A CLASSIFICATION OF TAUT, STEIN SURFACES WITH A PROPER $\mathbb{R}$-ACTION

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Abstract. We present a classification of 2-dimensional, taut, Stein manifolds with a proper $\mathbb{R}$-action. For such manifolds the globalization with respect to the induced local $\mathbb{C}$-action turns out to be Stein. As an application we determine all 2-dimensional taut, non-complete, Hartogs domains over a Riemann surface.

1. Introduction

Let the group $(\mathbb{R},+)$ act on a complex manifold X by biholomorphism. Then, by integrating the associated vector field one obtains a local action of $(\mathbb{C},+)$. For taut, Stein manifolds, the universal globalization with respect to such a local action is Hausdorff ([Ian]). That is, there exists a complex $\mathbb{C}$-manifold X^* containing X as an $\mathbb{R}$-invariant domain such that every $\mathbb{R}$-equivariant holomorphic map from X onto a complex $\mathbb{C}$-manifold extends $\mathbb{C}$-equivariantly on X^*. Recently C. Miebach and K. Oeljeklaus have shown that if X is 2-dimensional and the $\mathbb{R}$-action is proper, then the $\mathbb{C}$-action on X^* is also proper, implying that the globalization X^* can be regarded as a holomorphic principal $\mathbb{C}$-bundle over the Riemann surface $S := X^*/\mathbb{C}$ ([MiOe]).

Our main goal here is to present a classification of all such X, up to $\mathbb{R}$-equivariant biholomorphism. We first exploit the above bundle structure in order to give a more precise description of X^*. In the case when S is non compact, X^* is $\mathbb{C}$-equivariantly biholomorphic to $\mathbb{C} \times S$, where $\mathbb{C}$ acts by translations on the first factor. If the base S is compact, then it is hyperbolic and X^* turns out to be $\mathbb{C}$-equivariantly biholomorphic to a certain twisted bundle $(\mathbb{C} \times \Delta)/\Gamma$, where Δ is the unit disk in $\mathbb{C}$ and Γ is the group of deck transformations of the universal covering $\Delta \to S$. Then, by using a result of T. Ueda ([Ued]) as the main ingredient, we prove the following

Theorem. *Let X be a 2-dimensional, taut, Stein manifold with a proper $\mathbb{R}$-action. Then its universal globalization X^* is Stein.*

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Note that in the more general context of Stein $\mathbb{R}$-manifolds it is an open problem to determine whether X^* is always Stein or at least Hausdorff (cf. [HeIa], [CIT], [IST]). Once X^* is understood, we look at the realization of X as an $\mathbb{R}$-invariant domain of X^* and the following question turns out to be crucial. Given an upper semicontinuous function $a : \mathbb{C} \to \{-\infty\} \cup \mathbb{R}$, consider the ($\mathbb{R}$-invariant) subdomain of $\mathbb{C}^2$ defined by

$$\Omega_a := \{ (z, w) \in \mathbb{C}^2 \;:\; a(w) < \operatorname{Im} z \} .$$

Under which conditions on a is Ω_a taut ? Since taut domains in $\mathbb{C}^n$ are Stein, the function a is necessarily subharmonic. Moreover Ω_a cannot contain complex lines, therefore $a(w) > -\infty$ for all $w \in \mathbb{C}$.

Partial answers to this problem can be found, e.g., in [Yu] and [Gau]. Here the following necessary and sufficient condition is obtained by using tools of potential theory (Thm. 3.4).

Theorem. *The domain Ω_a is taut if and only if a is real valued, subharmonic, non-harmonic and continuous.*

This result put us in the position of showing that the 2-dimensional manifolds listed below are all taut and Stein.

Type CH *If S is compact hyperbolic, say $S = \Delta/\Gamma$, the models are certain twisted bundles $(H \times \Delta)/\Gamma$, with H a proper, $\mathbb{R}$-invariant, connected strip in $\mathbb{C}$.*

Type NCH *If S is non compact hyperbolic, the models are*

$$\{ (z, p) \in \mathbb{C} \times S : a(p) < \mathrm{Im}\ z < -b(p) \} ,$$

where a and b are subharmonic, continuous functions on S such that $a + b < 0$ and $\max\{a(p), b(p)\} > -\infty$ for all $p \in S$.

Type NCNH *If $S = \mathbb{C}$ or $S = \mathbb{C}^*$, the models are*

$$\{ (z, p) \in \mathbb{C} \times S \;:\; a(p) < \operatorname{Im} z \} \quad \text{or} \quad \{ (z, p) \in \mathbb{C} \times S \;:\; \operatorname{Im} z < -b(p) \} ,$$

with a, b subharmonic, non-harmonic, real valued, continuous functions on S.

On each such manifold let $\mathbb{R}$ act by translations on the first factor. Then the classification follows by proving that a 2-dimensional, taut, Stein manifold X with a proper $\mathbb{R}$-action is $\mathbb{R}$-equivariantly biholomorphic to a model as above and its type depends on compactness and hyperbolicity of the base S (cf. Thm. 6.1). We recall that in the non compact, simply connected case, a partial result is obtained in [MiOe], Theorem 6.3.

It is worth noting that X turns out to be homotopically equivalent to its base S. As a consequence, the corresponding type is strongly related to the topology of X. For instance, X is of type CH if and only if $H^2(X, \mathbb{Z}) \neq 0$ (cf. Sect. 6).

We also wish to recall that every taut manifold is Kobayashi hyperbolic, therefore its automorphism group is a Lie group acting properly on X (see [Kob], Thm. 5.4.2). It follows that there exists a proper $\mathbb{R}$-action on X if and only if the connected component of the identity in $Aut(X)$ is non compact (cf. [Hoc] p. 180, [MiOe] Lemma 6.3).

As an application of the above classification we determine all 2-dimensional, taut, non-complete Hartogs domains over a Riemann surface (Prop. 7.1). For a characterization of complete Hartogs domains see [ThDu], [Par].

The paper is organized as follows. In Section 2 we point out a characterization of taut manifolds and collect those results which are used in the sequel.
In Section 3 we characterize those domains of the form Ω_a which are taut (Thm. 3.4).
In Section 4 we study models of type CH and show that their globalization is Stein. We also prove that if the base S is compact, then X is $\mathbb{R}$-equivariantly biholomorphic to one of these models.
In Section 5 the analogous results are proved for models of type NCH and NCNH.
In Section 6 we point out that in most cases the type of X is determined by the topology of X (Cor. 6.3 and Rem. 6.4)
In Section 7 we classify 2-dimensional, taut, non-complete Hartogs domains over a Riemann surface.

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2. Preliminaries

By definition a complex manifold is taut if and only if every sequence of holomorphic maps $f_n : \Delta \to X$ admits a subsequence which is either converging uniformly on compact subsets or compactly divergent. If X is taut, then it is hyperbolic and a complete hyperbolic manifold is taut ([Kob], Thm. 5.1.3). We first recall a result of M. Abate and give a characterization of taut manifolds.

Theorem 2.1. ([Aba], Thm. 1.3) *Let X be a complex manifold and $X \cup \{\infty\}$ its Alexandroff compactification. Then X is hyperbolic if and only if $Hol(\Delta, X)$ is relatively compact in $C(\Delta, X \cup \{\infty\})$ with respect to the compact-open topology.*

Note that since $X \cup \{\infty\}$ is metrizable, the compact open topology of $C(\Delta, X \cup \{\infty\})$ coincides with the topology of uniform convergence on compact subsets.

Proposition 2.2. *For a complex manifold X the following conditions are equivalent.*

(i) *X is taut,*

(ii) *for every sequence of holomorphic maps $f_n : \Delta \to X$ such that $f_n(\zeta_0) \to x_0$ for some $\zeta_0 \in \Delta$ and $x_0 \in X$, there exists a subsequence converging uniformly on compact sets of Δ to a map in $Hol(\Delta, \mathbb{C})$.*

Proof. Condition (ii) is clearly satisfied if X is taut. Assume that (ii) holds true. We first show that X is hyperbolic. Recall that the the pseudo distance generated by the Kobayashi infinitesimal pseudo metric K_X on X coincides with the Kobayashi distance (see [Roy]). Then it is enough to show that for every compact subset C of X there exists a constant $c > 0$ such that $K_X(v) \geq c$ for every $p \in C$ and tangent vector $v \in T_pX$ such that $\|v\|_g = 1$, where g is a fixed hermitian metric on X.

Assume by contradiction that there exist a sequence x_n in C and unitary vectors $v_n \in T_{x_n}X$ such that $K_X(v_n) < 1/n$. Then, by definition of K_X, there exist holomorphic maps $f_n : \Delta \to X$ such that $f_n(0) = x_n$ and $f_n'(0) = \lambda_n v_n$, with $\lambda_n > n - 1$. Moreover, by compactness of C, up to subsequence $f_n(0)$ converges to an element x in C while $\|f_n'(0)\|_g \to +\infty$. On the other hand, condition (ii) implies that up to subsequence f_n converges uniformly on compact subsets to a holomorphic map from Δ to X, giving a contradiction. Thus X is Kobayashi hyperbolic.

Finally, let $X \cup \{\infty\}$ be the Alexandroff compactification of X. As a consequence of Theorem 2.1, up to subsequence every sequence of holomorphic maps $f_n : \Delta \to X$ converges uniformly on compact subsets either to the constant map of value ∞ or there exists $\zeta_0 \in \Delta$ and $x_0 \in X$ such that $f_n(\zeta_0) \to x_0$. In the latter case (ii) implies that there exists a subsequence converging uniformly on compact subsets of Δ to a map in $Hol(\Delta, \mathbb{C})$. Hence X is taut. $\square$

A different argument showing that condition (ii) implies hyperbolicity of X also in the context of complex spaces can be found in the proof of Thm. 5.1.6 in [Kob], p. 243. As a corollary to Proposition 2.2, one has

Corollary 2.3. *Let α be a plurisubharmonic, continuous function on a taut manifold X. Then the sublevel sets of α are taut.*

Proof. For $C \in \mathbb{R}$ consider the sublevel set $O_C = \{\, x \in X \;:\; \alpha(x) < C \,\}$ and let $f_n : \Delta \to O_C$ be a sequence of holomorphic maps such that $f_n(\zeta_0) \to x_0$, for some $\zeta_0 \in \Delta$ and $x_0 \in O_C$. Since X is taut, Proposition 2.2 applies to show that up to subsequence f_n converges uniformly on compact subsets to a holomorphic map $f : \Delta \to X$. Note that $\alpha \circ f(\zeta_0) < C$ and by continuity $\alpha \circ f \leq C$ on Δ. Then the maximum principle for plurisubharmonic functions implies that $\alpha \circ f < C$ on Δ, i.e. $f(\Delta) \subset O_C$. Finally the statement follows from Proposition 2.2. □

Next we recall two results due to D.D. Thai and N. L. Huong. ([ThHu], Lemma 3 and Cor. 4). For analogous statements where tautness is replaced by hyperbolicity or complete hyperbolicity, see [Kob], Thm 3.2.8.

Proposition 2.4. *Let X and Y be complex manifolds and $F : X \to Y$ a holomorphic map. If Y is taut and admits an open covering $\{U_j\}$ such that $F^{-1}(U_j)$ is taut for all j, then X is taut.*

Proposition 2.5. *Let X and Y be complex manifolds and $F : X \to Y$ be a holomorphic covering. Then Y is taut if and only if so is X.*

For later use we also collect the following well-known facts.

Lemma 2.6. *Let θ be a real, positive and closed $(1,1)$-current on a complex manifold X.*

(i) *If $H^1(X, \mathcal{O}) = H^2(X, \mathbb{R}) = 0$, then there exists a plurisubharmonic function τ on X such that $\theta = i\partial\bar{\partial}\tau$.*

(ii) *If X is compact Kähler and θ is exact then $\theta = 0$.*

(iii) *If $H^1(X, \mathbb{R}) = 0$ and τ is a pluriharmonic function on X, then there exists a holomorphic function $f : X \to \mathbb{C}$ such that $\operatorname{Im} f = \tau$.*

Proof. (i) follows from the proof of Prop. III 1.19 in [Dem]. For (ii) note that (i) implies that there exist a locally finite open covering $\{U_j\}$ of X and plurisubharmonic functions τ_j on U_j such that $\theta|_{U_j} = i\partial\bar{\partial}\tau_j$. Let ψ_j be a partition of unity associated to $\{U_j\}$, define $T := \sum_j \psi_j \tau_j$ and $\Theta := \theta - i\partial\bar{\partial}T$. Then for j_0 fixed one has $\Theta|_{U_{j_0}} = (\theta - i\partial\bar{\partial}T)|_{U_{j_0}} = i\partial\bar{\partial}\sum_j \psi_j(\tau_{j_0} - \tau_j)$. Since $\tau_{j_0} - \tau_j$ is pluriharmonic on $U_{j_0} \cap U_j$, it follows that Θ is a smooth, exact, real $(1,1)$-form on X. Then, the classical $\partial\bar{\partial}$-Lemma for compact Kähler manifolds (see e.g. [GrHa], Lemma 1.2, p. 148) implies that there exists a smooth function Q on X such that $\Theta = i\partial\bar{\partial}Q$. Hence $\theta = i\partial\bar{\partial}(Q + T)$ and Thm. I 3.31 in [Dem] implies that $Q + T$ is plurisubharmonic on X. Since X is compact, $Q + T$ is constant and consequently θ is zero. For (iii) see [Dem] Theorem I 5.16. □

Let us briefly recall the notion of globalization in the context of $\mathbb{R}$-manifolds. For further details and generalizations we refer to [Pal], [HeIa], [CIT] and [MiOe]. An $\mathbb{R}$-action by biholomorphisms on a complex manifold X induces a local holomorphic $\mathbb{C}$-action by integration of the associated holomorphic vector field. This is given by an open neighborhood Σ of the neutral section $\{0\} \times X$ in $\mathbb{C} \times X$ and a holomorphic map $\Phi : \Sigma \to X$, $(\lambda, x) \to \lambda \cdot x$, such that

(i) the set $\{\lambda \in \mathbb{C} \ : \ (\lambda, x) \in \Sigma\}$ is connected for all $x \in X$,

(ii) for all $x \in X$ one has $0 \cdot x = x$,

(iii) if $(\mu+\lambda, x) \in \Sigma$, $(\lambda, x) \in \Sigma$ and $(\mu, \lambda \cdot x) \in \Sigma$, then $(\mu+\lambda)\cdot x = \mu \cdot (\lambda \cdot x)$.

A possibly non-Hausdorff complex manifold with a global $\mathbb{C}$-action containing X as an $\mathbb{R}$-invariant domain is called a *globalization* of the local $\mathbb{C}$-action. By [HeIa], if X is holomorphically separable there exists a (unique) *universal* globalization X^*. That is, a globalization with the following universal property: for any $\mathbb{R}$-equivariant holomorphic map $f : X \to Y$ into a $\mathbb{C}$-manifold there exists a $\mathbb{C}$-equivariant holomorphic extension $f^* : X^* \to Y$.

For x in X^* let $\Sigma_x = \{ \lambda \in \mathbb{C} \ : \lambda \cdot x \in X \}$. Then Σ_x is $\mathbb{R}$-invariant, connected and there exist upper semicontinuous functions $\alpha, \beta : X^* \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\Sigma_x = \{ \lambda \in \mathbb{C} \ : \alpha(x) < \operatorname{Im} \lambda < -\beta(x) \} .$$

Note that α and β are $\mathbb{R}$-invariant and $\alpha + \beta < 0$. Moreover, an element x of X^* belongs to X if and only if $\alpha(x) < 0 < -\beta(x)$ (cf. [Ian]). Thus

$$X = \{x \in X^* : \alpha(x) < 0 \text{ and } \beta(x) < 0\}.$$

We recall the basic properties of α and β in the case when X is a taut, Stein manifold.

Lemma 2.7. *Let X be a taut, Stein $\mathbb{R}$-manifold. Then*

(i) *the functions α and β are continuous and plurisubharmonic,*

(ii) *for $\lambda \in \mathbb{C}$ and $x \in X^*$ one has*

$$\alpha(\lambda \cdot x) = -\operatorname{Im}(\lambda) + \alpha(x) \qquad \beta(\lambda \cdot x) = \operatorname{Im}(\lambda) + \beta(x) .$$

(iii) *the sum $\alpha + \beta$ is a negative, $\mathbb{C}$-invariant, plurisubharmonic, continuous function,*

(iv) *if the $\mathbb{R}$-action is proper, then* $\max(\alpha(x), \beta(x)) > -\infty$ *for all x in X^*.*

Proof. (i) Plurisubharmonicity of α and β in the case where X is a Stein $\mathbb{R}$-manifold is proved in [For]. Since X is also taut, such functions are continuous ([Ian], [MiOe], Prop. 3.2). (ii) is a direct consequence of the definition and (iii) follows from (i) and (ii). For (iv) note that properness of the $\mathbb{R}$-action implies that there are no fixed points. Therefore if $\alpha(x) = \beta(x) = -\infty$ for some x in

X, the (local) $\mathbb{C}$-orbit through x is biholomorphic either to $\mathbb{C}$ or to $\mathbb{C}^*$. Since X is taut, this gives a contradiction. Recalling that $X^* = \mathbb{C} \cdot X$, the result follows from (ii). $\square$

Finally we recall the following result of C. Miebach and K. Oeljeklaus (see [MiOe], Thm. 4.4) which is often used in the sequel.

Theorem 2.8. *Let X be a 2-dimensional, taut, Stein manifold with a proper $\mathbb{R}$-action. Then the $\mathbb{C}$-action on X^* is proper, i.e. X^* can be regarded as a holomorphic principal $\mathbb{C}$-bundle over the Riemann surface $S := X^*/\mathbb{C}$. In particular if S is non compact, then X^* is $\mathbb{C}$-equivariantly biholomorphic to $\mathbb{C} \times S$.*

Note that last part of the statement follows directly from the fact that on a non compact Riemann surface S the cohomology group $H^1(S, \mathcal{O})$ vanishes.

3. DISTINGUISHED $\mathbb{R}$-INVARIANT DOMAINS IN $\mathbb{C}^2$

Consider the domains of $\mathbb{C}^2$ of the form $\Omega_a = \{\, (z,w) \in \mathbb{C}^2 \,:\, a(w) < \operatorname{Im} z \,\}$, with $a : \mathbb{C} \to \{-\infty\} \cup \mathbb{R}$ an upper semicontinuous function. Note that $\mathbb{R}$ acts properly on Ω_a by translations on the first factor. The main result of this section is Theorem 3.4, where we determine necessary and sufficient conditions for Ω_a to be taut.

We already noted in the introduction that if Ω_a is taut, then a is real valued and subharmonic. Moreover $\mathbb{C}^2$ is the universal globalization of Ω_a, therefore by (i) of Lemma 2.7 the function $\alpha : \mathbb{C}^2 \to \{-\infty\} \cup \mathbb{R}$, given by $(z,w) \to a(w) - \operatorname{Im} z$, is continuous. As a consequence a necessarily belongs to

$$\mathcal{C} := \{\, \text{subharmonic, real valued, continuous functions on } \mathbb{C} \,\}.$$

Also note that if Ω_a is taut, then for all positive $\tau \in \mathbb{R}$ the domain $\Omega_{\tau a}$ is also taut, since the biholomorphism $\mathbb{C}^2 \to \mathbb{C}^2$, defined by $(z,w) \to (\tau z, w)$, maps Ω_a onto $\Omega_{\tau a}$. Thus the set of interest $\mathcal{F} := \{\, a \in \mathcal{C} \,:\, \Omega_a \text{ is taut} \,\}$ is a cone. Here we show that $\mathcal{F}$ coincides with $\{\, a \in \mathcal{C} \,:\, a \text{ is not harmonic} \,\}$. We need some preliminary lemmata.

Lemma 3.1. *Let $a \in \mathcal{C}$ and $(f_n, g_n) : \Delta \to \Omega_a$ be a sequence of holomorphic maps such that*

(i) *$f_n(\zeta_0) \to z_0$, for some $\zeta_0 \in \Delta$ and $z_0 \in \mathbb{C}$,*

(ii) g_n *converges uniformly on compact subsets of* Δ *to a holomorphic map* $g : \Delta \to \mathbb{C}$ *such that* $(z_0, g(\zeta_0)) \in \Omega_a$.

Then there exists a subsequence of f_n *converging uniformly on compact subsets of* Δ *to a holomorphic map* $f : \Delta \to \mathbb{C}$ *such that* $(f, g)(\Delta) \subset \Omega_a$.

Proof. Let U_1 be a relatively compact disk of Δ containing ζ_0. By condition (ii) the sequence g_n converges uniformly to g on the closure $\overline{U}_1$ of U_1. Then, for n large enough $f_n(U_1)$ is contained in the set

$$S_1 := \{\, z \in \mathbb{C} \ : \ \operatorname{Im} z > \min_{w \in \overline{U_1}} \{a(g(w))\} - 1 \,\},$$

which is biholomorphic to the unit disc of $\mathbb{C}$. In particular S_1 is taut, therefore there exists a subsequence $f_{n,1}$ of f_n converging uniformly on compact subsets of U_1 to a holomorphic map $f_1 : U_1 \to S_1$.

Complete U_1 to an increasing sequence of simply connected domains $\{U_k\}_{k\in\mathbb{N}}$ which exhaust Δ. By iterating the above argument, for each $k \in \mathbb{N}$ one obtains subsequences $\{f_{n,k}\}_{n\in\mathbb{N}}$ converging uniformly on compact subsets of U_k to holomorphic maps $f_k : U_k \to S_k$. Then the diagonal sequence $\{f_{j,j}\}_{j\in\mathbb{N}}$ converges uniformly on compact subsets of Δ.

Finally note that $a\circ g(\zeta_0) - \operatorname{Im} f(\zeta_0) < 0$ and by continuity $a\circ g - \operatorname{Im} f \le 0$ on Δ. Then, by the maximum principle for subharmonic functions, $a \circ g - \operatorname{Im} f < 0$ on Δ, i.e. $(f, g)(\Delta) \subset \Omega_a$.

□

Given a subharmonic function a on $\mathbb{C}$, denote by $M_{\zeta_0,r}(a)$ its mean value $\frac{1}{2\pi}\int_0^{2\pi} a(\zeta_0 + re^{i\theta})d\theta$.

Lemma 3.2. *For* a *in* $\mathcal{C}$ *the following conditions are equivalent.*

(i) $a \in \mathcal{F}$,

(ii) *for any sequence of holomorphic functions* $g_n : \Delta \to \mathbb{C}$ *satisfying*

(a) $g_n(\zeta_0) \to w_0$ *for some* $\zeta_0 \in \Delta$ *and* $w_0 \in \mathbb{C}$,

(b) *for every* $0 < r < 1 - |\zeta_0|$ *there exists* $M_r \in \mathbb{R}$ *such that* $M_{\zeta_0,r}(a \circ g_n) < M_r$ *for all* $n \in \mathbb{N}$,

there exists a subsequence converging uniformly on compact subsets of $\Delta_{1-|\zeta_0|}(\zeta_0)$.

Proof. Assume that Ω_a is taut and let g_n be a sequence as in (ii). For $n \in \mathbb{N}$ and $0 < r < 1 - |\zeta_0|$, denote by h_n the harmonic function on $\Delta_r(\zeta_0)$ which coincides with $a \circ g_n$ on the boundary of $\Delta_r(\zeta_0)$. Then $h_n(\zeta_0) = M_{\zeta_0,r}(a \circ g_n)$ and consequently, for n large enough, one has

$$a(w_0) - 1 < a(g_n(\zeta_0)) \le h_n(\zeta_0) < M_r.$$

As a consequence, up to subsequence $h_n(\zeta_0)$ converges to a real number y. Let $f_n : \Delta_r(\zeta_0) \to \mathbb{C}$ be the sequence of holomorphic functions defined by $\operatorname{Im} f_n = h_n + 1$ and $\operatorname{Re} f_n(\zeta_0) = 0$. Since $\operatorname{Im} f_n = h_n + 1 \geq a \circ g_n + 1 > a \circ g_n$, it follows that (f_n, g_n) defines a sequence of holomorphic maps from $\Delta_r(\zeta_0)$ to Ω_a.

Moreover $(f_n, g_n)(\zeta_0) \to (i(y+1), w_0) \in \Omega_a$ and Ω_a is taut. Then by Lemma 2.2 there exists a subsequence (f_n, g_n) converging uniformly on compact subsets of $\Delta_r(\zeta_0)$.

Let r_k be an increasing sequence of positive numbers converging to $1 - |\zeta_0|$ such that $r_1 = r$. The analogous argument as above shows that there exist subsequences $(f_{n,k}, g_{n,k})$ converging uniformly on compact subsets of $\Delta_{r_k}(\zeta_0)$. Then the diagonal subsequence $(f_{n,n}, g_{n,n})$ converges uniformly on compact subsets of $\Delta_{1-|\zeta_0|}(\zeta_0)$ and so does $g_{n,n}$. This implies (ii).

Conversely assume (ii) and let $(f_n, g_n) : \Delta \to \Omega_a$ be a sequence of holomorphic maps such that $(f_n, g_n)(\zeta_0) \to (z_0, w_0)$, for some ζ_0 in Δ and (z_0, w_0) in Ω_a. By Lemma 2.2 it is enough to show that, up to subsequence, (f_n, g_n) converges uniformly on compact subsets of Δ to some (f, g) with $(f, g)(\Delta) \subset \Omega_a$. Note that for $0 < r < 1 - |\zeta_0|$ and n large enough one has

$$\operatorname{Im} z_0 + 1 > \operatorname{Im} f_n(\zeta_0) = M_{\zeta_0,r}(\operatorname{Im} f_n) > M_{\zeta_0,r}(a \circ g_n)\,.$$

Thus, by assumption, up to subsequence g_n converges uniformly on compact sets of the disk $\Delta_{1-|\zeta_0|}(\zeta_0)$ and, by Lemma 3.1 so does f_n. Therefore for every point $\zeta \in \Delta_{1-|\zeta_0|}(\zeta_0)$ there exists a subsequence of (f_n, g_n) converging at ζ to an element of Ω_a. Then by constructing a finite chain of disks one shows that, up to subsequence, (f_n, g_n) converges at 0 to an element of Ω_a. Finally the analogous argument as above implies that, up to subsequence (f_n, g_n), converges uniformly on compact subsets of Δ to some (f, g) with $(f, g)(\Delta) \subset \Omega_a$. □

Lemma 3.3. *The cone $\mathcal{F}$ has the following properties.*

(i) *Harmonic functions do not belong to $\mathcal{F}$.*

(ii) *If $a \in \mathcal{C}$ is non constant and bounded from below, then $a \in \mathcal{F}$.*

(iii) *If $b \in \mathcal{C}$ and $c \in \mathcal{F}$ then $b + c \in \mathcal{F}$.*

Proof. (i) If a is harmonic, then $a = \operatorname{Im} f$ for some holomorphic $f : \mathbb{C} \to \mathbb{C}$. Then the biholomorphism of $\mathbb{C}^2$ defined by $(z, w) \to (z - f(w), w)$ maps Ω_a onto $\{\,(z, w) \in \mathbb{C}^2 \;:\; \operatorname{Im} z > 0\,\}$, which is not taut. Thus Ω_a is not taut.

For (ii) consider the restriction to Ω_a of the projection from $\mathbb{C}^2$ onto the first factor given by

$$p|_{\Omega_a} : \Omega_a \to p(\Omega_a)\,, \qquad (z, w) \to z\,.$$

Since a is bounded from below, the image $p(\Omega_a)$ is contained in the half plane $\{\operatorname{Im} z > \inf_{\mathbb{C}} a\,\}$, which is taut. Then, by Lemma 2.4, in order to prove that Ω_a

is taut it is enough to show that $(p|_{\Omega_a})^{-1}(U)$ is taut for every relatively compact open subset U in $p(\Omega_a)$.

For this, let M be the maximum of $\operatorname{Im} z$ on the closure of U and note that $(p|_{\Omega_a})^{-1}(U)$ is contained in $U \times \{ a < M \}$. Since a is not constant, it is not bounded. As a consequence $\{ a < M \}$ is a hyperbolic domain of $\mathbb{C}$. Thus it is taut and so is $U \times \{ a < M \}$. Finally, the image $(p|_{\Omega_a})^{-1}(U)$ is the zero sublevel set in $U \times \{ a < M \}$ of the subharmonic, continuous function $(z, w) \to a(w) - \operatorname{Im} z$. Thus it is taut by Corollary 2.3, concluding (ii).

For (iii) let $g_n : \Delta \to \mathbb{C}$ be a sequence of holomorphic maps such that $g_n(\zeta_0) \to w_0$ for some $\zeta_0 \in \Delta$, $w_0 \in \mathbb{C}$ and for every $0 < r < 1 - |\zeta_0|$ there exists a real number M_r such that $M_{\zeta_0,r}((b + c) \circ g_n) < M_r$ for all $n \in \mathbb{N}$. Then by Lemma 3.2 in order to show that $b + c$ belongs to $\mathcal{F}$, it is enough to find a subsequence of g_n converging uniformly on compact subsets of Δ. For $0 < r < 1 - |\zeta_0|$ and n large enough one has

$$M_r > M_{\zeta_0,r}((b + c) \circ g_n) \geq b(g_n(\zeta_0)) + M_{\zeta_0,r}(c \circ g_n) > b(w_0) - 1 + M_{\zeta_0,r}(c \circ g_n)\,.$$

Hence

$$M_{\zeta_0,r}(c \circ g_n) < M_r - b(w_0) + 1\,.$$

Since $c \in \mathcal{F}$, Lemma 3.2 implies that there exists a subsequence of g_n converging uniformly on compact subsets of Δ, as wished. $\square$

Theorem 3.4. *Let $a : \mathbb{C} \to \{-\infty\} \cup \mathbb{R}$ be an upper semicontinuous function. Then $\Omega_a := \{ (z, w) \in \mathbb{C}^2 \ : \ a(w) < \operatorname{Im} z\}$ is taut if and only if a is real valued, subharmonic, non-harmonic and continuous.*

Proof. We already noted at the beginning of the section that if Ω_a is taut, then a belongs to $\mathcal{C}$. Moreover, by (i) of the above lemma a is not harmonic, giving one implication.

Conversely, given $a \in \mathcal{C}$ non-harmonic we want to show that $a \in \mathcal{F}$. By (ii) and (iii) of the above lemma, it is enough to show that $a = b + c$, with $b,\ c \in \mathcal{C}$ and c non constant and bounded from below.

For this consider the positive measure $\mu = L(a)$, where $L(a)$ denotes the laplacian of a, and choose r big enough such that μ is non zero on $\Delta_r(0)$. Let $\chi_{\Delta_r(0)}$ be the characteristic function of $\Delta_r(0)$ and define $\mu_1 = (1 - \chi_{\Delta_r(0)})\mu$ and $\mu_2 = \chi_{\Delta_r(0)}\mu$, so that $\mu = \mu_1 + \mu_2$ gives a decomposition of μ as a sum of positive measures on $\mathbb{C}$. Note that μ_2 is non zero with compact support and consider the potential $c : \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$ associated to μ_2 defined by

$$c(w) := \frac{1}{2\pi} \int_{\mathbb{C}} \log(|w - \xi|) d\mu_2(\xi) = \frac{1}{2\pi} \int_{\Delta_r(0)} \log(|w - \xi|) d\mu_2(\xi)\,.$$

Then the laplacian $L(c)$ of c coincides with μ_2 (see e.g. [Kli], Prop. 4.1.2), therefore c is non constant and subharmonic.

Furthermore, the real $(1,1)$-current $\mu_1 d\xi d\overline{\xi}$ is closed and positive on $\mathbb{C}$, hence by (i) of Lemma 2.6 there exists a subharmonic function $\tilde{b} : \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$ such that $L(\tilde{b}) = \mu_2$. It follows that $L(\tilde{b} + c) = L(a)$ and consequently $a = \tilde{b} + c + h$, with h harmonic on $\mathbb{C}$. This implies that $\tilde{b} + c$ is continuous and real valued. Since $\tilde{b}$ and c are everywhere smaller than $+\infty$, they are also real valued. Moreover c is upper semicontinuous, $-\tilde{b}$ is lower semicontinuous and $c = -\tilde{b} + a - h$, with $a - h$ continuous. Thus $\tilde{b}$ and c are continuous subharmonic functions, i.e. they belong to $\mathcal{C}$, and $b := \tilde{b} + h \in \mathcal{C}$.

Finally note that the non constant function c is bounded from below. Indeed by definition of c, if w is not in $\Delta_{r+1}(0)$ then $c(w) \geq 0$. Since c is continuous, this implies that $c \geq \min\{0, m\}$, with $m := \min_{w \in \overline{\Delta_{r+1}(0)}}\{c(w)\}$. Then $a = b + c$ gives the desired decomposition. $\square$

4. Models with compact base

Let S be a compact hyperbolic Riemann surface, say $S = \Delta/\Gamma$, with Γ the subgroup in $\mathrm{Aut}(\Delta)$ of deck transformations of the universal covering $\Delta \to S$. Choose a non trivial group homomorphism $\Psi : \Gamma \to \mathbb{R}$ and let Γ act on $\mathbb{C} \times \Delta$ by $\gamma \cdot (z, w) := (z + \Psi(\gamma), \gamma \cdot w)$. Endow the quotient $(\mathbb{C} \times \Delta)/\Gamma$ with the $\mathbb{R}$-action defined by $t \cdot (z, w) := (z + t, w)$. We introduce the first class of models as $\mathbb{R}$-invariant subdomains of $(\mathbb{C} \times \Delta)/\Gamma$.

Type CH A model of type CH with compact hyperbolic base $S = \Delta/\Gamma$ is given by

$$(H \times \Delta)/\Gamma\,,$$

where H is a proper, $\mathbb{R}$-invariant, connected strip of $\mathbb{C}$. Up to $\mathbb{R}$-equivariant biholomorphism, we may assume that H is one of the strips $\{0 < \mathrm{Im}\, z\}$, $\{\mathrm{Im}\, z < 0\}$ or $\{0 < \mathrm{Im}\, z < C\}$, for some real positive C.

Proposition 4.1. *Let X be a model of type* CH *with base $S = \Delta/\Gamma$. Then*
(i) *the universal globalization of X is $(\mathbb{C} \times \Delta)/\Gamma$, which is Stein.*
(ii) *X is a taut, Stein manifold with a proper $\mathbb{R}$-action.*

Before proving the above proposition we need a preparatory lemma. Given a rank two holomorphic vector bundle E over a compact Riemann surface S, denote by P its (fiberwise) projectification and let $p : E \backslash S \to P$ be the canonical projection. Here S is identified with the zero section in E. Let $\sigma : S \to P$ be a holomorphic section of P and consider its image $C := \sigma(S)$. Recall that the normal bundle N of the curve C is given by $TP|_C/TC$ and it can be identified with the line bundle $\sigma^*(N)$ over S.

Regard the tautological line bundle $\mathcal{O}(-1)$ as a subbundle in $\pi^*(E)$, where $\pi : P \to S$ is the bundle projection. Then the holomorphic line bundle associated to σ is $L := \sigma^*(\mathcal{O}(-1))$ and can be identified with the subbundle of E given by $p^{-1}(C) \cup S$.

Lemma 4.2. *The normal bundle $\sigma^*(N)$ is isomorphic to $(E/L) \otimes L^*$.*

Proof. Consider the relative tangent bundle $T_{P/S} := \operatorname{Ker} d\pi$. We first note that N is isomorphic to the restriction $T_{P/S}|_C$ of such a bundle to C, since one has the short exact sequence of vector bundles over C

$$0 \to TC \to TP|_C \to T_{P/S}|_C \to 0\,,$$

where the third map is defined by $v \mapsto v - d\sigma \circ d\pi(v)$.

We first assume that L is trivial, i.e. it admits a non zero holomorphic section τ. Then one has the commutative diagram

$$\begin{tikzcd} E \setminus S \arrow[rr, "p"] & & P \\ & S \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] & \end{tikzcd}$$

and an exact sequence of vector bundles over $\tau(S)$

$$0 \to T_{L/S}|_{\tau(S)} \to T_{E/S}|_{\tau(S)} \to p^*(T_{P/S}|_C) \to 0\,,$$

where the third map is given by $v \mapsto dp(v)$. Since $p \circ \tau = \sigma$, by applying τ^* one obtains the following exact sequence of vector bundles over S

$$0 \to L \to E \to \sigma^*(T_{P/S}|_C) \to 0\,,$$

where we use the natural identification $\tau^*(T_{F/S}|_{\tau(S)}) \cong F$ for any vector subbundle F of E. Moreover, by recalling that N is isomorphic to $T_{P/S}|_C$, one obtains that $\sigma^*(N)$ is isomorphic to E/L, as wished.

Finally, if L is non trivial note that P can be regarded as the projectification $\mathbb{P}(E \otimes L^*)$ of $E \otimes L^*$. For this, let $\hat{p} : (E \otimes L^*) \setminus S \to \mathbb{P}(E \otimes L^*)$ be the canonical projection and let $\rho : U \to L^*$ be a local, never vanishing, holomorphic section of L^* defined on a domain U of S. Then the identification $P \to \mathbb{P}(E \otimes L^*)$ is locally defined by $p(w) \to \hat{p}(\, w \oplus \rho(\pi(p(w)))\,)$, for all $w \in E \setminus S$ such that $\pi(p(w)) \in U$. Since $\sigma(S) = \mathbb{P}(L)$, such an identification maps $\sigma(S)$ onto the projectification of the trivial line bundle $L \otimes L^*$. Then an analogous argument as above implies that $\sigma^*(N)$ is isomorphic to $(E \otimes L^*)/(L \otimes L^*)$ and by the exactness of the sequence of vector bundles over S

$$0 \to L \otimes L^* \to E \otimes L^* \to (E/L) \otimes L^* \to 0\,.$$

one has $(E \otimes L^*)/(L \otimes L^*) \cong (E/L) \otimes L^*$. □

Proof of Proposition 4.1 (i) Note that X is orbit-connected in $(\mathbb{C} \times \Delta)/\Gamma$. Then Lemma 1.5 in [CIT] implies that $X^* := (\mathbb{C} \times \Delta)/\Gamma$ is the universal globalization of X. Consider the $\mathbb{P}^1$-bundle $P := (\mathbb{P}^1 \times \Delta)/\Gamma$, where Γ act on $\mathbb{P}^1 \times \Delta$ by $\gamma \cdot ([z_1 : z_2], w) := ([z_1 + \Psi(\gamma) z_2 : z_2], \gamma \cdot w)$. Then X^* is embedded in P via the map

$$[z, w] \to [[z : 1], w] .$$

and the union of points at infinity defines the complex curve $C := \{ [[1 : 0], w] \in P \ : \ w \in \Delta \}$ which is biholomorphic to S. Indeed it can be regarded as the holomorphic section $\sigma : S \to P$, defined by $[w] \to [[1 : 0], w]$.

We wish to apply Theorem 1, p. 590 in [Ued] in order to obtain a suitable strictly plurisubharmonic function on $V_0 \setminus C$, for some open neighborhood V_0 of C in P. For this we first check that the normal bundle of C is trivial. Consider the rank two vector bundle over S defined by $E := (\mathbb{C}^2 \times \Delta)/\Gamma$, where Γ acts on $\mathbb{C}^2 \times \Delta$ by $\gamma \cdot ((z_1, z_2), w) := ((z_1 + \Psi(\gamma) z_2, z_2), \gamma \cdot w)$. Note that the line subbundle $L := \{ [(z_1, z_2), w] \in E \ : z_2 = 0 \}$ associated to the section σ is trivial. Indeed it admits the global section $[w] \to [(1, 0), w]$. Since P is the projectification of E, by Lemma 4.2 this implies that the normal bundle of $C := \sigma(S)$ is isomorphic to E/L. Moreover one has the short exact sequence of vector bundles over S

$$0 \to L \to E \to \mathbb{C} \times S \to 0 ,$$

where the third map is defined by $[(z_1, z_2), w] \to (z_2, [w])$. Therefore E/L is trivial and so is the normal bundle of C.

Next we check that the curve C is of type 1, in the sense of Definition p. 589 in [Ued]. For this choose an open covering $\{U_j\}$ of S such that there exist injective, local sections $s_j : U_j \to \Delta$ of the universal covering $\Delta \to S$. Define local trivializations of P by

$$\mathbb{P}^1 \times U_j \to P , \quad ([z_1 : z_2], p) \to [[z_1 : z_2], s_j(p)] .$$

Note that the curve C is locally defined by $\{z_2 = 0\}$ and in a neighborhood of C the intersection of two trivializations associated to the sections s_j and s_k is given by

$$[[1 : z_2], s_k(p)] = [[1 : z_2'], s_j(p)].$$

This implies that there exists $\gamma \in \Gamma$ such that $s_j(p) = \gamma \cdot s_k(p)$ and consequently

$$[[1 : z_2], s_k(p)] = [[1 : z_2'], \gamma \cdot s_k(p)] = [[1 - \Psi(\gamma) z_2' : z_2'], s_k(p)] .$$

Since Γ acts freely on Δ, it follows that $z_2 = z_2'/(1 - \Psi(\gamma) z_2')$ and

$$z_2 - z_2' = z_2'\left(\frac{1}{1 - \Psi(\gamma) z_2'} - 1\right) = (z_2')^2 \frac{\Psi(\gamma)}{1 - \Psi(\gamma) z_2'} = (z_2')^2 (\Psi(\gamma) + O(z_2')) .$$

In our setting the normal bundle of C is holomorphically trivial, therefore the locally constant maps $f_{jk} : U_j \cap U_k \to \mathbb{C}$, given by $p \to \Psi(\gamma)$, define a cocycle in $H^1(S, \mathcal{O})$ (cf. [Ued], p. 588).

Claim. The cocycle f_{jk} is cohomologous to zero if and only if Ψ is trivial.

Proof of Claim. By using the above defined sections $s_j : U_j \to \Delta$ one has local trivializations of X^* given by

$$\mathbb{C} \times U_j \to X^*\,, \quad (z,p) \to [z, s_j(p)]\,.$$

It follows that f_{jk} is the cocycle defining X^* as a holomorphic principal $\mathbb{C}$-bundle over S. Assume that there exists a holomorphic ($\mathbb{C}$-equivariant) trivialization $F : X^* \to \mathbb{C} \times S$. We can choose a ($\mathbb{C}$-equivariant) lifting $\tilde{F} : \mathbb{C} \times \Delta \to \mathbb{C} \times \Delta$ to the universal coverings such that $\tilde{F}(z,w) = (z + \tilde{f}(w), w)$, with $\tilde{f} : \Delta \to \mathbb{C}$ holomorphic. Moreover for every $\gamma \in \Gamma$ one has

$$\tilde{F}(\gamma \cdot (z,w)) = \gamma \cdot \tilde{F}(z,w) = (z + \tilde{f}(w), \gamma \cdot w)\,,$$

implying that $\tilde{f}(\gamma \cdot w) + \Psi(\gamma) = \tilde{f}(w)$. In particular

$$\tilde{f}(w) - \tilde{f}(\gamma \cdot w) = \Psi(\gamma) \in \mathbb{R}.$$

Hence $\operatorname{Im} \tilde{f}$ is Γ-invariant, therefore it pushes down to a harmonic function on $S := \Delta/\Gamma$. Then the compactness of S implies that $\operatorname{Im} \tilde{f}$ is constant and consequently $\tilde{f}$ is constant. Hence $\Psi(\gamma) = 0$ for all $\gamma \in \Gamma$, proving the claim.

Since Ψ is non-trivial by assumption, the cocycle f_{jk} is not cohomologous to zero, i.e. the curve C is of type 1. Then, by Theorem 1, p. 590 in [Ued] there exists an open neighborhood V_0 of C in P and a smooth, strictly plurisubharmonic function ρ defined on $V_0 \setminus C$ such that $\lim \rho(p) = \infty$ for p approaching C. In particular we may assume that ρ is positive.

Fix N large enough such that the domain $X_N := \{\,[z,w] \in (\mathbb{C} \times \Delta)/\Gamma \;:\; \operatorname{Im} z > N\}$ is contained in V_0. Note that X_N is Stein, since it admits the smooth, strictly plurisubharmonic exhaustion $\rho + \frac{1}{\operatorname{Im} z - N}$. Moreover for all $n \in \mathbb{N}$ the domains X_{N-n} are also Stein, being biholomorphic to X_N via a translation in the first factor. Furthermore X_{N-n} can be regarded as a sublevel set of the plurisubharmonic function $\operatorname{Im} z$, therefore it is Runge in $X_{N-(n+1)}$. Then $(\mathbb{C} \times \Delta)/\Gamma = \cup_n X_{N-n}$ is Stein by a classical result of K. Stein [Ste].

(ii) Note that X is an $\mathbb{R}$-invariant, locally Stein domain in the Stein, principal $\mathbb{C}$-bundle $X^* = (\mathbb{C} \times \Delta)/\Gamma$ over S. Thus the $\mathbb{R}$-action on X is proper and X is Stein by [DoGr]. Finally the universal covering of X is given by $H \times \Delta$, which is taut. Thus X is taut by Proposition 2.5. $\square$

Remark 4.3. It was pointed out to us by Christian Miebach that a similar strategy as above applies to show that every non trivial principal $\mathbb{C}$-bundle over a compact Riemann surface is Stein.

Remark 4.4. Let $F : (H \times \Delta)/\Gamma \to (H' \times \Delta)/\Gamma'$ be an $\mathbb{R}$-equivariant biholomorphism between two models of type CH and consider a holomorphic lifting $\tilde{F} : H \times \Delta \to H' \times \Delta$ to the universal covering spaces. We claim that $\tilde{F}(z,w) = (z + r, \tilde{\varphi}(w))$, where $r \in \mathbb{R}$ and $\tilde{\varphi} \in Aut(\Delta)$. In particular $H = H'$.

In order to prove this, note that $\tilde{F} = (\tilde{F}_1, \tilde{F}_2)$ is also $\mathbb{R}$-equivariant. Then, from the analytic continuation principle it follows that $\tilde{F}_1$ extends holomorphically on $\mathbb{C}^2$ and $\tilde{F}_1(z,w) = \tilde{F}_1(0,w) + z$ for all $(z,w) \in H \times \Delta$. Similarly, $\tilde{F}_2(z,w) = \tilde{F}_2(0,w)$ for all $(z,w) \in H \times \Delta$. As a consequence $\tilde{F}(z,w) = (z + f(w), \tilde{\varphi}(w))$, with $f : \Delta \to \mathbb{C}$ holomorphic and $\tilde{\varphi} \in Aut(\Delta)$.

Now recall that the actions of Γ and Γ' on $\mathbb{C} \times \Delta$ are given respectively by $\gamma \cdot (z,w) = (z + \Psi(\gamma), \gamma(w))$ and $\gamma' \cdot (z,w) = (z + \Psi'(\gamma'), \gamma'(w))$. Moreover, for every $\gamma \in \Gamma$ there exists $\gamma' \in \Gamma'$ such that $\tilde{F}(\gamma \cdot (z,w)) = \gamma' \cdot \tilde{F}(z,w)$. That is

$$(z + \Psi(\gamma) + f(\gamma(z)),\ \tilde{\varphi}(\gamma(w))) = (z + f(w) + \Psi'(\gamma'),\ \gamma'(\tilde{\varphi}(w)))$$

and consequently

$$\tilde{\varphi}\gamma\tilde{\varphi}^{-1} = \gamma' \quad \text{and} \quad f \circ \gamma - f = \Psi'(\tilde{\varphi}\gamma\tilde{\varphi}^{-1}) - \Psi(\gamma) \in \mathbb{R}\,.$$

In particular $\Gamma' = \tilde{\varphi}\Gamma\tilde{\varphi}^{-1}$, therefore $\tilde{\varphi}$ induces a biholomorphism $\varphi : \Delta/\Gamma \to \Delta/\Gamma'$. Moreover $\operatorname{Im} f$ is Γ-invariant. Then the analogous argument as in the claim in the proof of Proposition 4.1 implies that $f \equiv r$, with $r \in \mathbb{C}$. In particular $\Psi'(\tilde{\varphi}\gamma\tilde{\varphi}^{-1}) = \Psi(\gamma)$ and H, H' are either both of finite width or of infinite width. Assume that, e.g. $H = \{0 < \operatorname{Im} z < C\}$ and $H' = \{0 < \operatorname{Im} z < C'\}$. By applying $\tilde{F}$ to any $(z,w) \in H \times \Delta$ one sees that $0 < \operatorname{Im} z$ if and only if $0 < \operatorname{Im} z + \operatorname{Im} r$. This implies that $\operatorname{Im} r = 0$, i.e. that r is a real number and consequently $C = C'$. An analogous argument applies to the case when H has infinite width. $\square$

Let X be a 2-dimensional, taut, Stein manifold with a proper $\mathbb{R}$-action. By Theorem 2.8, the $\mathbb{C}$-action on X^* is proper and one can consider the associated holomorphic principal $\mathbb{C}$-bundle

$$\Pi : X^* \longrightarrow S := X^*/\mathbb{C}\,.$$

If S is compact, we show that X is $\mathbb{R}$-equivariantly biholomorphic to a model of type CH. Then Proposition 4.1 implies that the globalization X^* is Stein. We need a preliminary result. Let the functions α, β be defined as in Lemma 2.7.

Lemma 4.5. *If S is compact then α, respectively β, is either pluriharmonic or constantly equal to $-\infty$.*

Proof. Assume that α is not constantly equal to $-\infty$. Since by (ii) of Lemma 2.7 one has $\alpha(\lambda \cdot x) = -\operatorname{Im}(\lambda) + \alpha(x)$ for all $x \in X^*$ and $\lambda \in \mathbb{C}$, the real, positive $(1,1)$-current $i\partial\bar{\partial}\alpha$ is $\mathbb{C}$-invariant. Therefore it pushes down to a $(1,1)$-current θ on S such that $\Pi^*(\theta) = i\partial\bar{\partial}\alpha$. Note that θ is also positive.

Recall that all cohomology groups with values in the sheaf of smooth functions on S vanish. Thus X^* is trivial as a differentiable principal $\mathbb{C}$-bundle and the maps induced by Π in cohomology are isomorphisms. Since $i\partial\bar{\partial}\alpha$ is an exact current, this implies that θ is also an exact current. Then, from (ii) of Lemma

2.6 it follows that $\theta = 0$ and consequently $i\partial\bar\partial\alpha = \Pi^*(\theta) = 0$. Hence α is pluriharmonic. An analogous argument applies to show that if the function β is not constantly equal to $-\infty$, then it is is pluriharmonic. $\square$

Proposition 4.6. *Let X be a 2-dimensional, taut, Stein manifold with a proper $\mathbb{R}$-action and assume that $S := X^*/\mathbb{C}$ is compact. Then S is hyperbolic and X is $\mathbb{R}$-equivariantly biholomorphic to a model of type* CH. *In particular X^* is Stein.*

Proof. First note that S can not be biholomorphic to the Riemann sphere. Indeed $H^1(\mathbb{P}^1(\mathbb{C}), \mathcal{O}) = 0$, thus if $S = \mathbb{P}^1(\mathbb{C})$ then $X^* = \mathbb{C} \times \mathbb{P}^1(\mathbb{C})$. Moreover the functions $\mathbb{P}^1(\mathbb{C}) \to \mathbb{R} \cup \{\infty\}$, defined by $p \to \alpha(0,p)$ and $p \to \beta(0,p)$, are constant, being subharmonic on $\mathbb{P}^1(\mathbb{C})$. Since (ii) of Lemma 2.7 implies that $X = \{ (z,p) \in \mathbb{C} \times \mathbb{P}^1(\mathbb{C}) \; : \; \alpha(0,p) < \operatorname{Im} z < -\beta(0,p)\}$, it follows that X is the product of a strip in $\mathbb{C}$ and $\mathbb{P}^1(\mathbb{C})$. However X is Stein, therefore this is impossible.

Now let us show that S is hyperbolic. Consider the universal covering space $\pi : \tilde{X}^* \to X^*$ of X^* with deck transformation group Γ. The proper $\mathbb{C}$-action on X^* lifts to a proper $\mathbb{C}$-action on $\tilde{X}^*$, therefore $\tilde{X}^*$ is a principal $\mathbb{C}$-bundle over $\tilde{S} \cong \tilde{X}^*/\mathbb{C}$. Note that X^* and $\tilde{X}^*$ are trivial as differentiable principal $\mathbb{C}$-bundles over S and $\tilde{S}$, respectively. This implies that the Riemann surface $\tilde{S}$ is simply connected, therefore it is non compact and consequently $\tilde{X}^*$ is $\mathbb{C}$-equivariantly biholomorphic to $\mathbb{C} \times \tilde{S}$ (cf. Thm. 2.8). One has a commutative diagram of holomorphic maps

$$\begin{array}{ccc} \tilde{X}^* = \mathbb{C} \times \tilde{S} & \stackrel{\pi}{\to} & X^* = (\mathbb{C} \times \tilde{S})/\Gamma \\ \downarrow & & \downarrow \\ \tilde{S} & \stackrel{\hat\pi}{\to} & S = \tilde{S}/\Gamma \end{array} ,$$

where $\hat\pi$ is the universal covering of S with deck transformation group Γ.

By (iii) of Lemma 2.7, the sum $\alpha+\beta$ is $\mathbb{C}$-invariant, thus it can be regarded as a subharmonic function on S. Since S is compact, $\alpha + \beta$ is constant. Moreover polar sets have zero measure, therefore if $\alpha + \beta \equiv -\infty$ then either $\alpha \equiv -\infty$ or $\beta \equiv -\infty$. As a consequence $X = \{ x \in X^* \; : \; \alpha(x) < 0 \}$ or $X = \{ x \in X^* \; : \; \beta(x) < 0 \}$. On the other hand, if $\alpha + \beta = -C$ for some positive real number C, one has $X = \{x \in X^* : \alpha(x) < 0 < -\beta(x)\} = \{x \in X^* : -C < \alpha(x) < 0\}$.

First consider the case when $X = \{ x \in X^* \; : \; \alpha(x) < 0 \}$. Set $\tilde\alpha = \alpha \circ \pi$ and let $\tilde{X} := \pi^{-1}(X) = \{(z,p) \in \mathbb{C} \times \tilde{S} \; : \; \tilde\alpha(z,p) < 0\}$. Recall that α is pluriharmonic by Lemma 4.5, therefore so is $\tilde\alpha$. Since $\tilde{X}^*$ is simply connected, (iii) of Lemma 2.6 implies that there exists a holomorphic function $f : \tilde{X}^* \to \mathbb{C}$ such that $\operatorname{Im}(f) = \tilde\alpha$. Moreover, for all $(z,p) \in \tilde{X}^*$ one has

$$\tilde\alpha(z,p) = \alpha \circ \pi(z \cdot (0,p)) = \alpha(z \cdot \pi(0,p)) = \tilde\alpha(0,p) - \operatorname{Im} z \,,$$

therefore $f(z,p) = f(0,p) - z$. Then the map defined by

$$(z,p) \to (-f(z,p),p) = (z - f(0,p),p)$$

gives a $\mathbb{C}$-equivariant biholomorphism of $\mathbb{C} \times \tilde{S}$ and its restriction to $\tilde{X}$ defines an $\mathbb{R}$-equivariant biholomorphism onto $\{(z,p) \in \mathbb{C} \times \tilde{S} : 0 < \operatorname{Im} z\}$, which is simply connected. Thus $\tilde{X}$ can be regarded as the universal covering of X and since $\tilde{X}$ is taut by Proposition 2.5, this implies that $\tilde{S} \cong \Delta$, i.e. that S is hyperbolic.

An analogous argument applies to the cases when $X = \{\, x \in X^* \ : \ \beta(x) < 0 \,\}$ and $X = \{\, x \in X^* \ : \ -C < \alpha(x) < 0 \,\}$, showing that S is hyperbolic and that $\tilde{X}$ is $\mathbb{R}$-equivariantly biholomorphic to $H \times \Delta$, where H is given by $\{0 < \operatorname{Im} z\}$, $\{\operatorname{Im} z < 0\}$ or $\{0 < \operatorname{Im} z < C\}$, for some positive real C.

Identify the universal covering $\tilde{X}$ with $H \times \Delta$ and note that it is Γ-invariant in $\tilde{X}^* \cong \mathbb{C} \times \Delta$. In order to describe the Γ-action, observe that every γ in Γ is $\mathbb{C}$-equivariant, therefore there exists a holomorphic map $F_\gamma : \Delta \to \mathbb{C}$ such that

$$\gamma \cdot (z,w) = (z + F_\gamma(w), \gamma \cdot w)\,,$$

for all $(z,w) \in \mathbb{C} \times \Delta$. Since $\gamma(H \times \Delta) = H \times \Delta$, it follows that $\operatorname{Im} F_\gamma \equiv 0$ and consequently the holomorphic function F_γ is a real constant. Thus the Γ-action on $\mathbb{C} \times \Delta$ is given by $\gamma \cdot (z,w) = (z + \Psi(\gamma), \gamma \cdot w)$, where the group homomorphism $\Psi : \Gamma \to \mathbb{R}$ is defined by $\gamma \to F_\gamma$.

Finally note that $\operatorname{Ker} \Psi \neq \Gamma$. Otherwise one has $X = (H \times \Delta)/\Gamma = H \times (\Delta/\Gamma) = H \times S$. Since X is Stein and S is compact, this gives a contradiction. Thus X is $\mathbb{R}$-equivariantly biholomorphic to a model of type CH and X^* is Stein by (i) of Proposition 4.1. $\square$

Remark 4.7. Note that two models of type CH, one of the form $(H \times \Delta)/\Gamma$, with H of finite width, and one of the form $(H' \times \Delta)/\Gamma'$, with H' of infinite width, cannot be biholomorphic. Let Γ act on $H \times \Delta$ by $\gamma \cdot (z,w) = (z + \Psi(\gamma), \gamma(w))$ and let Γ' act on $H' \times \Delta$ by $\gamma' \cdot (z,w) = (z + \Psi'(\gamma'), \gamma'(w))$, where $\Psi : \Gamma \to \mathbb{R}$ and $\Psi' : \Gamma' \to \mathbb{R}$ are non trivial homomorphisms. Recall that, since the Riemann surfaces Δ/Γ and Δ/Γ' are compact hyperbolic, it follows that the elements of Γ and Γ' are all hyperbolic, i.e. they have two fixed point on the boundary of Δ (see [FaKr], Cor. 2, p. 216).

In particular every element of Γ which does not belong to $\operatorname{Ker} \Psi$ has 4 fixed points on the boundary of the universal covering Δ^2 of X, while an element of Γ' has either infinite or 2 fixed points. Here we are identifying $H \times \Delta$ and $H' \times \Delta$ with Δ^2 and using the fact that every element of $Aut(\Delta^2) \cong \mathbb{Z}_2 \ltimes (PSL(2,\mathbb{R}))^2$ extends bijectively on the closure of Δ^2 in $\mathbb{C}^2$.

Assume that there exists a biholomorphism $F : (H \times \Delta)/\Gamma \to (H' \times \Delta)/\Gamma'$. Then F lifts to a biholomorphism of the universal coverings $\tilde{F} : \Delta^2 \to \Delta^2$ which extends bijectively on the closure of Δ^2. Since $\tilde{F}\circ\gamma\circ\tilde{F}^{-1}\in\Gamma'$ for all $\gamma\in\Gamma$, this gives a contradiction. $\square$

5. MODELS WITH NON COMPACT BASE

Here we consider the models with base a non compact Riemann surface. Let us start with the hyperbolic case.

Type NCH Let S be a non compact hyperbolic Riemann surface. A model of type NCH with base S is given by

$$\{\,(z,p)\in\mathbb{C}\times S : a(p)<\operatorname{Im} z<-b(p)\,\}\,,$$

where a and b are subharmonic, continuous functions on S such that $a+b<0$ and $\max\{a(p),b(p)\}>-\infty$ for all $p\in S$.

Type NCNH A model of type NCNH with base $S=\mathbb{C}$ or $S=\mathbb{C}^*$ is given by

$$\{\,(z,p)\in\mathbb{C}\times S\ :\ a(p)<\operatorname{Im} z\,\}\quad\text{or}\quad\{\,(z,p)\in\mathbb{C}\times S\ :\ \operatorname{Im} z<-b(p)\,\}\,,$$

with a, b subharmonic, non-harmonic, real valued, continuous functions on S.

On each manifold as above let $\mathbb{R}$ act by translations on the first factor.

Proposition 5.1. *Let X be a model of type* NCH *or* NCNH *with base S. Then*

(i) *the universal globalization of X is $\mathbb{C}\times S$, which is Stein,*

(ii) *X is a taut, Stein manifold with a proper $\mathbb{R}$-action.*

Proof. (i) Note that X is orbit-connected in $\mathbb{C}\times S$. Then Lemma 1.5 in [CIT] implies that $\mathbb{C}\times S$ is the (Stein) universal globalization of X.

(ii) Since X is an $\mathbb{R}$-invariant submanifold in $\mathbb{C}\times S$, the $\mathbb{R}$-action on X is proper. Moreover, X is given as the sublevel set of plurisubharmonic functions defined on the product $\mathbb{C}\times S$, which is Stein. Thus X is Stein.

Finally we show that X is taut. For X a model of type NCH consider the projection $\Pi|_X : X\to S$, $(z,p)\to p$, onto the second factor. By Lemma 2.4 it is sufficient to prove that for every p in S there exists a neighborhood U of p in S such that $(\Pi|_X)^{-1}(U)$ is taut. Since $\max\{a(p),b(p)\}>-\infty$ we may assume that, e.g. $a(p)>-\infty$. By continuity $a>M$ on a neighbourhood U of p, for some real constant M. Then $(\Pi|_X)^{-1}(U)$ is contained in $H\times S$, with $H=\{z\in\mathbb{C}\ :\ M<\operatorname{Im} z\}$. Moreover the inverse image $(\Pi|_X)^{-1}(U)$ it is defined as a sublevel set of continuous plurisubharmonic functions, therefore it is taut by Corollary 2.3.

Assume now that X is a model of type NCNH. Note that if $S=\mathbb{C}^*$, then the universal covering $\tilde{X}$ of X is contained in $\mathbb{C}^2$ and it is also of type NCNH.

Moreover, by Proposition 2.5 the manifold X is taut if and only if so is $\tilde{X}$. Thus we may assume that $S = \mathbb{C}$. Since a domain of the form $\{ (z, w) \in \mathbb{C}^2 : \operatorname{Im} z < -b(w) \}$ is biholomorphic to Ω_b via the biholomorphism of $\mathbb{C}^2$ defined by $(z, w) \to (-z, w)$, all such models are taut by Theorem 3.4. $\square$

Remark 5.2. Let F be a $\mathbb{R}$-equivariant biholomorphism between two models of type NCH defined by $\{ (z,p) \in \mathbb{C} \times S \ : \ a(p) < \operatorname{Im} z < -b(p) \}$ and $\{ (z,p') \in \mathbb{C} \times S' \ : \ a'(p') < \operatorname{Im} z < -b'(p') \}$. Then $F(z,p) = (z + f(p), \varphi(p))$ where $f : S \to \mathbb{C}$ is holomorphic and $\varphi : S \to S'$ is a biholomorphism such that $a = (a' \circ \varphi - \operatorname{Im} f)$ and $b = (b' \circ \varphi + \operatorname{Im} f)$. An analogous statement holds for models of type NCNH. $\square$

Proposition 5.3. *Let X be a 2-dimensional, taut, Stein manifold with a proper $\mathbb{R}$-action and assume that the Riemann surface $S := X^*/\mathbb{C}$ is non compact. Then X^* is $\mathbb{C}$-equivariantly biholomorphic to $\mathbb{C} \times S$, which is Stein. Moreover, depending on hyperbolicity of S the manifold X is $\mathbb{R}$-equivariantly biholomorphic to a model of type either* NCH *or* NCNH.

Proof. Since S is non compact by assumption, the principal $\mathbb{C}$-bundle X^* is trivial (cf. Thm. 2.8), implying the first statement. Regard X as $\{ (z,p) \in \mathbb{C} \times S \ : \ \alpha(z,p) < 0 < -\beta(z,p) \}$ and define $a(p) := \alpha(0,p)$ and $b(p) := \beta(0,p)$. Since from (ii) of Lemma 2.7 it follows that $\alpha(z,p) = -\operatorname{Im} z + \alpha(0,p)$ and $\beta(z,p) = \operatorname{Im} z + \beta(0,p)$, one has

$$X = \{ (z,p) \in \mathbb{C} \times S \ : \ a(p) < \operatorname{Im} z < -b(p) \} .$$

Moreover the same lemma implies that a and b are subharmonic, continuous functions, $a + b < 0$ and $\max\{a(p), b(p)\} > -\infty$ for all $p \in S$. This concludes the case when S is hyperbolic.

For $S = \mathbb{C}$ or $S = \mathbb{C}^*$ we first note that $a + b$ is constant, being a subharmonic, negative function on S. We claim that $a + b \equiv -\infty$. Assume by contradiction that $a + b = -C$ for some positive C. Then $a = -b - C$ is harmonic and $X = \{ (z,p) \in \mathbb{C} \times S : a(p) < \operatorname{Im} z < a(p) + C \}$. In the case when $S = \mathbb{C}$, there exists a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that $\operatorname{Im} f = a$. Then the map $\zeta \to (f(\zeta) + iC/2, \zeta)$ is a non constant holomorphic map from $\mathbb{C}$ into X. Since X is taut, this gives a contradiction. If $S = \mathbb{C}^*$, one can show that $a + b \equiv -\infty$ by applying the analogous argument to the universal covering of X, which is taut by Proposition 2.5.

Thus $a + b \equiv -\infty$ and since the sets $\{a = -\infty\}$ and $\{b = -\infty\}$ have zero measure, either a or b are constantly equal to $-\infty$. Assume, e.g. that $b \equiv -\infty$. Since X is taut, a is necessarily real valued and the above argument also proves that a can not be harmonic. Thus X is $\mathbb{R}$-equivariantly biholomorphic to a model of type NCNH. $\square$

Corollary 5.4. *Let S be a non compact Riemann surface and consider the subdomain of $\mathbb{C} \times S$ defined by*

$$\Omega := \{ (z,p) \in \mathbb{C} \times S : a(p) < \operatorname{Im} z < -b(p) \} ,$$

where $a, b : S \to \{-\infty\} \cup \mathbb{R}$ are upper semicontinuous functions. Then Ω is taut if and only if it is a model of type NCH *or* NCNH.

Proof. First note that if Ω is taut, then it is Stein. For this consider the universal covering $Id \times \pi : \mathbb{C} \times \tilde{S} \to \mathbb{C} \times S$, where $\tilde{S} = \mathbb{C}$ or $\tilde{S} = \Delta$. Then Proposition 2.5 applies to show that the inverse image $(Id \times \pi)^{-1}(\Omega)$ is a taut domain of $\mathbb{C}^2$. Thus it is Stein by Thm. 5.4.1 in [Kob] and consequently it is locally Stein in $\mathbb{C} \times \tilde{S}$. It follows that Ω is locally Stein in $\mathbb{C} \times S$, which is Stein. Thus Ω is Stein by [DoGr].

Then an analogous argument as in the above proof applies to prove that Ω is a model of type NCH or NCNH. □

6. Homotopy of the models

Let us summarize the main results of the previous sections as follows (see Prop. 4.1, 4.6, 5.1 and 5.3).

Theorem 6.1. *Every model of type* CH, NCH *or* NCNH *is taut and Stein. Moreover a 2-dimensional, taut, Stein manifold with a proper $\mathbb{R}$-action is $\mathbb{R}$-equivariantly biholomorphic to one of them. In particular its universal globalization is Stein.*

Here we show that in most cases, but not all of them, the type of a 2-dimensional, taut, Stein manifold with a proper $\mathbb{R}$-action is uniquely determined by its topology.

Proposition 6.2. *Let X be a 2-dimensional, taut, Stein manifold with a proper $\mathbb{R}$-action Then X is homotopically equivalent to S.*

Proof. We first find a smooth global section of the restriction $\Pi|_X : X \to S$ of Π to X. In a given smooth, trivialization $\mathbb{C} \times S$ of X^* one has $X = \{(z,p) \in \mathbb{C} \times S \,:\, a(p) < \operatorname{Im} z < -b(p)\}$, with a and b continuous functions on S (maybe no longer subharmonic) with values in $\{-\infty\} \cup \mathbb{R}$. Moreover, by continuity of a and b one can choose a locally finite covering $\{U_j\}$ of S and real constants M_j such that $a < M_j < -b$ on each U_j. Thus the constant functions iM_j can be regarded as smooth local sections of $\Pi|_X$. Choose a smooth partition of unity $\{\psi_j\}$ subordinated to $\{U_j\}$. Then $\theta = \sum_j iM_j\psi_j$ defines a smooth global section of $\Pi|_X$, since $a(p) < \operatorname{Im} \theta < -b(p)$ on S.

Finally note that the map $X\times[0,1]\to X$ defined by $((z,p),t)\to(z+t(\theta(p)-z),\, p)$ is a homotopy equivalence, showing that S is a strong deformation retract of X. $\square$

Corollary 6.3. *Let X be a 2-dimensional, taut Stein manifold with a proper $\mathbb{R}$-action. Then*

(i) *X is $\mathbb{R}$-equivariantly biholomorphic to a model of type* CH *if and only if $H^2(X,\mathbb{Z})\neq 0$,*

(ii) *if $H^2(X,\mathbb{Z})=0$ and $\pi_1(X)$ is neither trivial, nor isomorphic to $\mathbb{Z}$, then X is $\mathbb{R}$-equivariantly biholomorphic to a model of type* NCH*,*

(iii) *if $\pi_1(X)=0$ or $\pi_1(X)=\mathbb{Z}$, then X is $\mathbb{R}$-equivariantly biholomorphic to a model of type* NCH (*type* NCNH) *if and only if it* (*does not*) *admits a non constant, bounded holomorphic $\mathbb{R}$-invariant function.*

Proof. (i) and (ii) are direct consequences of the above proposition. For (iii) note that an $\mathbb{R}$-invariant holomorphic function on X pushes down to a holomorphic function on S. Moreover, the assumption on the fundamental group implies that S is biholomorphic to one of the following domains $\mathbb{C}$, $\mathbb{C}^*$, Δ, Δ^* or an annulus. This implies the statement. $\square$

Remark 6.4. Let X be a taut, Stein surface such that either $\pi_1(X)=0$ or $\pi_1(X)=\mathbb{Z}$. Then, for different proper $\mathbb{R}$-actions, the manifold X may be $\mathbb{R}$-equivariantly biholomorphic to models of different types.

As an example consider the unbounded realization of the unit ball of $\mathbb{C}^2$ given by $X=\{\,(u,v)\in\mathbb{C}^2\ :\ |v|^2<\operatorname{Im}u\}$ and the two different $\mathbb{R}$-actions on X defined by

$$t\diamond(u,v):=(u+t,v),\qquad t*(u,v):=(u-2tv+it^2,v-it)\,.$$

Such actions appear in [FaIa] as normal forms of parabolic elements in the automorphism group of X. It is clear that the globalization with respect to the first action is $\mathbb{C}^2$ and its $\mathbb{C}$-quotient is $\mathbb{C}$.

Note that the second $\mathbb{R}$-action extends to a $\mathbb{C}$-action on $\mathbb{C}^2$ and a simple computation shows that $\mathbb{C}*X=\{\,(u,v)\in\mathbb{C}^2\ :\ \operatorname{Im}u>(\operatorname{Im}v)^2-(\mathrm{Re\,v})^2\,\}$. Moreover, one checks that X is orbit-connected in $\mathbb{C}*X$. Then Lemma 1.5 in [CIT] implies that $\mathbb{C}*X$ is the universal globalization with respect to the local $\mathbb{C}$-action on X induced by the second $\mathbb{R}$-action. Let $\mathbb{H}=\{\,z\in\mathbb{C}\ :\ 0<\mathrm{Im}z\}$. One has a $\mathbb{C}$-equivariant biholomorphism

$$\Psi:\mathbb{C}\times\mathbb{H}\to\mathbb{C}*X\,,\qquad(\lambda,u)\to\lambda*(u,0)=(u+i\lambda^2,-i\lambda)\,.$$

Therefore the $\mathbb{C}$-quotient of $\mathbb{C}*X$ is biholomorphic to $\mathbb{H}$. This completes the example in the case when the fundamental group is trivial.

A similar example with fundamental group isomorphic to $\mathbb{Z}$ is as follows. Since Ψ is a biholomorphism, $X' := \Psi^{-1}(X) = \{\, (u,v) \in \mathbb{C}^2 \;:\; \operatorname{Im} v > 2(\operatorname{Im} u)^2\}$ is also a model for the unit ball of $\mathbb{C}^2$. On this model the above actions look like

$$t \diamond (u,v) := (u, v+t), \qquad t * (u,v) := (u+t, v)\,.$$

Then the $\mathbb{Z}$-action on X' defined by $n \cdot (u,v) := (u+n, v+n)$ commutes with both the $\mathbb{R}$-actions. Thus such $\mathbb{R}$-actions push down to proper $\mathbb{R}$-actions on the quotient $X'/\mathbb{Z}$, whose fundamental group is $\mathbb{Z}$. Observe that $X'/\mathbb{Z}$ is taut by Prop. 2.5 and it is Stein by [FaIa]

Finally note that the restrictions $X' \to \mathbb{C}$ and $X' \to \mathbb{H}$ to X' of the projections of the associated holomorphic principal $\mathbb{C}$-bundles are $\mathbb{Z}$ equivariant. Thus they factorize to the restrictions $X'/\mathbb{Z} \to \mathbb{C}^*$ and $X'/\mathbb{Z} \to \Delta^* = \mathbb{H}/\mathbb{Z}$ to $X'/\mathbb{Z}$ of the projections of the holomorphic principal $\mathbb{C}$-bundles associated to the pushed down $\mathbb{R}$-actions on $X'/\mathbb{Z}$. Since the bases of this bundles are $\mathbb{C}^*$ and Δ^*, this shows that also in this case the type of X depends on the chosen $\mathbb{R}$-action. □

7. taut Hartogs domains

As an application of the given classification, we give necessary and sufficient conditions for tautness of (non-complete) Hartogs domains over a non compact Riemann surface S. A complete Hartogs domain over S is given by

$$\{\, (u,p) \in \mathbb{C} \times S \;:\; |u| < e^{-b(p)}\,\}\,,$$

with $b : S \to \mathbb{R} \cup \{-\infty\}$ an upper semicontinuous function. A non-complete Hartogs domain over S is given by

$$\{\, (u,p) \in \mathbb{C} \times S \;:\; e^{a(p)} < |u| < e^{-b(p)}\,\}\,,$$

where $a, b : S \to \mathbb{R} \cup \{-\infty\}$ are upper semicontinuous functions with $a + b < 0$.

We wish to determine under which conditions on a and b such domains are taut. A result of Thai-Duc ([ThDu]), which applies in a more general context, implies that a complete Hartogs domain is taut if and only if S is hyperbolic and b is a real valued, subharmonic, continuous function. The following proposition gives a characterization of non-complete Hartogs domains (cf. [Par] for related results).

Proposition 7.1. *Let $\Omega = \{(u,p) \in \mathbb{C} \times S \;:\; e^{a(p)} < |u| < e^{-b(p)}\}$ be a non-complete Hartogs domain over a non compact Riemann surface. Then Ω is taut if and only if either*

(i) *the Riemann surface* S *is hyperbolic, the functions* a, b *are continuos subharmonic and* $max(a(p), b(p)) > -\infty$ *for every* $p \in S$, *or*

(ii) *the Riemann surface* S *is not hyperbolic,* $b \equiv -\infty$ (*respectively* $a \equiv -\infty$) *and* a (*respectively* b) *is a subharmonic, non-harmonic, real-valued, continuous function.*

Proof. Consider the covering map $F : \mathbb{C} \times S \to \mathbb{C}^* \times S$ given by $(z, p) \to (e^{-iz}, p)$. Since the restriction of F to $F^{-1}(\Omega)$ is a covering, from proposition 2.5 it follows that $F^{-1}(\Omega)$ is taut if and only if so is Ω. Note that Ω is invariant under the S^1-action defined by $e^{i\theta} \cdot (u, p) := (e^{i\theta}u, p)$. As a consequence $\mathbb{R}$ acts properly on $F^{-1}(\Omega)$ by $t \cdot (z, p) := (z + t, p)$. Then Corollary 5.4 applies to show that $F^{-1}(\Omega)$ is taut if and only if it is a model of type NCH or NCNH, depending on hyperbolicity of S. This implies the statement. □

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