline{\sigma}^j_\alpha$ denotes the closure of σ^j_α in M. Let us denote by $\dot{\sigma}^j_\alpha = f^j_\alpha(\{x \in \mathbb{R}^j : \|x\| = 1\})$ the boundary of σ^j_α, for all j, α. A CW-complex is *regular* if f^j_α is a homeomorphism, for all j, α.

We denote by $\mathcal{E}_j(M) := \{\sigma^j_\alpha : \alpha \in \mathcal{A}_j\}$, $j = 0, 1, \dots, p$, the family of j-cells, and by $M^j := \cup_{k=0}^j \mathcal{E}_k(M)$, the j-skeleton of M. Then $C_j(M) := H_j(M^j, M^{j-1}, \mathbb{Z})$ is the (abelian) group of j-dimensional cellular chains, and is generated by the class of σ^j_α, $\alpha \in \mathcal{A}_j$. Let $\partial_j : C_j(M) \to C_{j-1}(M)$ be the boundary operator, which is the connecting homomorphism of the homology sequence of the triple (M^j, M^{j-1}, M^{j-2}). Let us choose an *orientation* of M, that is, a basis $\{\widehat{\sigma}^j_\alpha : \alpha \in \mathcal{A}_j\}$ of $C_j(M)$, $j \in \{0, \dots, p\}$, where each $\widehat{\sigma}^j_\alpha$ is (up to sign) the class of one (open) j-cell. We will usually identify the algebraic cell $\widehat{\sigma}^j_\alpha$ with the geometric cell σ^j_α, and denote by $-\sigma^j_\alpha$ the cell σ^j_α with the opposite orientation. Then the action of ∂_j on the chosen basis is given by $\partial_j \sigma^j_\alpha = \sum_{\beta\in\mathcal{A}_{j-1}} [\sigma^j_\alpha : \sigma^{j-1}_\beta]\sigma^{j-1}_\beta$, where $[\sigma^j_\alpha : \sigma^{j-1}_\beta] \in \mathbb{Z}$ depends on the chosen orientation and is called incidence number. If M is regular, $[\sigma^j_\alpha : \sigma^{j-1}_\beta] \in \{-1, 0, 1\}$, and $[\sigma^j_\alpha : \sigma^{j-1}_\beta] = 0 \iff \sigma^{j-1}_\beta \cap \overline{\sigma}^j_\alpha = \emptyset$. Let us recall that the orientation of the zero-cells is chosen in such a way that, for any 1-cell σ^1, $\sum_\alpha [\sigma^1, \sigma^0_\alpha] = 0$.

In the following we will consider only regular CW-complexes, unless otherwise stated.

A Hilbert norm on $C_j(M) \otimes_{\mathbb{Z}} \mathbb{C}$ is then defined as $\|c\|^2 := \sum_i |c_i|^2$ when $c = \sum_i c_i \cdot \sigma_i \in C_j(M) \otimes_{\mathbb{Z}} \mathbb{C}$. The Hilbert space $C^{(2)}_j(M) \equiv \ell^2(\mathcal{E}_j M)$ is the completion of $C_j(M) \otimes_{\mathbb{Z}} \mathbb{C}$ under this norm.

We can extend ∂_j to a densely defined linear operator $C^{(2)}_j(M) \to C^{(2)}_{j-1}(M)$. Then the *half–Laplace operators* $\Delta_{j\pm}$ are

$$\begin{aligned} \Delta_{j+} &:= \partial_{j+1}\partial^*_{j+1} \\ \Delta_{j-} &:= \partial^*_j \partial_j \end{aligned}$$

and the *Laplace operators* are $\Delta_j := \Delta_{j+} + \Delta_{j-}$. These are operators on $\ell^2(\mathcal{E}_j M)$ densely defined on $C_j(M) \otimes_{\mathbb{Z}} \mathbb{C}$.

Let us observe that ∂_j is a bounded operator under some condition.

Definition 2.1 (Bounded complex)**.** Let M be a regular CW-complex, denote by

$$\begin{aligned} V^+_j &:= \sup_{\sigma\in\mathcal{E}_j(M)} |\{\tau \in \mathcal{E}_{j+1}(M) : \dot\tau \supset \sigma\}| \\ V^-_j &:= \sup_{\sigma\in\mathcal{E}_j(M)} |\{\rho \in \mathcal{E}_{j-1}(M) : \rho \subset \dot\sigma\}|, \end{aligned}$$

where $|\cdot|$ denotes the cardinality. We say that M is a bounded complex if $V^\pm_j < \infty$, for all j.

Lemma 2.2. *Let M be a bounded regular CW-complex. Then $\partial_j : \ell^2(\mathcal{E}_j M) \to \ell^2(\mathcal{E}_{j-1} M)$ is bounded.*

Proof. If $c = \sum_i c_i \cdot \sigma_i$, setting $\alpha \sim \beta$ if there is $\rho \in \mathcal{E}_{j-1}(M)$ such that $\rho \subset \dot\sigma_\alpha \cap \dot\sigma_\beta$, we have

$$\begin{aligned} \|\partial_j c\|^2 &= \sum_{\alpha\sim\beta} \overline{c_\alpha} \cdot c_\beta \cdot (\partial_j\sigma_\alpha, \partial_j\sigma_\beta) \\ &\le \sum_{\alpha\sim\beta} |c_\alpha| \cdot |c_\beta| \cdot |(\partial_j\sigma_\alpha, \partial_j\sigma_\beta)| \\ &\le \frac{1}{2} V^-_j \sum_{\alpha\sim\beta} \Big(|c_\alpha|^2 + |c_\beta|^2\Big) \\ &\le (V^-_j)^2 V^+_{j-1} \|c\|^2. \end{aligned}$$

Indeed

$$\begin{aligned} |(\partial_j\sigma_\alpha, \partial_j\sigma_\beta)| &\le \sum_{\rho\subset\dot\sigma_\alpha\cap\dot\sigma_\beta} |[\sigma_\alpha : \rho]| \cdot |[\sigma_\beta : \rho]| \\ &\le |\{\rho \in \mathcal{E}_{j-1} : \rho \subset \dot\sigma_\alpha \cap \dot\sigma_\beta\}| \le V^-_j, \end{aligned}$$

while, for any $\alpha \in \mathcal{A}_j$, $|\{\beta \in \mathcal{A}_j : \beta \sim \alpha\}| \le V^-_j V^+_{j-1}$. $\square$

Lemma 2.3.
$$\partial^*_{j+1}\sigma = \sum_{\tau\in\mathcal{E}_{j+1}(M)} [\tau:\sigma]\tau\,.$$

Proof. Indeed, with $\tau \in \mathcal{E}_{j+1}(M)$,
$$(\tau, \partial^*_{j+1}\sigma) = (\partial_{j+1}\tau, \sigma) = \sum_{\sigma'\in\mathcal{E}_j(M)} [\tau:\sigma'](\sigma',\sigma) = [\tau:\sigma].$$
□

Proposition 2.4. *Let M be a bounded regular CW-complex. Then, for $\sigma\,,\sigma' \in \mathcal{E}_j(M)$, we have*
$$(\sigma\,,\Delta_{j+}\,\sigma') = \sum_{\tau\in\mathcal{E}_{j+1}(M)} [\tau:\sigma][\tau:\sigma'],$$
and
$$(\sigma\,,\Delta_{j-}\,\sigma') = \sum_{\tau\in\mathcal{E}_{j-1}(M)} [\sigma:\tau][\sigma':\tau].$$
In particular,
$$(\sigma, \Delta_{j+}\sigma) = |\,\{\tau \in \mathcal{E}_{j+1}(M) : \dot\tau \supset \sigma\}\,|,$$
$$(\sigma, \Delta_{j-}\sigma) = |\,\{\tau \in \mathcal{E}_{j-1}(M) : \tau \subset \dot\sigma\}\,|.$$

Proof. Straightforward computation. □

Remark 2.5. It follows that $\Delta_{j\pm}$ does not depend on the orientation of the $(j\pm1)$-cells, but only on the orientation of the j-cells.

3. Local Isomorphisms and Geometric Operators

In this section, we define geometric operators and prove that the Laplacians (absolute or relative to the boundary subcomplex) are geometric.

Definition 3.1 (Combinatorial distance)**.** Let M be a connected, regular, bounded CW-complex. Let σ, σ' be distinct cells in $\mathcal{E}_j(M)$. We set
(*i*) $d_-(\sigma,\sigma') = 1$, if there is $\rho \in \mathcal{E}_{j-1}(M)$ such that $\rho \subset \dot\sigma \cap \dot\sigma'$,
(*ii*) $d_+(\sigma,\sigma') = 1$, if there exists $\tau \in \mathcal{E}_{j+1}(M)$ such that $\sigma\cup\sigma' \subset \dot\tau$,
(*iii*) $d(\sigma,\sigma') = 1$, if either $d_-(\sigma,\sigma') = 1$ or $d_+(\sigma,\sigma') = 1$.
The distances d, d_-, d_+ between two general distinct cells σ and σ' are then defined as the minimum number of steps of length one needed to pass from σ to σ', and as $+\infty$ if such a path does not exist.

We say that $\mathcal{E}_j(M)$ is $d_\pm$-connected if $d_\pm(\sigma,\sigma') < +\infty$ for any $\sigma,\sigma' \in \mathcal{E}_j(M)$.

Proposition 3.2. *Let M be a p-dimensional, regular, bounded CW-complex.*
(*i*) *If $\mathcal{E}_j(M)$ is d_+-connected, then it is d_--connected.*
(*ii*) *Assume any j-cell is contained in the boundary of some $(j+1)$-cell, $j+1 \le p$. Then, if $\mathcal{E}_{j+1}(M)$ is d_--connected, then $\mathcal{E}_j(M)$ is d_+-connected.*

Proof. (*i*). Let us show that if $d_+(\sigma_0,\sigma_1) = 1$, σ_0, $\sigma_1 \in \mathcal{E}_j(M)$, then $d_-(\sigma_0,\sigma_1) \le V^-_{j+1} - 1$. Let $\tau \in \mathcal{E}_{j+1}(M)$ be such that $\sigma_0,\sigma_1 \subset \dot\tau$. Let $\{\sigma_i\}$ be a basis of j-cells oriented according to some orientation on $\dot\tau$, which is homeomorphic to the j-sphere. Then $\sum_i \sigma_i$ is the unique j-cycle (up to constant multiples) representing the non-trivial homology class, hence $\partial_j \sum_i \sigma_i = 0$. This corresponds to the fact

that any $(j-1)$-cell has non-trivial incidence number with exactly two j-cells, one incidence number being 1 and the other -1. Assume now there is a d_--connected component $\cup_k \sigma_{i_k}$ which is properly contained in the boundary of τ. Since $\sum_k \sigma_{i_k}$ is not a cycle, there exists a $(j-1)$-cell ρ such that $(\rho, \partial_j \sum_k \sigma_{i_k}) \neq 0$. Then there is exactly one j-cell, not belonging to $\cup_k \sigma_{i_k}$, having non-trivial incidence number with ρ. But this is impossible, since $\cup_k \sigma_{i_k}$ is d_--connected. Since the maximum number of j-faces of $\tau \in \mathcal{E}_{j+1}(M)$ is V^-_{j+1}, the thesis follows.
(ii) Let $\rho_1 \neq \rho_2 \in \mathcal{E}_j(M)$, $\sigma_1, \sigma_2 \in \mathcal{E}_{j+1}$ such that $\rho_i \subset \dot{\sigma}_i$. Then, since a d_--path from σ_1 to σ_2 gives rise to a d_+-path from ρ_1 to ρ_2, we have

$$d_+(\rho_1, \rho_2) \leq d_-(\sigma_1, \sigma_2) + 1.$$

□

If $\sigma \in \mathcal{E}_j(M)$, $r \in \mathbb{N}$, we write $B_r(\sigma) := \{\sigma' \in \mathcal{E}_j(M) : d(\sigma', \sigma) \leq r\}$.

Definition 3.3 (Finite propagation operators)**.** A bounded linear operator A on $\ell^2(\mathcal{E}_j M)$ has *finite propagation* $r = r(A) \geq 0$ if, for all $\sigma \in \mathcal{E}_j(M)$, $\mathrm{supp}(A\sigma) \subset B_r(\sigma)$ and $\mathrm{supp}(A^*\sigma) \subset B_r(\sigma)$.

Lemma 3.4. *Finite propagation operators form a *-algebra.*

Proof. The set of finite propagation operators is *-closed by definition. To prove that it is also an algebra, one can choose, for example,

$$r(\lambda A + B) = r(A) \vee r(B)\,, \qquad r(AB) = r(A) + r(B).$$

□

Given two CW-complexes M, N, a continuous map $f : M \to N$ is called *cellular* if $f(M^j) \subset N^j$, for all j; it induces linear maps $f_j : C_j(M) \otimes_{\mathbb{Z}} \mathbb{C} \to C_j(N) \otimes_{\mathbb{Z}} \mathbb{C}$ intertwining the boundary maps. The cellular map f is called *regular* if, for all j, $\sigma \in \mathcal{E}_j(M)$, there are k, $\tau \in \mathcal{E}_k(N)$ such that $f(\sigma) = \tau$, $f(\dot{\sigma}) = \dot{\tau}$; then, necessarily, $k \leq j$. We call f an *isomorphism* if it is a bijective regular map such that $[f_j \sigma^j_\alpha : f_{j-1}\sigma^{j-1}_\beta] = [\sigma^j_\alpha : \sigma^{j-1}_\beta]$, for all j, α, β. Then f is a homeomorphism and f_j is a linear isomorphism.
A *subcomplex* N of M is a closed subspace of M which is a union of (open) cells. We call N a *full* subcomplex if, for all j, $\sigma \in \mathcal{E}_j(M)$, $\dot{\sigma} \subset N$ imply $\sigma \subset N$.

To prove that a cell belongs to a full subcomplex, we will find it convenient in the sequel to refer to the following

Lemma 3.5. *Let N be a full subcomplex of the regular CW-complex M. Let $\tau \in \mathcal{E}_j(M)$ be such that, for all $\rho \in \mathcal{E}_{j-1}(\dot{\tau})$, one has $\rho \in N$. Then $\tau \in N$.*

Proof. As N is a subcomplex, it follows that $\dot{\tau} \subset N$; therefore $\tau \in N$, because N is full. □

Definition 3.6 (Local Isomorphisms and Geometric Operators)**.** A *local isomorphism* of the CW-complex M is a triple

$$\Big(s(\gamma)\,, r(\gamma)\,, \gamma\Big)$$

where $s(\gamma)\,, r(\gamma)$ are full subcomplexes of M and $\gamma : s(\gamma) \to r(\gamma)$ is an isomorphism.

For any $j = 0, \dots, \dim(M)$, the local isomorphism γ defines a *partial isometry* $V_j(\gamma) : \ell^2(\mathcal{E}_j M) \to \ell^2(\mathcal{E}_j M)$, by setting

$$V_j(\gamma)(\sigma) := \begin{cases} \gamma_j(\sigma) & \sigma \in \mathcal{E}_j(s(\gamma)) \\ 0 & \sigma \notin \mathcal{E}_j(s(\gamma)), \end{cases}$$

and extending by linearity. An operator $T \in B(\ell^2(\mathcal{E}_j M))$ is called *geometric* if there exists $r > 0$ such that T has finite propagation r and, for any local isomorphism γ, any $\sigma \in \mathcal{E}_j(M)$ such that $B_r(\sigma) \subset s(\gamma)$ and $B_r(\gamma\sigma) \subset r(\gamma)$, one has

$$TV_j(\gamma)\sigma = V_j(\gamma)T\sigma, \quad T^*V_j(\gamma)\sigma = V_j(\gamma)T^*\sigma\,.$$

Proposition 3.7. *Let M be a regular, bounded CW-complex. Then, for any j, geometric operators on $\ell^2(\mathcal{E}_j M)$ form a *-algebra. The half Laplacians $\Delta_{j\pm}$ belong to it.*

Proof. The first statement is obvious. Concerning the second, let us note that, since the complex is bounded, half Laplacians $\Delta_{j\pm}$ are bounded (cf. Lemma 2.2).

Let $\sigma, \sigma' \in \mathcal{E}_j(M)$, with $B_1(\sigma) \subset s(\gamma)$ and $B_1(\gamma\sigma) \subset r(\gamma)$. If $\sigma' \notin r(\gamma)$, from $\operatorname{supp}(\Delta_{j\pm}\sigma) \subset B_1(\sigma) \subset s(\gamma)$ and $\operatorname{supp}(\Delta_{j\pm}(\gamma_j\sigma)) \subset B_1(\gamma\sigma) \subset r(\gamma)$, we obtain

$$(\sigma', \Delta_{j\pm}\, V_j(\gamma)\sigma) = 0 = (\sigma', V_j(\gamma)\Delta_{j\pm}\, \sigma).$$

So, let us suppose that $\sigma' \in r(\gamma)$, so that $\sigma' = \gamma_j\sigma''$, for $\sigma'' \in s(\gamma)$ and

$$\begin{aligned}
(\sigma', \Delta_{j-}\, V_j(\gamma)\sigma) &= \sum_{\tau \in \mathcal{E}_{j-1}(M)} [\sigma' : \tau][\gamma_j\sigma : \tau] \\
&= \sum_{\tau' \in \mathcal{E}_{j-1}(M)} [\sigma' : \gamma_{j-1}\tau'][\gamma_j\sigma : \gamma_{j-1}\tau'] \\
&= \sum_{\tau' \in \mathcal{E}_{j-1}(M)} [\sigma'' : \tau'][\sigma : \tau'] \\
&= (V_j(\gamma)^*\sigma', \Delta_{j-}\, \sigma) = (\sigma', V_j(\gamma)\Delta_{j-}\, \sigma),
\end{aligned}$$

where the third equality comes from the incidence-preserving property of γ, and in the second equality we used the fact that the non-zero terms in the sum come from τ's which are "components" of the chain $\partial_j\gamma_j\sigma = \gamma_{j-1}\partial_j\sigma = \sum c_i\gamma_{j-1}\rho_i$, if $\partial_j\sigma = \sum c_i\rho_i$, so that $\tau = \gamma_{j-1}\rho_i$, for some i. By linearity we get that Δ_{j-} is geometric. As for Δ_{j+},

$$\begin{aligned}
(\sigma', \Delta_{j+}\, V_j(\gamma)\sigma) &= \sum_{\tau \in \mathcal{E}_{j+1}(M)} [\tau : \sigma'][\tau : \gamma_j\sigma] \\
&= \sum_{\tau' \in \mathcal{E}_{j+1}(M)} [\gamma_{j+1}\tau' : \sigma'][\gamma_{j+1}\tau' : \gamma_j\sigma] \\
&= \sum_{\tau' \in \mathcal{E}_{j+1}(M)} [\tau' : \sigma''][\tau' : \sigma] \\
&= (V_j(\gamma)^*\sigma', \Delta_{j+}\, \sigma) = (\sigma', V_j(\gamma)\Delta_{j+}\, \sigma),
\end{aligned}$$

where the third equality comes from the incidence-preserving property of γ, and in the second equality we used the fact that the non-zero terms in the sum come from τ's such that $[\tau : \gamma_j\sigma] \neq 0$, so that, for all $\rho \in \mathcal{E}_j(\dot\tau)$, we get $d(\rho, \gamma\sigma) = 1$, hence $\rho \in r(\gamma)$; from Lemma 3.5, $\tau \in r(\gamma)$, so there is $\tau' \in s(\gamma)$ such that $\tau = \gamma_{j+1}\tau'$. By linearity we get that Δ_{j+} is geometric. $\square$

We now consider a version of the boundary operators relative to the boundary subcomplex. This idea is due to Lott and Lück [25], who introduced relative invariants for covering CW-complexes. In this way, other non-trivial L^2-Betti numbers are available, as shown in Section 8.

Let M be a p-dimensional, regular, bounded CW-complex. We shall consider the $(p-1)$-dimensional *boundary subcomplex* ∂M, defined as follows:

(i) a $(p-1)$-cell of M is in ∂M if it is contained in at most one p-cell.
(ii) a j-cell of M is in ∂M if it is contained in a $(p-1)$-cell in ∂M.

Then ∂M is a regular bounded CW-complex.

Lemma 3.8. *Let N be a full subcomplex of M, and $\sigma_0 \in \mathcal{E}_j(N)$ be such that $B_k(\sigma_0) \subset N$. Then, for any $\tau_0 \in \mathcal{E}_{j+1}(N)$ such that $\sigma_0 \subset \dot{\tau}_0$, one has $B_\ell(\tau_0) \subset N$, for $\ell \leq \frac{k}{V^-_{j+1}-1}$.*

Proof. Let $\tau_1 \in \mathcal{E}_{j+1}(M)$ be such that $d(\tau_1, \tau_0) \leq \ell$. From the proof of Proposition 3.2 (i), for any $\sigma_1 \in \mathcal{E}_j(M)$, $\sigma_1 \subset \dot{\tau}_1$, one has $d(\sigma_1, \sigma_0) \leq \ell(V^-_{j+1}-1) \leq k$. Therefore $\sigma_1 \subset N$. As N is full, $\tau_1 \subset N$, and the thesis follows. □

Lemma 3.9. *Let γ be a local isomorphism, $\sigma \in \mathcal{E}_j(M)$ be such that $B_k(\sigma) \subset s(\gamma)$, $B_k(\gamma\sigma) \subset r(\gamma)$, where $k \geq (V^-_{p-1}-1)(V^-_{p-2}-1)\ldots(V^-_{j+1}-1)$. Then $\sigma \in \partial M$ iff $\gamma\sigma \in \partial M$.*

Proof. ($\Rightarrow$) Let $\sigma \in \mathcal{E}_{p-1}(\partial M)$. If there were $\tau \neq \tau' \in \mathcal{E}_p(M)$ such that $\gamma\sigma \subset \dot{\tau} \cap \dot{\tau}'$, then for all $\rho \in \mathcal{E}_{p-1}$, $\rho \subset \dot{\tau}$ we would get $d(\rho, \gamma\sigma) = 1$, hence $\rho \in r(\gamma)$; from Lemma 3.5, $\tau \in r(\gamma)$; analogously $\tau' \in r(\gamma)$. As γ_p preserves incidences and boundaries, $\sigma \subset \gamma^{-1}\dot{\tau} \cap \gamma^{-1}\dot{\tau}'$, which implies $\sigma \notin \partial M$, and we have reached a contradiction. Therefore, there is a unique $\tau \in \mathcal{E}_p(M)$ such that $\gamma\sigma \subset \dot{\tau}$, which means that $\sigma \in \partial M$.

If $\sigma \in \mathcal{E}_j(\partial M)$, there is $\tau \in \mathcal{E}_{p-1}(M) \cap \partial M$ such that $\sigma \subset \dot{\tau}$, and $\gamma\sigma \subset (\gamma\tau)^{\cdot} = \gamma(\dot{\tau})$. Then, from Lemma 3.8, $B_1(\tau) \subset \sigma(\gamma)$, and $B_1(\gamma\tau) \subset \rho(\gamma)$. From what has already been proved, $\gamma\tau \in \partial M$. Therefore $\gamma\sigma \in \partial M$, because ∂M is a subcomplex.

($\Leftarrow$) follows from the above applied to γ^{-1}. □

Let $\overline{\partial}_j \equiv \partial_j^{M,\partial M}$ be the boundary operator of the relative cellular complex $C_j(M, \partial M) := H_j(M^j \cup \partial M, M^{j-1} \cup \partial M, \mathbb{Z})$. As $C_j(M, \partial M) \cong \oplus_{\sigma \in \overline{\mathcal{E}}_j(M)} \mathbb{Z}\sigma$, where $\overline{\mathcal{E}}_j(M) := \{\sigma \in \mathcal{E}_j(M) : \sigma \cap \partial M = \emptyset\}$, we can identify $C_j^{(2)}(M, \partial M)$, the ℓ^2-completion of $C_j(M, \partial M) \otimes_{\mathbb{Z}} \mathbb{C}$, with $\ell^2(\overline{\mathcal{E}}_j(M))$, a closed subspace of $\ell^2(\mathcal{E}_j(M))$. Moreover we can consider $\overline{\partial}_j : C_j^{(2)}(M) \to C_{j-1}^{(2)}(M)$, $\overline{\partial}_j^* : C_{j-1}^{(2)}(M) \to C_j^{(2)}(M)$, by extending them to 0 on $C_j^{(2)}(M, \partial M)^\perp$ or $C_{j-1}^{(2)}(M, \partial M)^\perp$, respectively. Define $\overline{\Delta}_{j+} := \overline{\partial}_{j+1}\overline{\partial}_{j+1}^*$, $\overline{\Delta}_{j-} := \overline{\partial}_j^*\overline{\partial}_j$. Then

Lemma 3.10.
(i) *$\overline{\Delta}_{j\pm}\sigma = 0$, for $\sigma \in C_j^{(2)}(M, \partial M)^\perp$,*
(ii) *for $\sigma, \sigma' \in C_j^{(2)}(M, \partial M)$,*

$$(\sigma', \overline{\Delta}_{j+}\sigma) = \sum_{\tau \in \overline{\mathcal{E}}_{j+1}(M)} [\tau : \sigma][\tau : \sigma'], \qquad (\sigma', \overline{\Delta}_{j-}\sigma) = \sum_{\tau \in \overline{\mathcal{E}}_{j-1}(M)} [\sigma : \tau][\sigma' : \tau].$$

Proposition 3.11. *Let γ be a local isomorphism, $\sigma \in \mathcal{E}_j(M)$ be such that $B_k(\sigma) \subset s(\gamma)$, $B_k(\gamma\sigma) \subset r(\gamma)$, for some $k \geq 1 + \prod_{i=j+1}^{p-1}(V_i^- - 1)$. Then*

$$\overline{\Delta}_{j-}\, V_j(\gamma)\sigma = V_j(\gamma)\, \overline{\Delta}_{j-}\sigma, \qquad \overline{\Delta}_{j+}\, V_j(\gamma)\sigma = V_j(\gamma)\, \overline{\Delta}_{j+}\sigma\,.$$

Proof. Let us prove that, for any $\sigma' \in \overline{\mathcal{E}}_j(M)$, we have

$$(s', \overline{\Delta}_{j\pm}\, V_j(\gamma)\sigma) = (\sigma', V_j(\gamma)\, \overline{\Delta}_{j\pm}\sigma).$$

If $\sigma' \notin B_1(\gamma\sigma)$, the thesis is true. Indeed, from $\text{supp}(\overline{\Delta}_{j\pm}\sigma) \subset B_1(\sigma) \subset s(\gamma)$, and $\text{supp}(\overline{\Delta}_{j\pm}\gamma\sigma) \subset B_1(\gamma\sigma) \subset r(\gamma)$, it follows $(\sigma', \overline{\Delta}_{j\pm}V_j(\gamma)\sigma) = 0$, whereas, if $\sigma' \notin r(\gamma)$ we get $(\sigma', V_j(\gamma)\overline{\Delta}_{j\pm}\sigma) = 0$, while, if $\sigma' \in r(\gamma) \setminus B_1(\gamma\sigma)$, we get $(\sigma', V_j(\gamma)\overline{\Delta}_{j\pm}\sigma) = (\gamma_j^{-1}\sigma', \overline{\Delta}_{j\pm}\sigma) = 0$, as $d(\gamma_j^{-1}\sigma', \sigma) = d(\sigma', \gamma_j\sigma) > 1$. Therefore, we can assume $\sigma' \in B_1(\gamma\sigma)$. Moreover, if $\sigma \in \partial M$, so that $\gamma\sigma \in \partial M$ (by Lemma 3.9), we get $\overline{\Delta}_{j\pm}V_j(\gamma)\sigma = 0 = V_j(\gamma)\overline{\Delta}_{j\pm}\sigma$.

Therefore, we now assume $\sigma \notin \partial M$, $\sigma' \in B_1(\gamma\sigma)$. Then

$$(\sigma', \overline{\Delta}_{j-}\, V_j(\gamma)\sigma) = \sum_{\tau \in \overline{\mathcal{E}}_{j-1}(M)} [\sigma' : \tau][\gamma_j\sigma : \tau].$$

Let $\tau \in \overline{\mathcal{E}}_{j-1}(M)$, $\tau \subset (\gamma\sigma)\dot{} \cap \dot{\sigma}'$. Then, as in the first part of the proof of Proposition 3.7, there is $\tau' \in \mathcal{E}_{j-1}(s(\gamma))$ such that $\tau = \gamma\tau'$; moreover $\tau' \subset \dot{\sigma}$, as $[\sigma : \tau'] = [\gamma\sigma : \gamma\tau'] \neq 0$. Let us now show that $\tau \in \partial M \iff \tau' \in \partial M$; indeed, if $\tau \in \partial M$, then there is $\rho \in \mathcal{E}_j(\partial M)$ such that $\tau \subset \dot{\rho}$; therefore $d(\rho, \gamma\sigma) \leq 1$, and $B_{k-1}(\rho) \subset r(\gamma)$, so, from Lemma 3.9, it follows that $\rho' := \gamma_j^{-1}\rho \in \partial M$; then $[\rho' : \tau'] = [\rho : \tau] \neq 0$, hence $\tau' \subset \dot{\rho}'$, and $\tau' \in \partial M$. The other implication follows similarly. Therefore

$$\begin{aligned}
(\sigma', \overline{\Delta}_{j-}\, V_j(\gamma)\sigma) &= \sum_{\tau \in \overline{\mathcal{E}}_{j-1}(M)} [\sigma' : \tau][\gamma_j\sigma : \tau] \\
&= \sum_{\tau' \in \overline{\mathcal{E}}_{j-1}(M)} [\sigma' : \gamma_{j-1}\tau'][\gamma_j\sigma : \gamma_{j-1}\tau'] \\
&= \sum_{\tau' \in \overline{\mathcal{E}}_{j-1}(M)} [\gamma_j^{-1}\sigma' : \tau'][\sigma : \tau'] \\
&= (V_j(\gamma)^*\sigma', \overline{\Delta}_{j-}\sigma) = (\sigma', V_j(\gamma)\, \overline{\Delta}_{j-}\sigma).
\end{aligned}$$

As for $\overline{\Delta}_{j+}$, we get

$$(\sigma', \overline{\Delta}_{j+}\, V_j(\gamma)\sigma) = \sum_{\tau \in \overline{\mathcal{E}}_{j+1}(M)} [\tau : \sigma'][\tau : \gamma_j\sigma].$$

Let $\tau \in \mathcal{E}_{j+1}(M)$ be such that $\gamma\sigma \cup \sigma' \subset \dot{\tau}$; as for any $\rho \in \mathcal{E}_j(M)$, $\rho \subset \dot{\tau}$, it holds $d(\rho, \gamma\sigma) \leq 1$, so $\rho \in B_1(\gamma\sigma) \subset r(\gamma)$, from Lemma 3.5 we get $\tau \in r(\gamma)$; therefore there is $\tau' \in s(\gamma)$ such that $\tau = \gamma\tau'$. From Lemma 3.8 it follows $B_\ell(\tau) \subset r(\gamma)$, for

$\ell \leq k/(V_{j+1}^- - 1)$, and Lemma 3.9 gives us $\tau \in \partial M \iff \tau' \in \partial M$. Therefore

$$\begin{aligned}(\sigma', \overline{\Delta}_{j+} V_j(\gamma)\sigma) &= \sum_{\tau \in \overline{\mathcal{E}}_{j+1}(M)} [\tau : \sigma'][\tau : \gamma_j \sigma] \\ &= \sum_{\tau' \in \overline{\mathcal{E}}_{j+1}(M)} [\gamma_{j+1}\tau' : \sigma'][\gamma_{j+1}\tau' : \gamma_j \sigma] \\ &= \sum_{\tau' \in \overline{\mathcal{E}}_{j+1}(M)} [\tau' : \gamma_j^{-1}\sigma'][\tau' : \sigma] \\ &= (V_j(\gamma)^* \sigma', \overline{\Delta}_{j+}\sigma) = (\sigma', V_j(\gamma)\, \overline{\Delta}_{j+}\sigma).\end{aligned}$$

□

We have proved the following.

Proposition 3.12. *Let M be a p-dimensional, regular, bounded CW-complex. The relative half-Laplacians $\overline{\Delta}_{j\pm}$ are geometric operators.*

4. Self-similar CW-complexes

In this section we introduce self-similar complexes, and show that there is a natural trace state on the algebra of geometric operators.

If K is a subcomplex of M, we call *j-frontier* of K, and denote it by $\mathcal{F}(\mathcal{E}_j K)$, the family of cells in $\mathcal{E}_j K$ having distance 1 from the complement of $\mathcal{E}_j K$ in $\mathcal{E}_j(M)$.

Definition 4.1 (Amenable CW-Complexes)**.** A countably infinite CW-complex M is *amenable* if it is regular and bounded, and has an *amenable exhaustion*, namely, an increasing family of finite subcomplexes $\{K_n : n \in \mathbb{N}\}$ such that $\cup K_n = M$ and for all $j = 0, \dots, \dim(M)$,

$$\frac{|\mathcal{F}(\mathcal{E}_j K_n)|}{|\mathcal{E}_j K_n|} \to 0 \qquad \text{as } n \to \infty .$$

Definition 4.2 (Self-similar CW-Complexes)**.** A countably infinite CW-complex M is *self-similar* if it is regular and bounded, and it has an amenable exhaustion by full subcomplexes $\{K_n : n \in \mathbb{N}\}$ such that the following conditions (i) and (ii) hold:

(i) for all n there is a finite set of local isomorphisms $\mathcal{G}(n, n+1)$ such that, for all $\gamma \in \mathcal{G}(n, n+1)$, one has $s(\gamma) = K_n$,

$$\bigcup_{\gamma \in \mathcal{G}(n,n+1)} \gamma_j \Big(\mathcal{E}_j(K_n) \Big) = \mathcal{E}_j(K_{n+1}), \qquad j = 0, \dots, \dim(M)$$

and moreover if $\gamma, \gamma' \in \mathcal{G}(n, n+1)$ with $\gamma \neq \gamma'$

$$\mathcal{E}_j \gamma(K_n) \cap \mathcal{E}_j \gamma'(K_n) = \mathcal{F}(\mathcal{E}_j \gamma(K_n)) \cap \mathcal{F}(\mathcal{E}_j \gamma'(K_n)) , \quad j = 0, \dots, \dim(M). \tag{4.1}$$

(ii) We then define $\mathcal{G}(n, m)$, with $n < m$, as the set of all admissible products $\gamma_{m-1} \cdot \dots \cdot \gamma_n$, $\gamma_i \in \mathcal{G}(i, i+1)$, where admissible means that the range of γ_j is contained in the source of γ_{j+1}. We let $\mathcal{G}(n, n)$ consist of the identity isomorphism on K_n, and $\mathcal{G}(n) = \cup_{m \geq n} \mathcal{G}(n, m)$. We now define the $\mathcal{G}$-*invariant* j-frontier of K_n:

$$\mathcal{F}_{\mathcal{G}}(\mathcal{E}_j K_n) = \bigcup_{\gamma \in \mathcal{G}(n)} \gamma_j^{-1} \mathcal{F}(\mathcal{E}_j \gamma(K_n)),$$

and we ask that

$$\frac{|\mathcal{F}_{\mathcal{G}}(\mathcal{E}_j K_n)|}{|\mathcal{E}_j K_n|} \to 0 \qquad \text{as } n \to \infty\,.$$

Remark 4.3. We may replace the condition in (4.1) with the following

$$\mathcal{E}_j\gamma(K_n) \cap \mathcal{E}_j\gamma'(K_n) \subseteq B_r(\mathcal{F}(\mathcal{E}_j\gamma(K_n))) \cap B_r(\mathcal{F}(\mathcal{E}_j\gamma'(K_n)))\,, \quad j = 0, \ldots, \dim(M),$$

for a suitable $r > 0$. It is easy to see that all the theory developed below will remain valid.

Some examples of self-similar CW-complexes are given below, cf. Section 6 for more details on the construction.

Example 4.4. The Gasket graph in figure 1, the Lindstrom graph in figure 2, the Vicsek graph in figure 3 are examples of 1-dimensional self-similar complexes. The Carpet 2-complex in figure 4 is an example of a 2-dimensional self-similar CW-complex.

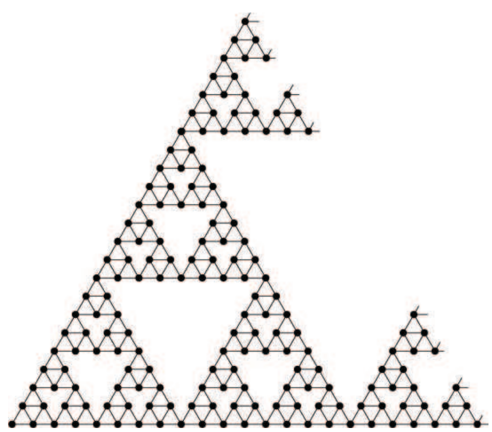

Figure 1. Gasket graph

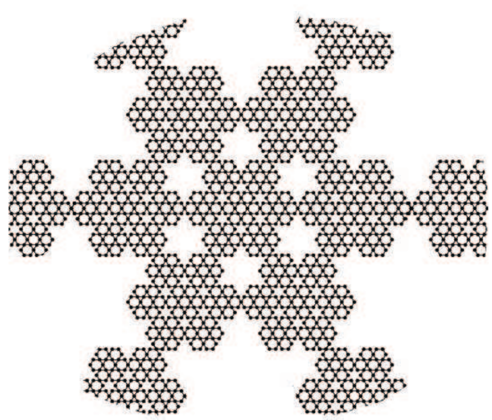

Figure 2. Lindstrom graph

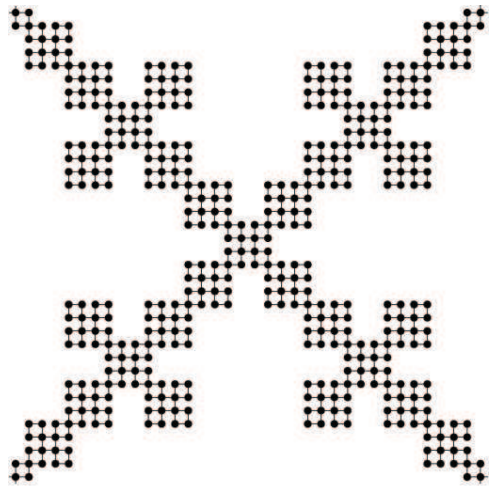

FIGURE 3. Vicsek graph

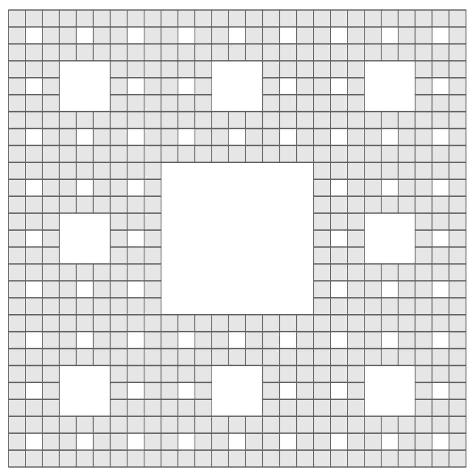

FIGURE 4. Carpet 2-complex

Theorem 4.5. *Let M be a self-similar CW-complex, $\mathcal{A}(\mathcal{E}_j M)$ the C^*–algebra given by the norm closure of the *-algebra of geometric operators. Then, on $\mathcal{A}(\mathcal{E}_j M)$ there is a well defined faithful trace state Φ_j given by*

$$\Phi_j(T) = \lim_n \frac{Tr\big(E(\mathcal{E}_j K_n)T\big)}{Tr\big(E(\mathcal{E}_j K_n)\big)}$$

where $E(\mathcal{E}_j K_n)$ is the orthogonal projection of $\ell^2(\mathcal{E}_j M)$ onto $\ell^2(\mathcal{E}_j K_n)$.

Proof. Fix $j \in \{0, \ldots, p\}$, and for a finite subset $N \subset \mathcal{E}_j M$ denote by $E(N) \in \mathcal{B}(\ell^2(\mathcal{E}_j M))$ the projection onto span$N$. Let us observe that, since N is an orthonormal basis for $\ell^2(N)$, then $Tr\big(E(N)\big) = |N|$.

First step: some combinatorial results.

a) Let $\mu \equiv \mu_j = \sup_{\sigma\in\mathcal{E}_jM} |B_1(\sigma)|$. First observe that μ is finite, since $\mu \leq V_j^+ + V_j^-$.

Then, since

$$B_{r+1}(\sigma) = \bigcup_{\sigma'\in B_r(\sigma)} B_1(\sigma'),$$

we get $|B_{r+1}(\sigma)| \leq |B_r(\sigma)|\mu$, giving $|B_r(\sigma)| \leq \mu^r$, $\forall \sigma \in \mathcal{E}_jM$, $r \geq 0$. As a consequence, for any finite set $\Omega \subset \mathcal{E}_jM$, we have $B_r(\Omega) = \cup_{\sigma'\in\Omega} B_r(\sigma')$, giving

$$|B_r(\Omega)| \leq |\Omega|\mu^r, \quad \forall r \geq 0. \tag{4.2}$$

b) Let us set $\Omega(n,r) = \mathcal{E}_jK_n \setminus B_r(\mathcal{F}_{\mathcal{G}}(\mathcal{E}_jK_n))$. Then, for any $\gamma \in \mathcal{G}(n)$, we have

$$\gamma_j\Omega(n,r) \subset \gamma_j\mathcal{E}_jK_n \subset \gamma_j\Omega(n,r) \cup B_r(\mathcal{F}_{\mathcal{G}}(\gamma_j\mathcal{E}_jK_n)).$$

Now assume $r \geq 1$. Then, the $\gamma_j\Omega(n,r)$'s are disjoint, for different γ's in $\mathcal{G}(n,m)$. Therefore,

$$|\mathcal{E}_jK_n| \leq |\Omega(n,r)| + |\mathcal{F}_{\mathcal{G}}(\mathcal{E}_jK_n)|\mu^r, \tag{4.3}$$

$$\left|\mathcal{E}_jK_m \setminus \bigcup_{\gamma\in\mathcal{G}(n,m)} \gamma_j\Omega(n,r)\right| \leq |\mathcal{G}(n,m)|\,|\mathcal{F}_{\mathcal{G}}(\mathcal{E}_jK_n)|\mu^r, \tag{4.4}$$

$$|\mathcal{G}(n,m)|\,|\Omega(n,r)| \leq |\mathcal{E}_jK_m| \leq |\mathcal{G}(n,m)|\,|\mathcal{E}_jK_n|. \tag{4.5}$$

Indeed, (4.3) and (4.5) are easily verified, whereas

$$\begin{aligned}\left|\mathcal{E}_jK_m \setminus \bigcup_{\gamma\in\mathcal{G}(n,m)} \gamma_j\Omega(n,r)\right| &= \left|\bigcup_{\gamma\in\mathcal{G}(n,m)} \gamma_j\mathcal{E}_jK_n \setminus \bigcup_{\gamma\in\mathcal{G}(n,m)} \gamma_j\Omega(n,r)\right| \\ &\leq \sum_{\gamma\in\mathcal{G}(n,m)} |\gamma_j[\mathcal{E}_jK_n \setminus \Omega(n,r)]| \\ &\leq |\mathcal{G}(n,m)|\,|B_r(\mathcal{F}_{\mathcal{G}}(\mathcal{E}_jK_n))| \\ &\leq |\mathcal{G}(n,m)|\,|\mathcal{F}_{\mathcal{G}}(\mathcal{E}_jK_n)|\,\mu^r.\end{aligned}$$

c) Let $\varepsilon_n = \dfrac{|\mathcal{F}_{\mathcal{G}}(\mathcal{E}_jK_n)|}{|\mathcal{E}_jK_n|}$, and recall that $\varepsilon_n \to 0$. Putting together (4.3) and (4.5) we get

$$|\mathcal{G}(n,m)|\,|\mathcal{E}_jK_n| - |\mathcal{G}(n,m)|\,|\mathcal{F}_{\mathcal{G}}(\mathcal{E}_jK_n)|\mu^r \leq |\mathcal{E}_jK_m| \leq |\mathcal{G}(n,m)|\,|\mathcal{E}_jK_n|,$$

which implies

$$1 - \varepsilon_n\mu^r \leq \frac{|\mathcal{E}_jK_m|}{|\mathcal{G}(n,m)|\,|\mathcal{E}_jK_n|} \leq 1. \tag{4.6}$$

Choosing n_0 such that, for $n > n_0$, $\varepsilon_n\mu^r \leq 1/2$, we obtain

$$0 \leq \frac{|\mathcal{G}(n,m)|\,|\mathcal{E}_jK_n|}{|\mathcal{E}_jK_m|} - 1 \leq 2\varepsilon_n\mu^r \leq 1. \tag{4.7}$$

Therefore, from (4.4), we obtain

$$\begin{aligned}\left|\mathcal{E}_jK_m \setminus \bigcup_{\gamma\in\mathcal{G}(n,m)} \gamma_j\Omega(n,r)\right| &\leq |\mathcal{G}(n,m)|\,|\mathcal{F}_{\mathcal{G}}(\mathcal{E}_jK_n)|\,\mu^r \\ &= |\mathcal{G}(n,m)|\,|\mathcal{E}_jK_n|\,\varepsilon_n\mu^r \leq 2\,|\mathcal{E}_jK_m|\,\varepsilon_n\mu^r.\end{aligned} \tag{4.8}$$

Second step: the existence of the limit for geometric operators.

a) By definition of $V_j(\gamma)$, we have, for $\gamma \in \mathcal{G}(n,m)$, $n<m$,

$$V_j^*(\gamma)V_j(\gamma) = E(\mathcal{E}_j K_n), \qquad V_j(\gamma)V_j^*(\gamma) = E\Big(\gamma_j\big(\mathcal{E}_j K_n\big)\Big).$$

Assume now $T \in \mathcal{B}(\ell^2(\mathcal{E}_j M))$ is a geometric operator with finite propagation r. Then,

$$TV_j(\gamma)E(\Omega(n,r)) = V_j(\gamma)TE(\Omega(n,r)) \qquad E(\gamma_j\Omega(n,r)) = V_j(\gamma)E(\Omega(n,r))V_j(\gamma)^*.$$

As a consequence,

$$\begin{aligned} &Tr\big(TE(\gamma_j(\Omega(n,r)))\big) \\ &\quad = Tr\big(TV_j(\gamma)E(\Omega(n,r))V_j(\gamma)^*\big) = Tr\big(V_j(\gamma)TE(\Omega(n,r))V_j(\gamma)^*\big) \\ &\quad = Tr\big(TE(\Omega(n,r))V_j(\gamma)^*V_j(\gamma)\big) = Tr\big(TE(\Omega(n,r))E(\mathcal{E}_jK_n)\big) \\ &\quad = Tr\big(TE(\Omega(n,r))\big). \end{aligned} \tag{4.9}$$

b) Let us show that the sequence is Cauchy:

$$\begin{aligned} &\left|\frac{TrTE(\mathcal{E}_jK_n)}{TrE(\mathcal{E}_jK_n)} - \frac{TrTE(\mathcal{E}_jK_m)}{TrE(\mathcal{E}_jK_m)}\right| \\ \leq &\frac{|TrT(E(\mathcal{E}_jK_n) - E(\Omega(n,r)))|}{|\mathcal{E}_jK_n|} + \frac{|TrT(E(\mathcal{E}_jK_m) - E(\cup_{\gamma\in\mathcal{G}(n,m)}\gamma_j\Omega(n,r)))|}{|\mathcal{E}_jK_m|} \\ + &\left|\frac{TrTE(\Omega(n,r))}{|\mathcal{E}_jK_n|} - \frac{|\mathcal{G}(n,m)|\,|\mathcal{E}_jK_n|}{|\mathcal{E}_jK_m|}\frac{TrTE(\Omega(n,r))}{|\mathcal{E}_jK_n|}\right| \\ \leq &\|T\|\left(\frac{|\mathcal{E}_jK_n\setminus\Omega(n,r)|}{|\mathcal{E}_jK_n|} + \frac{|\mathcal{E}_jK_m\setminus\cup_{\gamma\in\mathcal{G}(n,m)}\gamma_j\Omega(n,r)|}{|\mathcal{E}_jK_m|} + \left|1 - \frac{|\mathcal{G}(n,m)|\,|\mathcal{E}_jK_n|}{|\mathcal{E}_jK_m|}\right|\right) \\ \leq &5\|T\|\varepsilon_n\mu^r, \end{aligned}$$

where we used (4.9), in the first inequality, and (4.8), (4.7), in the second inequality.

Third step: Φ_j is a state on $\mathcal{A}(\mathcal{E}_j(M))$.

a) Let $T \in \mathcal{A}(\mathcal{E}_j(M))$, $\varepsilon > 0$. Now find a geometric operator T' such that $\|T-T'\| \leq \varepsilon/3$, and set $\phi_n(A) := \frac{TrAE(\mathcal{E}_jK_n)}{TrE(\mathcal{E}_jK_n)}$. Then choose n such that, for $m>n$, $|\phi_m(T') - \phi_n(T')| \leq \varepsilon/3$. We get

$$|\phi_m(T) - \phi_n(T)| \leq |\phi_m(T-T')| + |\phi_m(T') - \phi_n(T')| + |\phi_n(T-T')| \leq \varepsilon$$

namely $\lim\phi_n(T)$ exists.

b) The functional Φ_j is clearly linear, positive and takes value 1 at the identity, hence it is a state on $\mathcal{A}(\mathcal{E}_j(M))$.

Fourth step: Φ_j is a trace on $\mathcal{A}(\mathcal{E}_j(M))$.

Let A be a geometric operator with propagation r. Then

$$\begin{aligned} AE(\mathcal{E}_jK_n) &= E(B_r(\mathcal{E}_jK_n))AE(\mathcal{E}_jK_n), \\ E(\Omega(n,r))A &= E(\Omega(n,r))AE(\mathcal{E}_jK_n). \end{aligned}$$

Indeed, the first equality is easily verified. As for the second, we have

$$\Omega(n,r) \subset \mathcal{E}_jK_n \setminus B_r(\mathcal{F}(\mathcal{E}_jK_n)) = \{\sigma\in\mathcal{E}_jK_n : d(\sigma, M\setminus\mathcal{E}_jK_n) \geq r+2\},$$

so that

$$B_r(\Omega(n,r)) \subset \{\sigma \in \mathcal{E}_j K_n : d(\sigma, M \setminus \mathcal{E}_j K_n) \geq 2\} \subset \mathcal{E}_j K_n.$$

Since A^* has propagation r, we get

$$A^* E(\Omega(n,r)) = E(B_r(\Omega(n,r)) A^* E(\Omega(n,r)) = E(\mathcal{E}_j K_n) A^* E(\Omega(n,r)),$$

which proves the claim. Therefore,

$$\begin{aligned} AE(\mathcal{E}_j K_n) &= E(B_r(\mathcal{E}_j K_n) \setminus \Omega(n,r)) AE(\mathcal{E}_j K_n) + E(\Omega(n,r))A \\ &= E(B_r(\mathcal{E}_j K_n) \setminus \Omega(n,r)) AE(\mathcal{E}_j K_n) - E(\mathcal{E}_j K_n \setminus \Omega(n,r))A + E(\mathcal{E}_j K_n)A. \end{aligned}$$

Therefore, if $B \in \mathcal{A}(\mathcal{E}_j(M))$,

$$\begin{aligned} \phi_n([B,A]) &\leq \|A\| \, \|B\| \frac{|B_r(\mathcal{E}_j K_n) \setminus \Omega(n,r)| + |\mathcal{E}_j K_n \setminus \Omega(n,r)|}{|\mathcal{E}_j K_n|} \\ &\leq 2\|A\| \, \|B\| \varepsilon_n \mu^r, \end{aligned}$$

as $B_r(\mathcal{E}_j K_n) \setminus \Omega(n,r) \subset B_r(\mathcal{F}_{\mathcal{G}}(\mathcal{E}_j K_n))$. Taking the limit as $n \to \infty$ we get $\Phi_j([B,A]) = 0$. By continuity, the result holds for any $A, B \in \mathcal{A}(\mathcal{E}_j(M))$.

Fifth step: Φ_j is faithful.

Let A be a positive operator in $\mathcal{A}_j$. Then there exists $n_0 \in \mathbb{N}$ such that $Tr(E(\Omega(n_0, r))A) > 0$. Therefore,

$$\Phi_j(A) = \lim_n \frac{Tr(E(\mathcal{E}_j K_n)A)}{|\mathcal{G}(n_0,n)||\mathcal{E}_j K_{n_0}|} \geq \lim_n \frac{|\mathcal{G}(n_0,n)| Tr(E(\Omega(n_0,r))A)}{|\mathcal{G}(n_0,n)||\mathcal{E}_j K_{n_0}|} > 0,$$

where the equality follows by 4.6. □

In the following we use a different normalisation for the traces and, by giving up the state property, we obtain that the trace of the identity operator in $\mathcal{A}_j$ measures the relative volume of $\mathcal{E}_j(M)$. This simplifies the relations in Corollary 5.6 and Theorem 5.8.

Lemma 4.6. *Let M be a p-dimensional self-similar CW-complex. The following limits exist and are finite:*

$$\lim_n \frac{|\mathcal{E}_j(K_n)|}{|\mathcal{E}_p(K_n)|}, \quad 0 \leq j \leq p.$$

Proof. We show that the sequences are Cauchy. Indeed, by inequalities (4.7) in the proof of Theorem 4.5, we have, for $m > n$ and $j = 0, \dots, p$,

$$(1 + 2\varepsilon_n \mu)^{-1} |\mathcal{G}(n,m)||\mathcal{E}_j K_n| \leq |\mathcal{E}_j K_m| \leq |\mathcal{G}(n,m)||\mathcal{E}_j K_n|,$$

where the sequence $\varepsilon_n = \sup_{j=1,\dots,p} \frac{|\mathcal{F}_{\mathcal{G}}(\mathcal{E}_j K_n)|}{|\mathcal{E}_j K_n|}$ is infinitesimal and less than 1, and $\mu = \sup_j \sup_{\sigma \in \mathcal{E}_j M} |B_1(\sigma)|$. Therefore

$$(1 - 2\varepsilon_n \mu) \frac{|\mathcal{E}_j K_n|}{|\mathcal{E}_p K_n|} \leq \frac{|\mathcal{E}_j K_m|}{|\mathcal{E}_p K_m|} \leq (1 + 2\varepsilon_n \mu) \frac{|\mathcal{E}_j K_n|}{|\mathcal{E}_p K_n|}.$$

Hence, the sequence $\frac{|\mathcal{E}_j K_n|}{|\mathcal{E}_p K_n|}$ is bounded by some constant $M > 0$, and

$$\left| \frac{|\mathcal{E}_j K_m|}{|\mathcal{E}_p K_m|} - \frac{|\mathcal{E}_j K_n|}{|\mathcal{E}_p K_n|} \right| \leq 2M\mu\varepsilon_n.$$

The thesis follows. □

Definition 4.7. Let M be a p-dimensional self-similar CW-complex. On the C*-algebras $\mathcal{A}_j$ we shall consider the traces

$$Tr_j^{\mathcal{G}}(T) = \lim_n \frac{|\mathcal{E}_j(K_n)|}{|\mathcal{E}_p(K_n)|}\Phi_j(T) = \lim_n \frac{Tr\big(E(\mathcal{E}_j K_n)T\big)}{Tr\big(E(\mathcal{E}_p K_n)\big)}.$$

In this way, $Tr_j^{\mathcal{G}}(I)$ measures the relative volume of $\mathcal{E}_j(M)$ with respect to $\mathcal{E}_p(M)$.

5. L^2-Betti numbers and Novikov-Shubin numbers for self-similar CW-complexes

In this section, we define L^2-Betti numbers and Novikov-Shubin numbers for self-similar CW-complexes, prove various relations among them, and give a result on the Euler-Poincaré caracteristic of a complex.

Let M be a self-similar CW-complex, let Δ be one of the operators $\Delta_{j\pm}$, Δ_j, $\overline{\Delta}_{j\pm}$, $\overline{\Delta}_j$, and define

Definition 5.1 (L^2-Betti and Novikov-Shubin numbers)**.**

(*i*) $\beta(\Delta) := \lim_{t\to\infty} Tr_j^{\mathcal{G}}(e^{-t\Delta})$, the L^2-Betti number of Δ,

(*ii*) $\alpha(\Delta) := 2 \lim_{t\to\infty} \dfrac{\log\Big(Tr_j^{\mathcal{G}}(e^{-t\Delta}) - \beta(\Delta)\Big)}{-\log t}$, the Novikov-Shubin number of Δ,

and the lower and upper versions, if the above limits do not exist.

Then set

$$\begin{aligned}
\beta_j^{\pm}(M) &:= \beta(\Delta_{j\pm}),\\
\beta_j(M) &:= \beta(\Delta_j),\\
\beta_j^{\pm}(M,\partial M) &:= \beta(\overline{\Delta}_{j\pm}),\\
\beta_j(M,\partial M) &:= \beta(\overline{\Delta}_j),
\end{aligned}$$

and analogously for the Novikov-Shubin numbers.

Remark 5.2. (*i*) The L^2-Betti numbers and Novikov-Shubin numbers could have been defined also in terms of the spectral density function N_λ. This is usually defined in terms of spectral projections, which belong to the generated von Neumann algebra, hence, in our case, are not necessarily in the domain of the trace. However we may consider the spectral measure μ associated, via Riesz theorem, to the functional $\varphi \in C^0[0,\infty) \mapsto Tr_j^{\mathcal{G}}(\varphi(\Delta)) \in \mathbb{C}$, and then define $N_\lambda(\Delta) := \int_0^\lambda d\mu$. In terms of N_λ, L^2-Betti numbers are defined as $\beta(\Delta) = \lim_{\lambda\to 0} N_\lambda$, hence the two definitions clearly coincide. On the other hand, Novikov-Shubin numbers are defined as $\alpha(\Delta) = 2\lim_{\lambda\to 0} \dfrac{\log\left(N_\lambda - \beta(\Delta)\right)}{\log\lambda}$, hence the two definitions are related by a Tauberian theorem, as in [12].

(*ii*) By the observation above, L^2-Betti numbers can be written as $\mu(\{0\})$, or, equivalently, as the infimum of the traces of all continuous functional calculi of the Laplacian, with functions taking value 1 at 0, as in the original definition of John Roe [29].

(*iii*) We followed [25] for the definition of the relative L^2-invariants, even though we considered only the two cases of no boundary and of full boundary. It would be interesting to prove their Poincaré duality result in our context.

(iv) We have used the same normalizing sequence for each trace $Tr_j^{\mathcal{G}}$, see Definition 4.7, in order to compare L^2-Betti numbers. This will imply the relation in Theorem 5.8.

Lemma 5.3 (Hodge decomposition)**.** *The following decomposition holds true:*

$$\ell^2\big(\mathcal{E}_j M\big) = \overline{\mathrm{Im}\Delta_{j-}} \oplus \overline{\mathrm{Im}\Delta_{j+}} \oplus \ker\Delta_j \,.$$

Proof.

$$\overline{\mathrm{Im}\Delta_{j+}} = \big(\ker\partial_{j+1}\,\partial^*_{j+1}\big)^\perp = \big(\ker\partial^*_{j+1}\big)^\perp = \overline{\mathrm{Im}\,\partial_{j+1}} \subseteq \ker\partial_j = \ker\partial^*_j\,\partial_j = \big(\mathrm{Im}\Delta_{j-}\big)^\perp .$$

Then since

$$\big(\mathrm{Im}\Delta_{j+}\big)^\perp \cap \big(\mathrm{Im}\Delta_{j-}\big)^\perp = \ker\partial_j \cap \ker\partial^*_{j+1} = \ker\Delta_j$$

the thesis follows. □

Theorem 5.4. *With the notation above, we have the relations:*

$$\begin{aligned}
\beta_j(M) &= \beta_j^+(M) + \beta_j^-(M) - Tr_j^{\mathcal{G}}(I),\\
\beta_j(M,\partial M) &= \beta_{j+}(M,\partial M) + \beta_{j-}(M,\partial M) - Tr_j^{\mathcal{G}}(I),\\
\alpha_j(M) &= \min\{\alpha_{j+}(M)\,,\alpha_{j-}(M)\},\\
\alpha_j(M,\partial M) &= \min\{\alpha_{j+}(M,\partial M)\,,\alpha_{j-}(M,\partial M)\}.
\end{aligned}$$

Proof. By the orthogonality of the ranges we have $\Delta_{j+}\,\Delta_{j-} = 0$. Hence $\big(\Delta_{j+} + \Delta_{j-}\big)^n = \Delta^n_{j+} + \Delta^n_{j-}$ from which we get $e^{-t\Delta_j} = e^{-t\Delta_{j+}} + e^{-t\Delta_{j-}} - I$. Now the thesis easily follows. The proof for the relative invariants is analogous. □

Proposition 5.5. $Tr^{\mathcal{G}}_{j-1}((\partial_j\partial_j^*)^k) = Tr^{\mathcal{G}}_j((\partial_j^*\partial_j)^k),\ k\in\mathbb{N}$

Proof. Let us set $\Omega_n := \mathcal{E}_{j-1}K_n \setminus \mathcal{F}_{\mathcal{G}}(\mathcal{E}_{j-1}K_n)$. First we note that

$$E(\Omega_n)\partial_j = E(\Omega_n)\partial_j E(\mathcal{E}_j K_n),$$

indeed they coincide on the range of $E(\mathcal{E}_jK_n)$, and both vanish on its kernel. Analogously $\partial_jE(\mathcal{E}_jK_n) = E(B_1(\mathcal{E}_{j-1}K_n))\partial_jE(\mathcal{E}_jK_n)$.
Let us note that if $\partial_j = V_j|\partial_j|$ denotes the polar decomposition, and $A\in\mathcal{B}(\ell^2(\mathcal{E}_j(M)))$, then $Tr(\partial_jA\partial_j^*) = Tr(|\partial_j|A|\partial_j|)$. Then

$$Tr(E(\Omega_n)(\partial_j\partial_j^*)^k) = Tr(E(\Omega_n)\partial_jE(\mathcal{E}_jK_n)\partial_j^*(\partial_j\partial_j^*)^{k-1}),$$

and

$$\begin{aligned}
Tr(E(\mathcal{E}_jK_n)(\partial_j^*\partial_j)^k) &= Tr(|\partial_j|E(\mathcal{E}_jK_n)(\partial_j^*\partial_j)^{k-1}|\partial_j|)\\
&= Tr(\partial_jE(\mathcal{E}_jK_n)(\partial_j^*\partial_j)^{k-1}\partial_j^*)\\
&= Tr(E(B_1(\mathcal{E}_{j-1}K_n))\partial_jE(\mathcal{E}_jK_n)\partial_j^*(\partial_j\partial_j^*)^{k-1}).
\end{aligned}$$

Since $B_1(\mathcal{E}_{j-1}K_n)\setminus\Omega_n \subset B_1(\mathcal{F}_{\mathcal{G}}(\mathcal{E}_{j-1}K_n))$, we obtain

$$\begin{aligned}
&|Tr(E(\mathcal{E}_jK_n)(\partial_j^*\partial_j)^k) - Tr(E(\Omega_n)(\partial_j\partial_j^*)^k)|\\
&\quad\le Tr(E(B_1(\mathcal{F}_{\mathcal{G}}(\mathcal{E}_{j-1}K_n))))\,\big\|\partial_jE(\mathcal{E}_jK_n)\partial_j^*(\partial_j\partial_j^*)^{k-1}\big\|\\
&\quad\le \mu\,\big\|\partial_j\partial_j^*\big\|^k\,|\mathcal{E}_{j-1}(K_n)|\varepsilon_n,
\end{aligned}$$

where $\varepsilon_n = \dfrac{|\mathcal{F}_{\mathcal{G}}(\mathcal{E}_{j-1}K_n)|}{|\mathcal{E}_{j-1}K_n|}$ and $|B_1(\mathcal{F}(\mathcal{E}_{j-1}K_n))| \leq \mu|\mathcal{E}_{j-1}K_n|\varepsilon_n$. Finally,

$$\begin{aligned}
&\Big|\frac{Tr(E(\mathcal{E}_{j-1}K_n)(\partial_j\partial_j^*)^k)}{|\mathcal{E}_pK_n|} - \frac{Tr(E(\mathcal{E}_jK_n)(\partial_j^*\partial_j)^k)}{|\mathcal{E}_pK_n|}\Big| \\
&\leq |\mathcal{E}_pK_n|^{-1}\Big(Tr(E(\mathcal{F}_{\mathcal{G}}(\mathcal{E}_{j-1}K_n))(\partial_j\partial_j^*)^k) \\
&+ |Tr(E(\mathcal{E}_jK_n)(\partial_j^*\partial_j)^k) - Tr(E(\Omega_n)(\partial_j\partial_j^*)^k)|\Big) \\
&\leq \frac{|\mathcal{E}_{j-1}K_n|}{|\mathcal{E}_pK_n|}(\mu+1)\left\|\partial_j\partial_j^*\right\|^k\varepsilon_n \to 0, \text{ as } n\to\infty,
\end{aligned}$$

by Lemma 4.6. □

Corollary 5.6. *For any continuous bounded function $f : [0,\infty) \to \mathbb{C}$ vanishing at zero, one has $Tr^{\mathcal{G}}_{j-1}(f(\partial_j\partial_j^*)) = Tr^{\mathcal{G}}_j(f(\partial_j^*\partial_j))$. In particular*

$$Tr^{\mathcal{G}}_{j-1}(e^{-t\partial_j\partial_j^*}) - Tr^{\mathcal{G}}_{j-1}(I) = Tr^{\mathcal{G}}_j(e^{-t\partial_j^*\partial_j}) - Tr^{\mathcal{G}}_j(I).$$

Therefore

$$\beta^+_{j-1}(M) - Tr^{\mathcal{G}}_{j-1}(I) = \beta^-_j(M) - Tr^{\mathcal{G}}_j(I), \qquad \alpha^+_{j-1}(M) = \alpha^-_j(M),$$
$$\beta^+_{j-1}(M,\partial M) - Tr^{\mathcal{G}}_{j-1}(I) = \beta^-_j(M,\partial M) - Tr^{\mathcal{G}}_j(I), \qquad \alpha^+_{j-1}(M,\partial M) = \alpha^-_j(M,\partial M).$$

Proof. The proof for $\alpha(\Delta_{j\pm})$ and $\beta(\Delta_{j\pm})$ follows directly by the previous results. All the arguments above may be rephrased for the relative invariants, giving the corresponding equality. □

Remark 5.7. Let us recall that in [25] Novikov-Shubin numbers have been associated to the boundary operator ∂, namely depend on an index varying from 1 to the dimension p of the complex. As a consequence of Corollary 5.6, there are only p independent Novikov-Shubin numbers in our framework too.

Concerning L^2-Betti numbers, they have been defined in [25] as Γ-dimensions of the L^2-homology, hence coincide with the trace of the kernel of the full Laplacians. The relations proved in this section show that the $\beta_j^{\pm}$'s are completely determined by the β_j's. Moreover such relations imply a further identity which is the basis of the following Theorem.

Theorem 5.8 (Euler-Poincaré characteristic)**.** *Let M be a p-dimensional self-similar CW-complex, with exhaustion $\{K_n\}$. Then*

$$\chi^{\mathcal{G}}(M) := \sum_{j=0}^{p}(-1)^j\beta_j(M) = \lim_n \frac{\chi(K_n)}{|\mathcal{E}_p(K_n)|}.$$

Proof. Let us first observe that, by using Theorem 5.4 and Corollary 5.6, we get

$$\begin{aligned}
\chi^{\mathcal{G}}(M) &= \beta_0^+(M) + \sum_{j=1}^{p-1}(-1)^j(\beta_j^+(M) + \beta_j^-(M) - Tr^{\mathcal{G}}_j(I)) + (-1)^p\beta_p^-(M) \\
&= \sum_{j=0}^{p-1}(-1)^j\beta_j^+(M) + (-1)^pTr^{\mathcal{G}}_p(I) + \sum_{j=1}^{p}(-1)^j\beta^+_{j-1}(M) - \sum_{j=1}^{p}(-1)^jTr^{\mathcal{G}}_{j-1}(I) \\
&= \sum_{j=0}^{p}(-1)^jTr^{\mathcal{G}}_j(I).
\end{aligned}$$

On the other hand,

$$\sum_{j=0}^{p}(-1)^j Tr_j^{\mathcal{G}}(I)) = \lim_n \sum_{j=0}^{p}(-1)^j \frac{|\mathcal{E}_j(K_n)|}{|\mathcal{E}_p(K_n)|} = \lim_n \frac{\chi(K_n)}{|\mathcal{E}_p(K_n)|}.$$

The thesis follows. □

Example 5.9. (i) For the Gasket graph of figure 1 we get $|VK_n| = \frac{1}{2}3^n + \frac{3}{2}$ and $|EK_n| = 3^n$, so that $\chi^{\mathcal{G}}(X) = -\frac{1}{2}$.
(ii) For the Vicsek graph of figure 3 we get $|VK_n| = 3\cdot 5^n + 1$ and $|EK_n| = 4\cdot 5^n$, so that $\chi^{\mathcal{G}}(X) = -\frac{1}{4}$.
(iii) For the Lindstrom graph of figure 2 we get $|VK_n| = 4\cdot 7^n + 2$ and $|EK_n| = 6\cdot 7^n$, so that $\chi^{\mathcal{G}}(X) = -\frac{1}{3}$.

6. Prefractals as CW-complexes

We say that a j-dimensional polyhedron in some Euclidean space $\mathbb{R}^m$ is *strictly convex* if it is convex and any $(j-1)$-hyperplane contains at most one of its $(j-1)$-dimensional faces.

Definition 6.1. A polyhedral complex is a regular CW complex whose topology is that of a closed subset of some Euclidean space $\mathbb{R}^m$ and whose j-cells are flat strictly convex j-polyhedra in $\mathbb{R}^m$.

The following Proposition motivates the name of the boundary subcomplex.

Proposition 6.2. *Let M be a p-dimensional polyhedral complex in $\mathbb{R}^p$. Then the boundary subcomplex ∂M gives a CW structure on the boundary of M, seen as a subspace of $\mathbb{R}^p$.*

Proof. Clearly, given a $(p-1)$-cell σ of M and a point $x \in \sigma$, we may find $\varepsilon > 0$ such that the ball $B(x, \varepsilon)$ in $\mathbb{R}^p$ is contained in $\sigma \cup \tau_1 \cup \tau_2$, if σ is contained in the two distinct p-cells τ_1, τ_2, and is not contained in M (indeed half of it is in the complement of M with respect to $\mathbb{R}^p$), if σ is contained in only one p-cell τ. This proves the thesis. □

Proposition 6.3. *Let M be a p-dimensional polyhedral complex, $j = 1, \ldots, p$. If σ, σ' are distinct cells in $\mathcal{E}_{j-1}(M)$, there exists at most one polyhedron $\tau \in \mathcal{E}_j(M)$ such that $(\sigma, \partial_j\tau)(\sigma', \partial_j\tau) \neq 0$.*
If $\sigma, \sigma' \in \mathcal{E}_j(M)$, there exists at most one polyhedron $\rho \in \mathcal{E}_{j-1}(M)$ such that $(\partial_j\sigma, \rho)(\partial_j\sigma', \rho) \neq 0$.

Proof. Let $\sigma \neq \sigma' \in \mathcal{E}_{j-1}(M)$ and C denote the convex hull of $\sigma \cup \sigma'$. If C has dimension $j-1$, a τ as above would have two faces in the same $(j-1)$-plane, against the strict convexity.
If C has dimension greater or equal to j, two different $\tau, \tau' \in \mathcal{E}_j(M)$ containing both σ and σ' in their boundaries would contain C also, implying $\tau = \tau'$.
Finally, if $\sigma, \sigma' \in \mathcal{E}_j(M)$, and there were $\rho, \rho' \in \mathcal{E}_{j-1}(M)$ as above, they would belong to the $(j-1)$-plane separating σ and σ', namely σ and σ' would have two faces in the same $(j-1)$-plane, against strict convexity. □

Our main examples of self-similar CW-complexes will be a special class of polyhedral complexes, namely prefractal complexes.

Let us recall that a self-similar fractal K in $\mathbb{R}^p$ is determined by contraction similarities $w_1, \dots, w_q$ as the unique (compact) solution of the fixed point equation $K = WK$, where W is a map on subsets defined as $WA = \bigcup_{j=1}^{q} w_j A$.

The fractal K satisfies the *open set condition* with open set U if

$$w_j U \subset U, \quad w_j U \cap w_i U = \emptyset, \quad i \neq j.$$

Assume now we are given a self-similar fractal in $\mathbb{R}^p$ determined by similarities $w_1, \dots, w_q$, with the same similarity parameter and satisfying Open Set Condition for a bounded open set whose closure is a strictly convex p-dimensional polyhedron $\mathcal{P}$. If σ is a multiindex of length n, we let $w_\sigma := w_{\sigma_n} \circ \cdots \circ w_{\sigma_1}$ and assume also that $w_\sigma \mathcal{P} \cap w_{\sigma'} \mathcal{P}$ is a (facial) subpolyhedron of both $w_\sigma \mathcal{P}$ and $w_{\sigma'} \mathcal{P}$, $|\sigma| = |\sigma'|$. Finally we choose an infinite multiindex I and denote by $I|n$ its n-th truncation. We construct a polyhedral CW-complex as follows. First set

$$K_n := w_{I|n}^{-1} W^n \mathcal{P} = \bigcup_{|\sigma|=n} w_{I|n}^{-1} w_\sigma \mathcal{P}.$$

Lemma 6.4. *K_n is a finite polyhedral complex satisfying the following properties:*

$$\forall j < p, \ \sigma \in \mathcal{E}_j(K_n), \ \exists \tau \in \mathcal{E}_p(K_n) : \sigma \subset \dot{\tau}, \tag{6.1}$$

$$K_n \subset K_{n+1}. \tag{6.2}$$

Proof. Observe that $w_{I|n}^{-1} w_\sigma \mathcal{P} \cap w_{I|n}^{-1} w_{\sigma'} \mathcal{P} = w_{I|n}^{-1}(w_\sigma \mathcal{P} \cap w_{\sigma'} \mathcal{P})$, hence is a common facial subpolyhedron, namely K_n has a natural structure of polyhedral complex. Property (6.1) is obvious. Let us now prove (6.2).

$$K_{n+1} = w_{I|n}^{-1} w_{I_{n+1}}^{-1} \left(\bigcup_{j=1}^{q} w_j \right) W^n \mathcal{P} \supset w_{I|n}^{-1} W^n \mathcal{P} = K_n.$$

□

Corollary 6.5. *$M = \cup_{n \in \mathbb{N}} K_n$ is a polyhedral complex satisfying property (6.1). $\{K_n\}$ is an exhaustion for M. M is called a prefractal complex.*

Proof. Obvious. □

Proposition 6.6. *A prefractal complex is a regular bounded CW-complex.*

Proof. Regularity is obvious by construction.

Let us estimate V_j^+. Any $\sigma \in \mathcal{E}_j$ is contained in some copy of the fundamental polyhedron $\mathcal{P}$, therefore any $\tau \in \mathcal{E}_{j+1}$ such that $\dot{\tau} \supset \sigma$ will be contained in the same copy of $\mathcal{P}$ or in some neighboring copy. As a consequence, we may estimate the number of such τ's with the product of $|\mathcal{E}_{j+1}(\mathcal{P})|$ times the maximum number of disjoint copies of $\mathcal{P}$ having a j-cell in common. Since such copies are contained in a ball of radius diam($\mathcal{P}$), their number may be estimated e.g. by the ratio of the volume of the ball of radius diam($\mathcal{P}$) and the volume of $\mathcal{P}$.

As for V_j^-, again any $\sigma \in \mathcal{E}_j$ is contained in some copy of the fundamental polyhedron $\mathcal{P}$, therefore any $\rho \in \mathcal{E}_{j-1}$ such that $\rho \subset \dot{\sigma}$ will be contained in the same copy of $\mathcal{P}$. The number of such ρ's is majorised by $|\mathcal{E}_{j-1}(\mathcal{P})|$. □

In order to show that M is a self-similar complex, we shall prove that K_n is a regular exhaustion satisfying Definition 4.2.

Lemma 6.7. *Assume K is a p-dimensional polyhedral complex in $\mathbb{R}^p$ satisfying property (6.1), and we have $\sigma_i \in \mathcal{E}_i(K)$, $i = j_0, \dots, j_1$, $j_1 < p$, with $\sigma_i \subset \dot{\sigma}_{i+1}$. Then there exist $\sigma'_i \in \mathcal{E}_i(K)$, $i = j_0, \dots, j_1 + 1$, such that*
(α) $\sigma'_i \subset \dot{\sigma}'_{i+1}$, $i = j_0, \dots, j_1$,
(β) $\sigma_{i-1} \subset \dot{\sigma}'_i$, $i = j_0 + 1, \dots, j_1 + 1$,
(γ) *$ph(\sigma'_i \cup \sigma_{j_1}) = \sigma'_{j_1+1}$, $i = j_0, \dots, j_1$, where $ph(E)$ denotes the polyhedral hull, namely the minimal polyhedron in K (if it exists) containing $E \subset K$.*

Proof. The proof will be done by descending induction on i, starting from $j_1 + 1$. Take any $\sigma'_{j_1+1} \supset \sigma_{j_1}$, which exists, because of (6.1). Assume the statement for $k+1$. If $k > j_0$, as $\sigma_{k-1} \subset \dot{\sigma}_k$, $\sigma_k \subset \dot{\sigma}'_{k+1}$, there is (one and only one) $\sigma'_k \subset \dot{\sigma}'_{k+1}$, such that $\sigma_{k-1} \subset \dot{\sigma}'_k$, and $\sigma'_k \neq \sigma_k$, by regularity of the CW-complex. If $k = j_0$, simply take σ'_k as any k-dimensional face in the boundary of σ'_{k+1}, distinct from σ_k. Let us observe that, for $i = j_0$, property (β) is empty. This gives (α) and (β). Since σ'_{k+1} is strictly convex, and σ_k, σ'_k are two distinct k-dimensional faces, we have $\mathrm{ph}(\sigma'_k \cup \sigma_k) = \sigma'_{k+1}$. Then $\mathrm{ph}(\sigma_{j_1} \cup \sigma'_k) = \mathrm{ph}(\sigma_{j_1} \cup \sigma_k \cup \sigma'_k) = \mathrm{ph}(\sigma_{j_1} \cup \sigma'_{k+1}) = \sigma'_{j_1+1}$, which is property ($\gamma$). $\square$

Theorem 6.8. *Any prefractal complex M is a self-similar polyhedral complex.*

Proof. We already proved regularity and boundedness. Let us now observe that

$$K_{n+1} = \bigcup_{j=1}^{q} w^{-1}_{I|n+1} w_j W^n \mathcal{P} = \bigcup_{\ell=1}^{q} \gamma^n_\ell K_n,$$

where $\gamma^n_\ell = w^{-1}_{I|n+1} w_\ell w_{I|n}$ are local isomorphisms. Moreover, by OSC,

$$\gamma^n_k K_n = w^{-1}_{I|n+1} w_k W^n \mathcal{P} \subset w^{-1}_{I|n+1} w_k \mathcal{P}.$$

Therefore, $\gamma^n_k K_n \cap \gamma^n_\ell K_n \subset w^{-1}_{I|n+1}(w_k \mathcal{P} \cap w_\ell \mathcal{P})$, which is contained in some affine hyperplane π; hence $\gamma^n_k K_n \cap \gamma^n_\ell K_n$ has no p-dimensional polyhedra. Let σ_{j_0} be a j_0-dimensional polyhedron in $\gamma^n_k K_n \cap \gamma^n_\ell K_n$, and σ_i, $i = j_0, \dots, j_1$, be a maximal family of polyhedra in $\gamma^n_k K_n \cap \pi$ such that $\sigma_i \subset \dot{\sigma}_{i+1}$, $i = j_0, \dots, j_1 - 1$. Now apply the lemma with $K = \gamma^n_k K_n$, and get $\sigma'_i \in \mathcal{E}_i(\gamma^n_k K_n)$, $i = j_0, \dots, j_1 + 1$, with $\mathrm{ph}(\sigma'_i \cup \sigma_{j_1}) = \sigma'_{j_1+1}$. Then $\sigma'_{j_0} \not\subset \pi$, otherwise $\pi \supset \mathrm{ph}(\sigma'_{j_0} \cup \sigma_{j_1}) = \sigma'_{j_1+1}$, against the maximality. Therefore $\sigma'_{j_0} \notin \mathcal{E}_{j_0}(\gamma^n_\ell K_n)$. Moreover, since σ_{j_0}, $\sigma'_{j_0} \subset \dot{\sigma}'_{j_0+1}$, we have $d_+(\sigma_{j_0}, \sigma'_{j_0}) = 1$, namely $\sigma_{j_0} \in \mathcal{F}(\mathcal{E}_{j_0}(\gamma^n_\ell K_n))$. Similarly, one proves $\sigma_{j_0} \in \mathcal{F}(\mathcal{E}_{j_0}(\gamma^n_k K_n))$. Therefore, $\gamma^n_k K_n \cap \gamma^n_\ell K_n = \mathcal{F}(\gamma^n_k K_n) \cap \mathcal{F}(\gamma^n_\ell K_n)$. $\square$

Remark 6.9. The construction above can be easily generalised to the case of translation limit fractals [14, 15, 16, 17, 18].

We now study some properties of polyhedral complexes, which are valid in particular for the prefractal complexes.

Definition 6.10. We say that $\Delta_{j\pm}$ is a *graph-like Laplacian* if there exists a suitable orientation of M such that the off-diagonal entries of the matrix associated with $\Delta_{j\pm}$, in the corresponding orthonormal basis, belong to $\{0, -1\}$.

Theorem 6.11. *Assume M is a p-dimensional polyhedral complex in $\mathbb{R}^m$, $m \geq p$. Then Δ_{j+} is graph-like if and only if $j = 0$.*

Proof. If $\sigma, \sigma' \in \mathcal{E}_0(M)$, then $(\sigma, \Delta_{0+}\sigma') = \sum_\alpha [\tau^1_\alpha, \sigma][\tau^1_\alpha, \sigma']$. By the Proposition 6.3, the sum consists of at most one non-vanishing summand, corresponding to some 1-cell τ. By the choice we made for the orientation on the 0-cells, the sum of $[\tau^1_\alpha, \sigma]$ and $[\tau^1_\alpha, \sigma']$ is 0, hence the product is -1.

Let $j > 0$ and choose $\tau \in \mathcal{E}_{j+1}(M)$. Since $j + 1 \geq 2$, τ has at least three distinct faces $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{E}_j(M)$. Setting $[\tau, \sigma_j] = \lambda_j = \pm 1$, we obtain, for $j \neq k$, $\big(\sigma_i, \Delta_{j+}\sigma_k\big) = \lambda_i \cdot \lambda_k$. Therefore the product

$$\big(\sigma_1, \Delta_{j+}\sigma_2\big) \cdot \big(\sigma_2, \Delta_{j+}\sigma_3\big) \cdot \big(\sigma_3, \Delta_{j+}\sigma_1\big) = 1$$

so that the three off diagonal matrix elements will never be all equal to -1. □

Lemma 6.12. *Assume $\Delta_{(j+1)-}$ is not diagonal. Then Δ_{j-} is not graph-like.*

Proof. By assumption, there exist $\tau_1, \tau_2 \in \mathcal{E}_{j+1}(M)$ such that $(\partial_{j+1}\tau_1, \partial_{j+1}\tau_2) = (\tau_1, \Delta_{(j+1)-}\tau_2) \neq 0$, namely there exist $\sigma_3 \subset \dot\tau_1 \cap \dot\tau_2$. If $\rho \subset \dot\sigma_3$, $\partial_j(\partial_{j+1}\tau_i) = 0$ implies the existence of $\sigma_i \subset \dot\tau_i$ for which $\rho \in \dot\sigma_i$. Setting $\lambda_i = [\sigma_i, \rho]$, $i = 1, 2, 3$, $(\sigma_i, \Delta_{j-}\sigma_k) = \lambda_i\lambda_k$, the proof goes on as in Theorem 6.11. □

In order to prove a general result on the possibility of Δ_{j-} to be graph-like, we shall exclude some trivial cases.

Definition 6.13. We say that an operator A acting on $\ell^2(\mathcal{E}_j(M))$ is *irreducible* if the only self-adjoint idempotent multiplication operators commuting with A are 0 and I. We say that a (connected) polyhedral complex is q-*irreducible* if q is the maximum number such that Δ_{j+} is irreducible for any $j < q$, and Δ_{j-} is irreducible for any $j \leq q$.

Theorem 6.14. *Let M be a q-irreducible p-dimensional polyhedral complex in $\mathbb{R}^p$, $j \leq q$. Then Δ_{j-} graph-like implies $j = q$. If $q = p$ the above implication is indeed an equivalence.*

Proof. If $j < q$, Δ_{j-} is not graph-like by Lemma 6.12. If $q = p$ we may choose the orientation for p-cells according to a given orientation of $\mathbb{R}^p$. Then, if $\tau, \tau' \in \mathcal{E}_p(M)$ have a face in common, i.e. $(\tau, \Delta_{p-}\tau') \neq 0$, such face receives opposite orientations from the embeddings in $\dot\tau$, resp. $\dot\tau'$, i.e. $(\tau, \Delta_{p-}\tau') = (\partial_p\tau, \partial_p\tau') = -1$. □

Definition 6.15. If M is a p-irreducible polyhedral complex in $\mathbb{R}^p$, by the previous Theorem, a graph G is associated to $\overline{\Delta}_{p-}$, and is constructed as follows. The set of vertices of G is $\mathcal{E}_p(M)$, while $\sigma, \sigma' \in \mathcal{E}_p(M)$ are adjacent *iff* there is $\rho \in \mathcal{E}_{p-1}(M)$ such that $\rho \subset \dot\sigma, \dot\sigma'$. We call G the dual graph of M.

7. Computation of the Novikov-Shubin numbers for fractal graphs

Let us observe that a 1-dimensional regular CW-complex is the same as a simple graph, and boundedness means bounded degree. Recall that a simple graph $G = (VG, EG)$ is a collection VG of objects, called *vertices*, and a collection EG of unordered pairs of distinct vertices, called *edges*. We call a self-similar 1-dimensional CW-complex simply a self-similar graph.

The results in this section will allow us to calculate $\alpha_0 = \alpha_1$ of some self-similar graphs, and also α_p of a p-irreducible prefractal complex in $\mathbb{R}^p$, in the sense of definition 6.13.

In the rest of this section, G is a countably infinite graph with bounded degree. We denote by Δ the Laplacian on 0-cells (points), hence $\Delta = C - A$, where C is a diagonal operator, with $C(x,x) = c(x)$ the number of edges starting from the point x, and A is the adjacency matrix. Let P be the transition operator, i.e. $p(x,y)$ is the transition probability from x to y of the simple random walk on G. Let $\mathcal{A}_0$ be a C*-algebra of operators, acting on $\ell^2(G) = \ell^2(\mathcal{E}_0(G))$, which contains Δ, and possesses a finite trace τ_0.

We can also consider the Hilbert space $\ell^2(G,c)$ with scalar product $(v,w)_c = \sum_{x\in G} c(x)\overline{v}_x w_x$. On this space the transition operator P is selfadjoint, and the Laplace operator is defined as $\Delta_c = I - P$. Since G has bounded degree, C is bounded from above by a multiple of the identity, and, since G is connected, it is bounded from below by the identity. Also, the two spaces $\ell^2(G)$ and $\ell^2(G,c)$ coincide as topological vector spaces, with the obvious identification. With this identification we have $(u,\Delta v) = (u,\Delta_c v)_c$. We may also identify operators on the two spaces, hence the C*-algebras $\mathcal{A}_0$, resp. $\mathcal{A}_{0,c}$, acting on $\ell^2(G)$, resp. $\ell^2(G,c)$, can be identified as topological algebras. We use this identification to carry the trace τ_0 onto $\mathcal{A}_{0,c}$.

Theorem 7.1. *Let μ be the maximal degree of G. Then*

$$\tau_0\left(\frac{1}{1+\mu t\Delta_c}\right) \le \tau_0\left(\frac{1}{1+t\Delta}\right) \le \tau_0\left(\frac{1}{1+t\Delta_c}\right) \qquad t \ge 0.$$

Proof. Let us consider the positive self-adjoint operator $Q = C^{-1/2}\Delta C^{-1/2}$ on $\ell^2(G)$. Since C and Δ belong to $\mathcal{A}_0$, $Q \in \mathcal{A}_0$ too. Since $A = CP$, we have $Q = C^{1/2}(I-P)C^{-1/2}$. Clearly $Q \le Q^{1/2}CQ^{1/2} \le \mu Q$, hence, by operator monotonicity,

$$\tau_0\left(\frac{1}{1+\mu tQ}\right) \le \tau_0\left(\frac{1}{1+tQ^{1/2}CQ^{1/2}}\right) \le \tau_0\left(\frac{1}{1+tQ}\right) \qquad t \ge 0.$$

Now observe that, for sufficiently small t,

$$\begin{aligned}\tau_0\left(\frac{1}{1+tQ}\right) &= \sum_{n=0}^{\infty}(-t)^n\tau_0(Q^n)\\ &= \sum_{n=0}^{\infty}(-t)^n\tau_0(\Delta_c^n) = \tau_0\left(\frac{1}{1+t\Delta_c}\right).\end{aligned}$$

Since both the left and the right hand side are analitic functions for $t \ge 0$, they coincide for any $t \ge 0$. Analogously,

$$\begin{aligned}\tau_0\left(\frac{1}{1+tQ^{1/2}CQ^{1/2}}\right) &= \sum_{n=0}^{\infty}(-t)^n\tau_0((Q^{1/2}CQ^{1/2})^n)\\ &= \sum_{n=0}^{\infty}(-t)^n\tau_0(\Delta^n) = \tau_0\left(\frac{1}{1+t\Delta}\right).\end{aligned}$$

The result follows. □

In order to prove the main result of this section, we need a Tauberian theorem. It is a quite simple modification of a theorem of de Haan and Stadtmüller, cf. [5] thm. 2.10.2, and, on the same book, also thm. 1.7.6, by Karamata, showing that the bound $\alpha < 1$ below is a natural one.

Definition 7.2.

(i) Let us denote by $OR(1)$ the space of positive, non increasing functions f on $[0,\infty)$ such that $\exists T>0, c>0, \alpha\in(0,1)$ such that

$$\frac{f(\lambda t)}{f(t)} \leq c\lambda^{-\alpha}, \qquad \forall\lambda>0,\ t\geq T. \tag{7.1}$$

(ii) If f,g are functions on $[a,+\infty)$ we write $f\asymp g$, $t\to\infty$ if $\exists T\geq a, k>1$ such that

$$k^{-1}g(t)\leq f(t)\leq kg(t), \qquad t\geq T. \tag{7.2}$$

Remark 7.3. If the functions are bounded and defined on $[0,\infty)$, we may equivalently assume $T=0$ both in eq. (7.1) and (7.2), possibly changing the constants c and k.

Let us now denote by $\hat f$ the Laplace transform of f,

$$\hat f(t) = t\int_0^\infty e^{-ts}f(s)\ ds.$$

Lemma 7.4. *Let f be a positive bounded function. Then $f\in OR(1)$ iff $\hat f(1/\cdot)\in OR(1)$. In this case*

$$f\asymp \hat f(1/\cdot), \qquad t\to\infty.$$

Proof. Let us notice that since f is bounded also $\hat f$ is bounded, hence, according to Remark 7.3, the properties above should hold for all $t\geq 0$. Now observe that

$$\hat f(s) = \int_0^\infty e^{-y}f(y/s)\ dy,$$

hence

$$\hat f(1/s) = \int_0^\infty e^{-y}f(sy)\ dy \geq f(xs)\int_0^x e^{-y}\ dy, \tag{7.3}$$

giving

$$f(t)\leq \frac{e^x}{e^x-1}\hat f(x/t), \forall x,t. \tag{7.4}$$

Now assume $f\in OR(1)$.
By (7.3) we get $\dfrac{\hat f(1/t)}{f(t)} = \displaystyle\int_0^\infty e^{-\lambda}\frac{f(\lambda t)}{f(t)}\ d\lambda$. Therefore, splitting the domain of integration and using property $OR(1)$, we get

$$\frac{\hat f(1/t)}{f(t)} \leq c\int_0^1 \lambda^{-\alpha}e^{-\lambda}\ d\lambda + \int_1^\infty e^{-\lambda}\ d\lambda \leq c\Gamma(1-\alpha)+1.$$

This, together with (7.4) for $x=1$, implies $f\asymp \hat f(1/\cdot)$, $t\to\infty$.
Moreover,

$$\frac{\hat f(1/(\lambda t))}{\hat f(1/t)} \leq c'\frac{f(\lambda t)}{\hat f(1/t)} \leq c''\frac{f(\lambda t)}{f(t)} \leq c'''\lambda^{-\alpha},$$

showing that $\hat{f}(1/\cdot) \in OR(1)$.
Now assume $\hat{f}(1/\cdot) \in OR(1)$. Then, by (7.4) with $x = 1$,

$$\begin{aligned}\hat{f}(s) &= \int_0^a e^{-y} f(y/s)\, dy + \int_a^\infty e^{-y} f(y/s)\, dy \\ &\leq \frac{e}{e-1} \int_0^a e^{-y} \hat{f}(s/y)\, dy + f(a/s) \int_a^\infty e^{-y}\, dy \\ &\leq c\frac{e}{e-1} \hat{f}(s) \int_0^a y^{-\alpha} e^{-y}\, dy + f(a/s).\end{aligned}$$

Choosing a sufficiently small, we get $c\frac{e}{e-1}\int_0^a y^{-\alpha}e^{-y}\, dy \leq 1/2$, hence $\hat{f}(s) \leq 2f(a/s)$. Now, using (7.4) with $x = a$, we get

$$\frac{f(\lambda t)}{f(t)} \leq \frac{2e^a}{e^a - 1} \frac{\hat{f}\left(\frac{a}{\lambda t}\right)}{\hat{f}\left(\frac{a}{t}\right)} \leq \frac{2c' e^a}{e^a - 1} \lambda^{-\alpha},$$

namely $f \in OR(1)$. The proof is complete. □

Corollary 7.5. *Let $(\mathcal{A}, \tau)$ be a C^*-algebra with a finite trace, A be a positive element of $\mathcal{A}$. Then $\tau(e^{-tA}) \in OR(1)$ iff $\tau((1+tA)^{-1}) \in OR(1)$. In this case $\tau(e^{-tA}) \asymp \tau((1+tA)^{-1})$, $t \to \infty$.*

Proof. Let us consider, in the von Neumann algebra of the GNS representation of τ, the function $N(t) = \tau(e_{[0,t)}(A))$. Then $f(t) := \tau(e^{-tA}) = \int_0^\infty e^{-ts}\, dN(s)$, hence its Laplace transform is

$$\hat{f}(x) = x \int_0^\infty dN(s) \int_0^\infty dt\; e^{-t(s+x)} = \int_0^\infty \frac{x}{s+x}\, dN(s) = \tau\left(\frac{1}{1+x^{-1}A}\right).$$

The result now follows from the Lemma above. □

Now we come back to the Laplacians on G.

Corollary 7.6. *Let G be a countably infinite connected graph with bounded degree. Let Δ, resp. Δ_c, be the homological, resp. probabilistic Laplacian. Let $\mathcal{A}_0$ be a C^*-algebra of operators, acting on $\ell^2(G) = \ell^2(\mathcal{E}_0(G))$, which contains Δ (hence Δ_c), and possesses a finite trace τ_0. Consider the functions $\vartheta(t) = \tau_0(e^{-t\Delta})$, $\vartheta_c(t) = \tau_0(e^{-t\Delta_c})$. Then $\vartheta \in OR(1)$ iff $\vartheta_c \in OR(1)$. In this case $\vartheta \asymp \vartheta_c$ for $t \to \infty$.*

Proof. Assume $\vartheta \in OR(1)$. Then, by Corollary 7.5, $\vartheta(t) \asymp \tau_0((1+t\Delta)^{-1})$, $t \to \infty$ and $\tau_0((1+t\Delta)^{-1}) \in OR(1)$. Therefore, recalling Theorem 7.1, and denoting by μ the maximum degree of G, we get

$$1 \leq \frac{\tau_0((1+t\Delta_c)^{-1})}{\tau_0((1+t\Delta)^{-1})} \leq \frac{\tau_0((1+\mu^{-1}t\Delta)^{-1})}{\tau_0((1+t\Delta)^{-1})} \leq c\mu^\alpha,$$

namely $\tau_0((1+t\Delta_c)^{-1}) \asymp \tau_0((1+t\Delta)^{-1})$. Therefore $\tau_0((1+t\Delta_c)^{-1}) \in OR(1)$. Applying Corollary 7.5 again we get $\vartheta_c \in OR(1)$ and $\vartheta \asymp \vartheta_c$. The converse implication is proved analogously. □

Now we relate the large n asymptotics of the probability of returning to a point in n steps with the large time heat kernel asymptotics. Since for bipartite graphs the probability is zero for odd n, the estimates are generally given in terms of the sum of the n-step plus the $(n+1)$-step return probability. In order to match the above treatment we shall use a suitable mean for the return probability, namely the trace of the n-th power of the transition operator P.

First we need the auxiliary function described in the following

Lemma 7.7. *Let us denote by φ_γ, $\gamma > 0$, the function*

$$\varphi_\gamma(x) := e^x x^{-\gamma} \int_0^x e^{-t} d(t^\gamma),\ x \geq 0.$$

Then φ_γ extends to the entire function

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \binom{n+\gamma}{n}^{-1},\ x \in \mathbb{C}. \tag{7.5}$$

Proof. Let us observe that the power series in (7.5) is an entire function φ satisfying

$$\begin{cases} \varphi' = (1 - \frac{\gamma}{x})\varphi + \frac{\gamma}{x} \\ \varphi(0) = 1\,. \end{cases}$$

It is easy to see that φ_γ is the unique solution of the differential equation in $(0, +\infty)$ which tends to 1 when $x \to 0^+$. □

Theorem 7.8. *Let G be a countably infinite connected graph with bounded degree. Let $\mathcal{A}_0$ be a C^*-algebra of operators, acting on $\ell^2(G) = \ell^2(\mathcal{E}_0(G))$, which contains the homological Laplacian Δ and possesses a finite trace τ_0. If $\tau_0(P^n + P^{n+1}) \asymp n^{-\gamma}$, then $\vartheta_c(t) \asymp t^{-\gamma}$, $t \to \infty$.*

Proof. Let us observe that

$$e^{-x} x^\gamma \varphi_\gamma(x) \to \gamma\Gamma(\gamma), \quad x \to +\infty,$$

and that

$$\binom{n+\gamma}{n} n^{-\gamma} \to 1, \quad n \to +\infty.$$

On the one hand, we have

$$\begin{aligned} \vartheta_c(t) = e^{-t}\tau_0(e^{tP}) = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \tau_0(P^n) \leq K e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} n^{-\gamma} \\ \leq K' e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \binom{n+\gamma}{n}^{-1} = K' e^{-t} \varphi_\gamma(t) \leq K'' t^{-\gamma}. \end{aligned}$$

On the other hand, setting $\psi(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\tau_0(P^n)$, we get $\vartheta_c(t) = e^{-t}\psi(t)$, and $2\vartheta_c(t) + \vartheta_c'(t) = e^{-t}(\psi(t) + \psi'(t))$. Let us note that

$$\psi'(t) = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} \tau_0(P^n) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \tau_0(P^{n+1}),$$

therefore

$$\psi(t) + \psi'(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \tau_0(P^n + P^{n+1}) \geq c \sum_{n=0}^{\infty} \frac{t^n}{n!} n^{-\gamma} \geq c' e^t t^{-\gamma}.$$

Finally, since ϑ_c' is negative,

$$2\vartheta_c \geq 2\vartheta_c(t) + \vartheta_c'(t) = e^{-t}(\psi(t) + \psi'(t)) \geq c' t^{-\gamma}.$$

The thesis follows. □

Corollary 7.9. *Let G be a countably infinite connected graph with bounded degree. Let $\mathcal{A}_0$ be a C^*-algebra of operators, acting on $\ell^2(G) = \ell^2(\mathcal{E}_0(G))$, which contains Δ, and possesses a finite trace τ_0. Assume $\tau_0(P^n + P^{n+1}) \asymp n^{-\gamma}$, for $\gamma > 0$.*

(*i*) *If $\gamma \in (0,1)$, then the Novikov-Shubin number $\alpha(G) = 2\gamma$.*

(*ii*) *If $\gamma > 0$ and G has constant degree, then $\alpha(G) = 2\gamma$.*

Proof. (*i*) it follows from Corollary 7.6 and Theorem 7.8.

(*ii*) it follows from Theorem 7.8 and the observation that if the degree is constantly equal to μ, then $\tau_0(e^{-t\Delta}) = \tau_0(e^{-\mu t\Delta_c})$. □

Corollary 7.10. *Let G be a countably infinite connected graph with bounded degree. Let $\mathcal{A}_0$ be a C^*-algebra of operators, acting on $\ell^2(G) = \ell^2(\mathcal{E}_0(G))$, which contains Δ, and possesses a finite trace τ_0. Denote by $p_n(x,y)$ the (x,y)-element of the matrix P^n, which is the probability that a simple random walk started at x reaches y in n steps. Assume that there are $\gamma \in (0,1)$, $c_1, c_2 > 0$ such that, for all $x \in G$, $n \in \mathbb{N}$,*

$$
\begin{aligned}
p_n(x,x) &\leq c_2 n^{-\gamma} \\
p_n(x,x) + p_{n+1}(x,x) &\geq c_1 n^{-\gamma}.
\end{aligned} \tag{7.6}
$$

Then the Novikov-Shubin number $\alpha(G) = 2\gamma$.

Proof. It is just a restatement of the previous Corollary. □

As Novikov-Shubin numbers of covering manifolds are large scale invariants, one expects that graphs which are asymptotically close should have the same Novikov-Shubin number. We show that this happens in case of roughly isometric graphs.

Definition 7.11. Let G_1, G_2 be infinite graphs with bounded degree. A map $\varphi : G_1 \to G_2$ is called a *rough isometry* if there are a, b, $M > 0$ such that

(*i*) $a^{-1}d_1(x,y) - b \leq d_2(\varphi(x), \varphi(y)) \leq ad_1(x,y) + b$, for $x, y \in G_1$,

(*ii*) $d_2(\varphi(G_1), y) \leq M$, for $y \in G_2$.

Then G_1 and G_2 are said to be rough isometric.

Observe that being rough isometric is an equivalence relation.

Theorem 7.12. *Let G_1, G_2 be rough isometric, infinite graphs with bounded degree. For $j = 1, 2$, let $\mathcal{A}_j$ be a C^*-algebra of operators, acting on $\ell^2(G_j) = \ell^2(\mathcal{E}_0(G_j))$, which contains the Laplace operator Δ_j of the graph G_j, and possesses a finite trace τ_j. Assume G_1 satisfies (7.6), then G_2 does as well. As a consequence, $\alpha_0(G_1) = \alpha_0(G_2)$.*

Proof. It is a consequence of [11] Theorem 3.1, [4] Lemma 1.1, and [20] Theorem 5.11. □

8. Examples

In this section, we compute the Novikov-Shubin numbers of some explicit examples.

Our first class of examples is that of nested fractal graphs, for more details on the construction see Section 6 and [20].

Assume we are given a nested fractal K in $\mathbb{R}^p$ determined by similarities $w_1, \ldots, w_q$, with the same similarity parameter, and let S be the Hausdorff dimension of K in the resistance metric [22]. Let M be the nested fractal graph based on K.

Theorem 8.1. *Let K, S, and M be as above. Then (7.6) hold for M, with $\gamma = \frac{S}{S+1} \in (0,1)$. Therefore, $\alpha_0(M) = 2\gamma$.*

Proof. The thesis follows from Corollary 4.13 in [20], and Corollary 7.10 above. $\square$

Example 8.2. Using the previous Theorem we can compute α_0 for some self-similar graphs coming from fractal sets as in Section 6. Moreover, since $\beta_0 = 0$ by the estimate in Theorem 7.8, we get, by Corollary 5.8, $\beta_1 = -\chi$, the latter being explicitely computed in Example 5.9.

For the Gasket graph in figure 1 we obtain $\alpha_0 = \frac{2\log 3}{\log 5}$, see [3], $\beta_0 = 0$, and $\beta_1 = \frac{1}{2}$.

For the Vicsek graph in figure 3 we obtain $\alpha_0 = \frac{2\log 5}{\log 15}$, see [21], $\beta_0 = 0$, and $\beta_1 = \frac{1}{4}$.

For the Lindstrom graph in figure 2 we obtain $\alpha_0 = \frac{2\log 7}{\log 12.89027}$ computed numerically, see [23], $\beta_0 = 0$, and $\beta_1 = \frac{1}{3}$.

Our second class of examples is given by the following

Proposition 8.3. *Let M be a p-irreducible prefractal complex in $\mathbb{R}^p$, let G be the dual graph of M, as in Definition 6.15, and assume that (7.6) hold on G. Then the Novikov-Shubin number $\alpha_p(M, \partial M) = 2\gamma$.*

Proof. It is a consequence of Corollary 7.10. $\square$

Example 8.4. Let us consider the 2-dimensional complex M in figure 5. We want

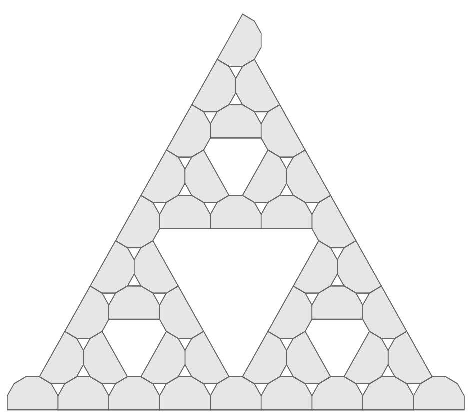

Figure 5. Dodecagon 2-complex

to compute its second relative Novikov-Shubin number $\alpha_2(M, \partial M)$. If we extend the definition of selfsimilar CW-complex as explained in Remark 4.3, it is easy to see that the complex in figure 5 is self-similar and its dual graph, defined in Definition 6.15, coincides with the Gasket graph considered in figure 1. Therefore, by Proposition 8.3, $\alpha_2(M, \partial M) = \frac{2\log 3}{\log 5}$.

Example 8.5. The Carpet 2-complex M in figure 4 is an example of a 2-dimensional self-similar CW-complex. Barlow [3] associates to M the graph G_1 in figure 6, which, by [3] Theorem 3.4, satisfies the estimates in Corollary 7.10, with $\gamma = \frac{\log 8}{\log(8\rho)}$, where $\rho \in [\frac{7}{6}, \frac{3}{2}]$, while computer calculations suggest that $\rho \cong 1.251$.

The dual graph of M, as in Definition 6.15, is the graph G_2 in figure 7, also associated to M by Barlow and Bass in [2]. It is easy to see that the graphs G_1 and G_2 are roughly-isometric. Therefore, by Theorem 7.12, $\alpha_2(M, \partial M) = \alpha(\overline{\Delta}_{2-}) = 2\gamma$ (so that $\alpha_2(M, \partial M) \in [1.67, 1.87]$).

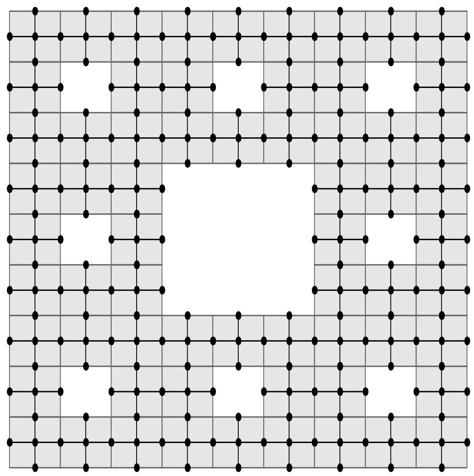

Figure 6. Carpet Graph (following Barlow)

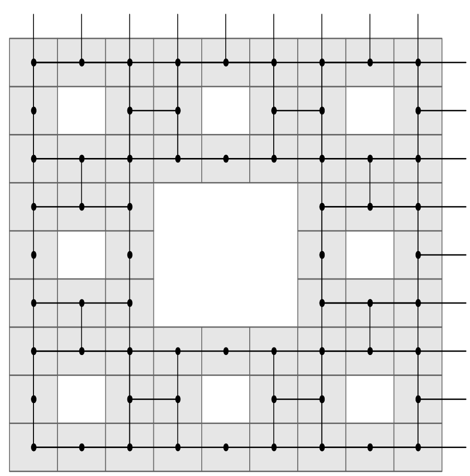

Figure 7. Dual Carpet Graph

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