# ORBIT STRUCTURE OF A DISTINGUISHED INVARIANT, STEIN DOMAIN IN THE COMPLEXIFICATION OF A HERMITIAN SYMMETRIC SPACE

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Abstract. We carry out a detailed study of $\Xi^+$, a distinguished $G$-invariant Stein domain in the complexification of an irreducible Hermitian symmetric space $G/K$. The domain $\Xi^+$ contains the crown domain $\Xi$ and is naturally diffeomorphic to the anti-holomorphic tangent bundle of $G/K$. The unipotent parametrization of $\Xi^+$ introduced in [KrOp08] and [Kro08] suggests that $\Xi^+$ also admits the structure of a twisted bundle $G \times_K \mathcal{N}^+$, with fiber a nilpotent cone $\mathcal{N}^+$. Here we give a complete proof of this fact and use it to describe the $G$-orbit structure of $\Xi^+$ via the $K$-orbit structure of $\mathcal{N}^+$. In the tube case, we also single out a Stein, $G$-invariant domain contained in $\Xi^+ \setminus \Xi$ which is relevant in the classification of envelopes of holomorphy of invariant subdomains of $\Xi^+$.

## 1. Introduction

Let $G/K$ be a non-compact, irreducible, Riemannian symmetric space. Its Lie group complexification $G^{\mathbb{C}}/K^{\mathbb{C}}$ is a Stein manifold and left translations by elements of $G$ are holomorphic transformations of $G^{\mathbb{C}}/K^{\mathbb{C}}$. In [AkGi90], Akhiezer and Gindikin introduced the crown domain $\Xi$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$, with the aim of determining a complex $G$-manifold whose analytic properties would reflect the harmonic analysis of $G/K$ and the representation theory of $G$. Since then its complex analytic properties have been extensively studied by several authors.

In the Hermitian case, Krötz and Opdam recently introduced two Stein $G$-invariant domains $\Xi^+$ and $\Xi^-$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$, with $\Xi^+ \cap \Xi^- = \Xi$, which are maximal with respect to properness of the $G$-action on $G^{\mathbb{C}}/K^{\mathbb{C}}$. The relevance of $\Xi$ and of the domains $\Xi^+$ and $\Xi^-$ for the representation theory of $G$ was underlined in Theorem 1.1 in [Kro08]. Here we carry out a detailed analysis of the $G$-orbit structure of the domain $\Xi^+$. Since $\Xi^+$ and $\Xi^-$ are $G$-equivariantly anti-biholomorphic, such analysis applies to $\Xi^-$ as well.

Let $G/K$ be an irreducible Hermitian symmetric space and let $G^{\mathbb{C}}/Q$ be its compact dual symmetric space, which is denoted by $\overline{G^{\mathbb{C}}/Q}$ when endowed with the opposite complex structure. The complexification $G^{\mathbb{C}}/K^{\mathbb{C}}$ admits an equivariant holomorphic embedding as the open dense $G^{\mathbb{C}}$-orbit

$$G^{\mathbb{C}}/K^{\mathbb{C}} \cong G^{\mathbb{C}} \cdot x_0 \subset G^{\mathbb{C}}/Q \times \overline{G^{\mathbb{C}}/Q}$$

through $x_0 := (eQ, eQ) \in G^{\mathbb{C}}/Q \times \overline{G^{\mathbb{C}}/Q}$, with the $G^{\mathbb{C}}$-action defined by

$$g \cdot (x, y) := (g \cdot x, \sigma(g) \cdot y).$$

Here $\sigma$ denotes the conjugation of $G^{\mathbb{C}}$ with respect to $G$. Let $\pi_1 : G^{\mathbb{C}}/Q \times \overline{G^{\mathbb{C}}/Q} \to G^{\mathbb{C}}/Q$ be the projection onto the first factor. The $G$-invariant domain $\Xi^+$ is defined by

$$\Xi^+ := (\pi_1)^{-1}(D) \cap G^{\mathbb{C}} \cdot x_0,$$

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where $D := G \cdot eQ$ is the Borel embedding of $G/K$ in $G^{\mathbb{C}}/Q$. It contains the crown domain as the subset $D \times \overline{D}$ and the $G$-action on $\Xi^+$ is proper.

The above definition leads to a natural $G$-equivariant diffeomorphism between the anti-holomorphic tangent bundle of $G/K$ and $\Xi^+$, via the map

$$G \times_K \mathfrak{p}^{0,1} \to \Xi^+, \qquad [g, Z] \mapsto g \exp Z \cdot x_0 .$$

Also note that $\Xi^+$ and $\Xi^- := \pi_2^{-1}(\overline{D}) \cap G^{\mathbb{C}} \cdot x_0$. are $G$-equivariantly anti-biholomorphic, since the $G$-equivariant anti-biholomorphism

$$G^{\mathbb{C}}/Q \times \overline{G^{\mathbb{C}}/Q} \to G^{\mathbb{C}}/Q \times \overline{G^{\mathbb{C}}/Q}\,, \qquad (x,\, y) \to (y,\, x)\,,$$

maps $\Xi^+$ onto $\Xi^-$.

An alternative construction of the domain $\Xi^+$ was given in [Kro08] and [KrOp08], via its unipotent parametrization. In the notation of Section 2, let $\lambda_1, \ldots, \lambda_r$ be long strongly orthogonal real restricted roots, and let $E_j \in \mathfrak{g}^{\lambda_j}$, for $j = 1, \ldots, r$, be root vectors normalized as in (5) and Definition 2.2. Consider the closed hyperoctant

$$\Lambda_r^{\llcorner} := \mathrm{span}_{\mathbb{R}_{\geq 0}}\{E_1, \ldots, E_r\}$$

and the subcone $\mathcal{N}^+ := \mathrm{Ad}_K \Lambda_r^{\llcorner}$ of the nilpotent cone of $\mathfrak{g}$. Then

$$\Xi^+ = G \exp i \bigoplus_j (-1, \infty) E_j \cdot x_0 = G \exp i \Lambda_r^{\llcorner} \cdot x_0 .$$

It was also suggested that the map

$$\psi \colon G \times_K \mathcal{N}^+ \to \Xi^+, \quad [g, X] \mapsto g \exp iX \cdot x_0$$

is a $G$-equivariant homeomorphism.

The first goal in this paper is to give a complete and selfcontained proof of this fact. The main difficulty is to show that the map $\psi$ is open. This is not a priori obvious because at every point in the slice $\exp i\Lambda_r^{\llcorner} \cdot x_0 \subset \Xi^+$, lying on a singular $G$-orbit, the tangent spaces to the orbit and to the slice itself do not span the whole tangent space to $\Xi^+$.

Consider the $K$-invariant fiber $P := \exp \mathfrak{p}^{0,1} \cdot x_0$ in the domain $\Xi^+ \cong G \times_K \mathfrak{p}^{0,1}$. We first use a topological argument (Lemma 5.2) to show that our goal is equivalent to show that the projection

$$\Lambda_r^{\llcorner} \to P/K, \quad X \mapsto G \exp iX \cdot x_0 \cap P\,,$$

is proper. Next, we check that such a projection is proper by using a novel decomposition inside $G^{\mathbb{C}}$ relating a unipotent element $\exp iX$, with $X \in \Lambda_r^{\llcorner}$, to an element in $\exp Z\, K^{\mathbb{C}}$, with $Z \in \mathfrak{p}^{0,1}$, lying on the same $G$-orbit (see Lemma 5.5 and Thm. 5.7). Possibly, a similar argument leads to a characterization of smooth twisted bundles in the context of proper $G$-actions on differentiable manifolds considered by R. S. Palais and C.-L. Terng in [PaTe87].

In view of the bundle structure defined by $\psi$, the $G$-orbit structure of $\Xi^+$ is completely determined by the $\mathrm{Ad}_K$-orbit structure of the nilpotent cone $\mathcal{N}^+$. In Section 6 we show that a fundamental domain for the action of the Weyl group $W_K(\Lambda_r^{\llcorner})$ on the hyperoctant $\Lambda_r^{\llcorner}$ is a perfect slice for the $K$-action on the cone $\mathcal{N}^+$ and hence it determines a perfect slice for the $G$-action on $\Xi^+$. Moreover, one has a one-to-one correspondence between the orbit strata of the $W_K(\Lambda_r^{\llcorner})$-action on the closed hyperoctant $\Lambda_r^{\llcorner}$ and the orbit strata of the $G$-action on $\Xi^+$.

The second goal of the paper is to describe some $G$-invariant subdomains of $\Xi^+$ which are relevant for a classification of envelopes of holomorphy of $G$-invariant subdomains of $\Xi^+$. It was observed in [GeIa08] that in the rank-one case, beside the crown $\Xi$, the domain $\Xi^+$ contains another distinguished $G$-invariant subdomain with the peculiarity that its boundary contains no principal $G$-orbits of $G^{\mathbb{C}}/K^{\mathbb{C}}$ (i.e. closed orbits of maximal dimension).

In the tube case $SL(2,\mathbb{R})/SO(2,\mathbb{R})$, such a subdomain $S^+$ arises from the compactly causal structure of a symmetric $G$-orbit in the semisimple boundary of $\Xi$ and it is Stein. It turns out that every Stein, invariant, proper subdomain of $\Xi^+$ is either contained in $\Xi$ or in $S^+$. In the non-tube case $SU(n,1)/U(n)$, such a subdomain $\Omega^+$ is not Stein and contains no invariant Stein subdomains. It follows that every Stein, invariant, proper subdomain of $\Xi^+$ is contained in $\Xi$.

Here we prove that the domains $S^+$ and $\Omega^+$ have higher rank analogues inside $\Xi^+$. In a forthcoming paper we will show that, like in the rank-one case, every Stein invariant proper subdomain of $\Xi^+$ is contained either in $\Xi$ or in $S^+$, in the tube case, while it is contained in $\Xi$ in the non-tube case. We will also characterize the envelopes of holomorphy of $G$-invariant domains in $\Xi^+$.

The paper is organized as follows. In Section 2 we set up the notation and collect some basic facts about Hermitian symmetric spaces. In Section 3 we recall the definition of the domain $\Xi^+$ and of its unipotent model. In Section 4 we define the Weyl group $W_K(\Lambda_r^\llcorner)$ of the cone $\Lambda_r^\llcorner$ and relate it to the Weyl group $W_K(\mathfrak{a})$. In Section 5 we prove that the map

$$\psi\colon G\times_K\mathcal{N}^+\to\Xi^+,\quad [g,X]\mapsto g\exp iX\cdot x_0$$

is a $G$-equivariant homeomorphism. In Section 6 we give an alternative proof of the above fact for the symmetric spaces $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ and $Sp(2,\mathbb{R})/U(2)$, by using global $G$-invariant functions on $G^{\mathbb{C}}/K^{\mathbb{C}}$. In Section 7 we study the $G$-orbit structure of $\Xi^+$ by means of the $\mathrm{Ad}_K$-orbit structure of $\Lambda_r^\llcorner$. Finally, in Section 8 we determine some distinguished $G$-invariant domains in $\Xi^+$.

## 2. Preliminaries

Let $G/K$ be an irreducible Hermitian symmetric space of the non-compact type. We may assume $G$ to be a connected, non-compact, real simple Lie group contained in its simple, simply connected universal complexification $G^{\mathbb{C}}$, and $K$ to be a maximal compact subgroup of $G$. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of $G$ and $K$ respectively. Denote by $\theta$ both the Cartan involution of $G$ with respect to $K$ and the derived involution of $\mathfrak{g}$. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the corresponding Cartan decomposition. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$. The *rank* of $G/K$ is by definition $r=\dim\mathfrak{a}$. The adjoint action of $\mathfrak{a}$ decomposes $\mathfrak{g}$ as

$$\mathfrak{g}=\mathfrak{a}\oplus Z_{\mathfrak{k}}(\mathfrak{a})\oplus\bigoplus_{\alpha\in\Delta(\mathfrak{g},\mathfrak{a})}\mathfrak{g}^\alpha,$$

where $Z_{\mathfrak{k}}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, the joint eigenspace $\mathfrak{g}^\alpha=\{X\in\mathfrak{g}\mid[H,X]=\alpha(H)X,\ H\in\mathfrak{a}\}$ is the $\alpha$-restricted root space and $\Delta(\mathfrak{g},\mathfrak{a})$ consists of those $\alpha\in\mathfrak{a}^*$ for which $\mathfrak{g}^\alpha\neq\{0\}$. A set of simple roots $\Pi_{\mathfrak{a}}$ in $\Delta(\mathfrak{g},\mathfrak{a})$ uniquely determines a set of positive restricted roots $\Delta^+(\mathfrak{g},\mathfrak{a})$ and an Iwasawa decomposition of $\mathfrak{g}$

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n},\qquad\text{where}\quad\mathfrak{n}=\bigoplus_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{a})}\mathfrak{g}^\alpha.$$

The restricted root system of a Lie algebra $\mathfrak{g}$ of Hermitian type is either of type $C_r$ (if $G/K$ is of tube type) or of type $BC_r$ (if $G/K$ is not of tube type) (cf. [Moo64]), i.e. there exists a basis $\{e_1,\dots,e_r\}$ of $\mathfrak{a}^*$ for which

$$\Delta(\mathfrak{g},\mathfrak{a})=\{\pm2e_j,\ 1\le j\le r,\ \pm e_j\pm e_k,\ 1\le j\neq k\le r\},\qquad\text{for type } C_r,$$

$$\Delta(\mathfrak{g},\mathfrak{a})=\{\pm e_j,\ \pm2e_j,\ 1\le j\le r,\ \pm e_j\pm e_k,\ 1\le j\neq k\le r\},\qquad\text{for type } BC_r.$$

Since $\mathfrak{g}$ admits a compact Cartan subalgebra $\mathfrak{t}\subset\mathfrak{k}\subset\mathfrak{g}$, there exists a set of $r$ long strongly orthogonal restricted roots $\{\lambda_1,\dots,\lambda_r\}$ (such that $\lambda_j\pm\lambda_k\notin\Delta(\mathfrak{g},\mathfrak{a})$, for

$j \neq k$), which are restrictions of *real* roots with respect to a maximally split $\theta$-stable Cartan subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ extending $\mathfrak{a}$. Choosing as simple roots

$$\Pi_{\mathfrak{a}} = \{e_1 - e_2, \dots, e_{r-1} - e_r, 2e_r\}, \quad \text{for type } C_r, \tag{1}$$

$$\Pi_{\mathfrak{a}} = \{e_1 - e_2, \dots, e_{r-1} - e_r, e_r\}, \quad \text{for type } BC_r. \tag{2}$$

one has

$$\lambda_1 = 2e_2, \dots, \lambda_r = 2e_r. \tag{3}$$

In both cases, the Weyl group $W_K(\mathfrak{a}) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is isomorphic to the group of signed permutations of $\{e_1, \dots, e_r\}$, and therefore of $\{\lambda_1, \dots, \lambda_r\}$. Denote by $W_K(\mathfrak{a})^+$ the subgroup of $W_K(\mathfrak{a})$ isomorphic to the the group of ordinary permutations of $\{e_1, \dots, e_r\}$ (it is the subgroup generated by the reflections in the first $r-1$ simple restricted roots). Let $\{A_1, \dots, A_r\}$ be the dual basis of $\{e_1, \dots, e_r\}$. The action of $W_K(\mathfrak{a})$ and of $W_K(\mathfrak{a})^+$ on $\mathfrak{a}$ is by signed permutations and by ordinary permutations of $\{A_1, \dots, A_r\}$, respectively.

For $j = 1, \dots, r$, choose $E_j \in \mathfrak{g}^{\lambda_j}$ such that the $\mathfrak{sl}(2)$-triple

$$\{E_j,\ \theta E_j,\ A_j := [\theta E_j, E_j]\} \tag{4}$$

is normalized as follows

$$[A_j, E_j] = 2E_j, \quad [A_j, \theta E_j] = -2\theta E_j. \tag{5}$$

Since the roots $\{\lambda_1, \dots, \lambda_r\}$ are strongly orthogonal and $\mathfrak{g}$ admits a compact Cartan subalgebra, the vectors $\{A_1, \dots, A_r\}$ form an orthogonal basis of $\mathfrak{a}$ (with respect to the restriction of the Killing form) and

$$[E_j, E_k] = [E_j, \theta E_k] = 0, \quad [A_j, E_k] = \lambda_k(A_j) E_k = 0, \quad \text{for } j \neq k. \tag{6}$$

In other words, the above $\mathfrak{sl}(2)$-triples commute with each other.

Observe that relations (5) and (4) determine the vectors $E_j$ only up to sign, while on the other hand the vectors $A_j$ are independent of those signs. Next, we are going to show that, *once a complex structure $J_0$ of $G/K$ is fixed, there is a unique choice of the vectors $E_j$, which is compatible with* $J_0$ (see Definition 2.2 below).

Identify $\mathfrak{p}$ with the tangent space to $G/K$ at the base point $eK$. An invariant complex structure on $G/K$ is uniquely determined by its restriction to $\mathfrak{p}$, and it is given by $J_0 := \mathrm{ad}_{Z_0}|_{\mathfrak{p}}$, where $Z_0$ is an element in the one-dimensional center of $\mathfrak{k}$. Once a complex structure is fixed, one can show that $J_0$ and $-J_0$ are the only invariant complex structures on $G/K$.

Let $\mathfrak{t} \subset \mathfrak{k}$ be a compact Cartan subalgebra of $\mathfrak{g}$ and let $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ denote the root system of $\mathfrak{g}^{\mathbb{C}}$ under the adjoint action by $\mathfrak{t}^{\mathbb{C}}$. A root $\alpha \in \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ is said to be *compact* if the root space $\mathfrak{g}^{\alpha}$ lies in $\mathfrak{k}^{\mathbb{C}}$ and *non-compact* if it lies in $\mathfrak{p}^{\mathbb{C}}$. There is a choice of positive roots in $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ for which the positive non-compact roots satisfy $\alpha(-iZ_0) = 1$ (see [KoWo65]).
Under the above choice, the holomorphic tangent space

$$\mathfrak{p}^{1,0} = \{W \in \mathfrak{p}^{\mathbb{C}} \mid J_0(W) = iW\}$$

is spanned by the root spaces of the non-compact positive roots.

Now, to the vectors $\{E_1, \dots, E_r\}$ one can associate a compact Cartan subalgebra of $\mathfrak{g}$

$$\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{c},$$

where $\mathfrak{c} := \mathrm{span}_{\mathbb{R}}\{T_1, \dots, T_r\}$, with $T_j := E_j + \theta E_j$, and $\mathfrak{s}$ is a Cartan subalgebra of $Z_{\mathfrak{k}}(\mathfrak{a})$, and vectors in $\mathfrak{p}^{\mathbb{C}}$

$$W_j := \frac{1}{2}\left((E_j - \theta E_j) - iA_j\right), \qquad W_{-j} = \overline{W}_j\,. \tag{7}$$

**Lemma 2.1.**
(i) *For* $j = 1, \dots, r$ *the triples* $\{W_j, W_{-j}, T_j\}$ *generate* $r$ *commuting complex Lie subalgebras of* $\mathfrak{g}^{\mathbb{C}}$*, isomorphic to* $\mathfrak{sl}(2, \mathbb{C})$*.*
(ii) *For* $j = 1, \dots, r$*, the vectors* $W_j$ *span the root spaces* $\mathfrak{g}^{\widetilde{\lambda}_j}$*, where* $\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_r$ *are the strongly orthogonal, non-compact, imaginary roots in* $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$*, defined by*

$$\begin{aligned} &\widetilde{\lambda}_j(T_j) = 2i \\ &\widetilde{\lambda}_j(T_k) = 0 \qquad \text{if } j \neq k \\ &\widetilde{\lambda}_j|_{\mathfrak{s}} = 0\,. \end{aligned}$$

*Proof.* (i) One can easily verify that for $j = 1, \dots, r$

$$[T_j, W_j] = 2iW_j, \quad [T_j, W_{-j}] = -2iW_{-j}, \quad [W_j, W_{-j}] = -iT_j, \tag{8}$$

and for $j \neq k$

$$[W_j, W_k] = [W_j, W_{-k}] = 0, \qquad [T_j, W_k] = [T_j, W_{-k}] = 0\,. \tag{9}$$

(ii) Since $Z_{\mathfrak{k}}(\mathfrak{a})$ acts trivially on the *one-dimensional* restricted root spaces $\mathfrak{g}^{\pm\lambda_j}$, for every $S \in \mathfrak{s}$ one has

$$[S, W_j] = [S, W_{-j}] = 0, \qquad j = 1, \dots, r.$$

This, together with relations (8) and (9), shows that the $W_{\pm j}$ span the root spaces $\mathfrak{g}^{\pm\widetilde{\lambda}_j}$ for the adjoint action of $\mathfrak{t}^{\mathbb{C}}$ on $\mathfrak{g}^{\mathbb{C}}$. Moreover, the roots $\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_r$ are strongly orthogonal in $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, and they are imaginary (i.e. they assume purely imaginary values on $\mathfrak{t}$). Finally, they are non-compact roots, since the root vectors $W_{\pm j}$ lie in $\mathfrak{p}^{\mathbb{C}}$. □

**Definition 2.2.** *We say that the choice of the vectors* $E_j$ *is compatible with the complex structure* $J_0$ *if one of the following equivalent sets of conditions is fulfilled*
(i) $\widetilde{\lambda}_j(-iZ_0) = 1$,
(ii) $[-iZ_0, W_j] = W_j$,
(iii) $W_j \in \mathfrak{p}^{1,0}$,
*for all* $j = 1, \dots, r$.

**Remark 2.3.** Observe that changing the sign of a vector $E_j$ corresponds to changing the sign of $T_j = E_j + \theta E_j$ and likewise of the root $\widetilde{\lambda}_j$. As a result, the vector

$$\frac{1}{2}\left((-E_j - (-\theta E_j)) - iA_j\right) \in \mathfrak{g}^{-\widetilde{\lambda}_j}$$

no longer lies in $\mathfrak{p}^{1,0}$.

We conclude this discussion by expressing the "compatibility condition" of Definition 2.2 entirely in terms of the $\mathfrak{sl}(2)$-triples $\{E_j, \theta E_j, A_j\}$. Observe that the central element $Z_0 \in Z(\mathfrak{k})$ lies in every compact Cartan subalgebra of $\mathfrak{g}$. In particular it lies in $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{c}$ and can be written as

$$Z_0 = S + \sum_{j=1}^{r} a_j T_j, \qquad \text{for } S \in \mathfrak{s},\ a_j \in \mathbb{R}. \tag{10}$$

**Lemma 2.4.** *The choice of the vectors* $E_j$ *is compatible with the complex structure* $J_0$ *if one of the following equivalent conditions is fulfilled:*
(i) $Z_0 = S + \frac{1}{2}\sum T_j$,

(ii) *the action of* $\mathrm{ad}_{Z_0}$ *on* $\mathfrak{p}$ *satisfies*

$$[Z_0, E_j - \theta E_j] = A_j, \quad [Z_0, A_j] = -(E_j - \theta E_j), \quad \text{for } j = 1, \ldots, r.$$

*In particular, it defines a complex structure on each* $\mathfrak{p}_j := \mathrm{span}_{\mathbb{R}}\{A_j, E_j - \theta E_j\}$.

*Proof.* Let $W_j$, for $j = 1, \ldots, r$, be the vectors defined in (7) and $Z_0$ the vector in (10). One easily verifies that

$$[Z_0, W_j] = a_j(A_j + i(E_j - \theta E_j)).$$

Hence conditions (i) of Definition 2.2 hold, i.e.

$$[Z_0, W_j] = iW_j, \quad j = 1, \ldots, r,$$

if and only if $a_j = \frac{1}{2}$, for all $j$, as wished.

For the equivalence of (i) and (ii), observe that the algebra $Z_{\mathfrak{k}}(\mathfrak{a})$ acts trivially on the one-dimensional restricted root spaces $\mathfrak{g}^{\lambda_j}$ and $\mathfrak{g}^{-\lambda_j}$, and therefore on the $\mathfrak{sl}(2)$-triples defined in (4). Then relations (5) and (6) yield

$$[Z_0, E_j - \theta E_j] = 2a_j A_j \quad \text{and} \quad [Z_0, 2a_j A_j] = -4a_j^2(E_j - \theta E_j),$$

showing that $\mathrm{ad}_{Z_0}$ stabilizes the subspaces $\mathfrak{p}_j$. Finally, one has that

$$[Z_0, E_j - \theta E_j] = A_j, \quad [Z_0, A_j] = -(E_j - \theta E_j)$$

if and only if $a_j = \frac{1}{2}$, for all $j = 1, \ldots, r$. □

**Remark 2.5.** A geometric interpretation of Definition 2.2 and Lemma 2.4 is the following: the compatibility conditions on the vectors $E_j$ guarantee that the $r$-dimensional polydisk associated to the $r$ commuting $\mathfrak{sl}(2)$ triples in $\mathfrak{g}$ is holomorphically embedded in the Hermitian symmetric space $G/K$.

More precisely, consider the lie algebra homomorphism $\mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}$ mapping $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to $E_j$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to $A_j$. This induces an embedding of symmetric spaces $SL(2, \mathbb{R})/SO(2, \mathbb{R}) \to G/K$. Endow $SL(2, \mathbb{R})/SO(2, \mathbb{R})$ with the unique invariant complex structure defined by $\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then such an embedding is holomorphic if and only if the sign of the vector $E_j$ is compatible. Otherwise it is anti-holomorphic.

By the above discussion and Koranyi-Wolf's Theorem (see Thm. A.3.5 in [HiOl97], p.256), one has the following characterization of $Z_0$.

**Proposition 2.6.** *Fix the vectors* $E_j$ *as in Definition 2.2. Then the following conditions are equivalent*

(i) $G/K$ *is of tube type, i.e.* $\Delta(\mathfrak{a}, \mathfrak{g})$ *si reduced of type* $C_r$,

(ii) $Z_0 = \frac{1}{2}\sum_j T_j$.

## 3. The domain $\Xi^+$.

Let $G/K$ be an irreducible Hermitian symmetric space of the non-compact type. Let $J_0$ be the complex structure of $\mathfrak{p}$, and let $\mathfrak{p}^{1,0}$ and $\mathfrak{p}^{0,1}$ be the $\pm i$-eigenspaces of $J_0$ in $\mathfrak{p}^{\mathbb{C}}$. Set $P := \exp \mathfrak{p}^{0,1}$ and $Q := K^{\mathbb{C}}P$. Then $Q$ is a maximal parabolic subgroup of $G^{\mathbb{C}}$, the quotient $G^{\mathbb{C}}/Q$ is the compact dual symmetric space of $G/K$ and the $G$-equivariant map

$$G/K \to G^{\mathbb{C}}/Q, \qquad g \to g \cdot eQ$$

defines an open holomorphic embedding of $G/K$ as the $G$-orbit $D := G \cdot eQ$.

Denote by $\sigma$ the antiholomorpic involution of $G^{\mathbb{C}}$ defining $G$. Then $\sigma(P) = \exp \mathfrak{p}^{1,0}$ and $\sigma(Q) = K^{\mathbb{C}}\sigma(P)$ is the opposite parabolic subgroup, which satisfies $Q \cap \sigma(Q) = K^{\mathbb{C}}$. Denote by $\overline{G^{\mathbb{C}}/Q}$ the compact dual symmetric space endowed with the opposite complex structure, i.e. the complex structure which makes the $G$-equivariant map

$$\overline{G^{\mathbb{C}}/Q} \to G^{\mathbb{C}}/\sigma(Q), \quad gQ \to \sigma(gQ) = \sigma(g)\sigma(Q)$$

a biholomorphism. Let $G^{\mathbb{C}}$ act on $G^{\mathbb{C}}/Q \times \overline{G^{\mathbb{C}}/Q}$ by

$$g \cdot (x, y) := (g \cdot x, \sigma(g) \cdot y),$$

and set $x_0 := (eQ, eQ)$. Then the map

$$G^{\mathbb{C}}/K^{\mathbb{C}} \to G^{\mathbb{C}}/Q \times \overline{G^{\mathbb{C}}/Q}, \qquad g \mapsto g \cdot x_0$$

defines an open dense $G^{\mathbb{C}}$-equivariant holomorphic embedding of $G^{\mathbb{C}}/K^{\mathbb{C}}$ into the product $G^{\mathbb{C}}/Q \times \overline{G^{\mathbb{C}}/Q}$, as the orbit through $x_0$. Let $\pi_1 : G^{\mathbb{C}}/Q \times \overline{G^{\mathbb{C}}/Q} \to G^{\mathbb{C}}/Q$ denote the projection onto the first factor. The domain $\Xi^+$ is defined as follows

$$\Xi^+ := \pi_1^{-1}(D) \cap G^{\mathbb{C}} \cdot x_0.$$

As $\Xi^+$ is a subdomain of $G^{\mathbb{C}} \cdot x_0$, it can be regarded as an open $G$-invariant domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$.

Recall that the anti-holomorphic tangent bundle of $G/K$ is $G$-equivariantly diffeomorphic to the twisted bundle $G \times_K \mathfrak{p}^{0,1}$. The following fact holds true.

**Lemma 3.1.** *The domain $\Xi^+$ is diffeomorphic to the anti-holomorphic tangent bundle of $G/K$ via the map*

$$\phi \colon G \times_K \mathfrak{p}^{0,1} \to \Xi^+, \quad (g, Z) \to g \exp Z \cdot x_0.$$

*Proof.* Let $L$ be Lie group, let $H$ be closed subgroup of $L$ and let $X$ be an $L$-manifold. Assume there exists a differentiable $L$-equivariant map $f \colon X \to L/H$. Then the fiber $F := f^{-1}(eH)$ is an embedded $H$-manifold and it is a standard fact that the map

$$L \times_H F \to X, \quad [g, x] \mapsto g \cdot x$$

is an $L$-equivariant diffeomorphism (see, e.g. [DuKo00], p. 102).

Since the isotropy subgroup of $eQ$ in $G^{\mathbb{C}}/Q$ is $Q = K^{\mathbb{C}}P = PK^{\mathbb{C}}$ and the isotropy subgroup of $x_0$ in $G^{\mathbb{C}}/Q \times \overline{G^{\mathbb{C}}/Q}$ is $K^{\mathbb{C}}$, the fiber $F := \pi_1^{-1}(eQ)$ is given by $P \cdot x_0$. As a consequence the map $\mathfrak{p}^{0,1} \to F$, defined by $Z \to \exp Z \cdot x_0$, is a biholomorphism. Now the statement follows from the above remark. $\square$

It should be pointed out that the above map is just a diffeomorphism and not a biholomorphism, for the simple reason that the symmetric space $G/K$ is a complex submanifold of its antiholomorphic tangent bundle (embedded as the zero section), while it is a totally real submanifold of $\Xi^+$.

Also note that $\Xi^+$ and $\Xi^- := \pi_2^{-1}(\overline{D}) \cap G^{\mathbb{C}} \cdot x_0$ are $G$-equivariantly anti-biholomorphic, since the $G$-equivariant anti-biholomorphism

$$G^{\mathbb{C}}/Q \times \overline{G^{\mathbb{C}}/Q} \to G^{\mathbb{C}}/Q \times \overline{G^{\mathbb{C}}/Q}\,, \qquad (x,\,y) \to (y,\,x)\,,$$

maps $\Xi^+$ onto $\Xi^-$. Also note that the restriction of such a map to $G^{\mathbb{C}} \cdot x_0 \cong G^{\mathbb{C}}/K^{\mathbb{C}}$ coincides with the anti-holomorphic $G$-equivariant involution induced by $\sigma$.

An alternative construction of the domain $\Xi^+$ was given in [Kro08], p.286, and [KrOp08], Sect.8, via the unipotent parametrization. More precisely, in the notation of Section 2, choose vectors $E_j \in \mathfrak{g}^{\lambda_j}$, for $j = 1, \dots, r$, compatible with the complex structure $J_0$ of $G/K$ (see Definition 2.2). Define

$$\Lambda_r := \mathrm{span}_{\mathbb{R}}\{E_1, \dots, E_r\} \quad \text{and} \quad \Lambda_r^{\llcorner} := \mathrm{span}_{\mathbb{R}_{\geq 0}}\{E_1, \dots, E_r\}. \tag{11}$$

Then

$$\Xi^+ = G \exp i \bigoplus_{j=1}^{r} (-1, \infty) E_j \cdot x_0 = G \exp i \Lambda_r^{\llcorner} \cdot x_0.$$

After defining the subcone $\mathcal{N}^+ := \mathrm{Ad}_K \Lambda_r^{\llcorner}$ of the nilpotent cone of $\mathfrak{g}$, it was suggested that the map

$$\psi \colon G \times_K \mathcal{N}^+ \to \Xi^+, \quad [g, X] \mapsto g \exp iX \cdot x_0$$

is a $G$-equivariant homeomorphism. We give a complete proof of this fact in Section 5.

## 4. The Weyl group $W_K(\Lambda_r)$

Resume the notation of Section 2. For $j = 1, \dots, r$, choose vectors $E_j \in \mathfrak{g}^{\lambda_j}$ compatible with the complex structure $J_0$ of $G/K$ (see Definition 2.2), and define $\Lambda_r$ and $\Lambda_r^{\llcorner}$ as in (11).

Consider the Adjoint action of $K$ on $\mathfrak{g}$ and define

$$Z_K(\Lambda_r) := \{k \in K \ : \ \mathrm{Ad}_k X = X,\ X \in \Lambda_r\}, \quad N_K(\Lambda_r) := \{k \in K \ : \ \mathrm{Ad}_k \Lambda_r = \Lambda_r\},$$

$$W_K(\Lambda_r) := N_K(\Lambda_r)/Z_K(\Lambda_r)\,.$$

**Lemma 4.1.**
(i) $Z_K(\Lambda_r) = Z_K(\mathfrak{a})$.
(ii) $N_K(\Lambda_r)$ *is a subgroup of* $N_K(\mathfrak{a})$*, implying that* $W_K(\Lambda_r)$ *is a subgroup of* $W_K(\mathfrak{a})$.
(iii) *As a subgroup of* $W_K(\mathfrak{a})$*, the group* $W_K(\Lambda_r)$ *coincides with* $W_K(\mathfrak{a})^+$*, acting on* $\mathfrak{a}$ *by permutations of* $\{A_1, \dots, A_r\}$*. Moreover,* $W_K(\Lambda_r)$ *acts on* $\Lambda_r$ *by permutations of* $\{E_1, \dots, E_r\}$.

*Proof.* (i) Let $k \in Z_K(\Lambda_r)$ and $X \in \Lambda_r$ be arbitrary elements. Then $\mathrm{Ad}_k X = X$ implies $\mathrm{Ad}_k \theta X = \theta X$ and $\mathrm{Ad}_k[\theta X, X] = [\theta X, X]$. Since $\mathfrak{a}$ is generated by the vectors $A_j = [\theta E_j, E_j]$, the inclusion $Z_K(\Lambda_r) \subset Z_K(\mathfrak{a})$ holds true.
In order to show the opposite one, observe that every restricted root space is invariant under the Adjont action of $Z_K(\mathfrak{a})$ on $\mathfrak{g}$. Since the Adjoint action of $K$ is isometric with respect to the inner product $B_\theta(X, Y) := B(X, \theta Y)$, for $X, Y \in \mathfrak{g}$, and the root spaces $\mathfrak{g}^{\pm\lambda_j}$ are one-dimensional, one has that $\mathrm{Ad}_k(E_j) = \pm E_j$, for $j = 1, \dots, r$. We claim that $\mathrm{Ad}_k E_j = E_j$, for every $k \in Z_K(\mathfrak{a})$ and $j = 1, \dots, r$. Let $W_j = \frac{1}{2}\big((E_j - \theta E_j) - iA_j\big)$ be the vector defined in (7). Recall that

$$[-iZ_0, W_j] = \pm W_j,$$

depending on whether the choice of $E_j$ is compatible with the complex structure determined by $Z_0$ (see Definition 2.2). If $k$ is an arbitrary element in $Z_K(\mathfrak{a})$, by applying $\mathrm{Ad}_k$ to both terms in the above equation, we obtain

$$[-iZ_0, \mathrm{Ad}_k W_j] = \pm \mathrm{Ad}_k W_j,$$

where

$$\mathrm{Ad}_k W_j = \frac{1}{2}\big(\mathrm{Ad}_k(E_j - \theta E_j) - i\mathrm{Ad}_k A_j\big) = \frac{1}{2}\big((\mathrm{Ad}_k E_j - \theta(\mathrm{Ad}_k E_j)) - iA_j\big).$$

Then Remark 2.3 now implies that indeed for $j = 1, \ldots, r$

$$\mathrm{Ad}_k(E_j) = E_j, \qquad \text{for } k \in Z_K(\mathfrak{a}).$$

(ii) Let $X \in \Lambda_r$ and $k \in N_K(\Lambda_r)$ be arbitrary elements. Then $\mathrm{Ad}_k X = Y$, for some $Y \in \Lambda_r$, and likewise $\mathrm{Ad}_k \theta X = \theta Y$ and $\mathrm{Ad}_k[\theta X, X] = [\theta Y, Y]$. Since $\mathfrak{a}$ is generated by the vectors $A_j = [\theta E_j, E_j]$, there is an inclusion $N_K(\Lambda_r) \subset N_K(\mathfrak{a})$. Since $Z_K(\mathfrak{a}) = Z_K(\Lambda_r)$, there is an induced inclusion of finite groups $W_K(\Lambda_r) \hookrightarrow W_K(\mathfrak{a})$.

(iii) We already showed that $W_K(\Lambda_r) \subset W_K(\mathfrak{a})$. Next we show that $W_K(\Lambda_r)$ contains the subgroup $W_K(\mathfrak{a})^+$. Recall that the subgroup $W_K(\mathfrak{a})^+$ acts on $\mathfrak{a}$ by permutations of $A_1, \ldots, A_r$ and on $\mathfrak{a}^*$ by permutations of the basis vectors $e_1, \ldots, e_r$ defined in Section 2. As a result, the corresponding elements in $K$ permute the root spaces $\mathfrak{g}^{\lambda_1}, \ldots, \mathfrak{g}^{\lambda_r}$ and thus normalize $\Lambda_r$. This proves the inclusion

$$W_K(\mathfrak{a})^+ \subset W_K(\Lambda_r).$$

In order to prove equality, assume by contradiction that there exists $k \in N_K(\Lambda_r)$ lying in $W_K(\mathfrak{a}) \setminus W_K(\mathfrak{a})^+$. Since $W_K(\mathfrak{a})$ acts on $\mathfrak{a}$ by signed permutations of $A_1, \ldots, A_r$, this means that there exist indices $j, h \in \{1, \ldots, r\}$ for which $\mathrm{Ad}_k(A_j) = -A_h$. By applying $\mathrm{Ad}_k$ to both terms of the relation $[A_j, E_j] = 2E_j$, we obtain

$$[A_h, \mathrm{Ad}_k E_j] = -2\mathrm{Ad}_k E_j.$$

We also have $[A_l, \mathrm{Ad}_k E_j] = 0$, for all $l \neq h$: indeed, we can write

$$[A_l, \mathrm{Ad}_k E_j] = \mathrm{Ad}_k[\mathrm{Ad}_{k^{-1}} A_l, E_k]$$

and, since $k$ normalizes $\mathfrak{a}$, we have that $\mathrm{Ad}_{k^{-1}} A_l \in \{\pm A_m\}$, for some $m \neq j$. Thus

$$\mathrm{Ad}_k[\mathrm{Ad}_{k^{-1}} A_l, E_j] = \mathrm{Ad}_k[\pm A_m, E_j] = 0,$$

as claimed. It follows that $\mathrm{Ad}_k E_j \in \mathfrak{g}^{-\lambda_h}$, contradicting the assumption that $k$ normalizes $\Lambda_r$. So $W_K(\mathfrak{a})^+ = W_K(\Lambda_r)$, as claimed.

To prove that $W_K(\Lambda_r)$ acts on $\Lambda_r$ by permutations of $E_1, \ldots, E_r$, assume by contradiction that there exists $k \in N_K(\Lambda_r)$ and indices $h, j \in \{1, \ldots, r\}$ such that

$$\mathrm{Ad}_k E_j = -E_h,$$

and consequently

$$\mathrm{Ad}_k \theta E_j = -\theta E_h, \quad \mathrm{Ad}_k A_j = A_h.$$

From the compatibility condition

$$[-iZ_0, W_j] = W_j$$

one obtains then

$$[-iZ_0, \mathrm{Ad}_k W_j] = \mathrm{Ad}_k W_j$$

where

$$\mathrm{Ad}_k W_j = \frac{1}{2}(\mathrm{Ad}_k(E_j - \theta E_j) - i\mathrm{Ad}_k A_j) = \frac{1}{2}(-(E_h - \theta E_h) - iA_h).$$

But this contradicts Remark 2.3. In conclusion,

$$\mathrm{Ad}_k E_j = E_h,$$

and $W_K(\Lambda_r)$ acts on $\Lambda_r^{\llcorner}$ by permutations of $E_1, \ldots, E_r$ , as claimed. $\square$

**Corollary 4.2.** *As a consequence of the previous lemma, the group $W_K(\Lambda_r)$ preserves the subset $\Lambda_r^{\llcorner}$. Hence*

$$W_K(\Lambda_r) := N_K(\Lambda_r)/Z_K(\Lambda_r) = N_K(\Lambda_r^{\llcorner})/Z_K(\Lambda_r^{\llcorner})\,.$$

## 5. The domain $\Xi^+$ as a nilpotent cone bundle

Consider the nilpotent cone in $\mathfrak{g}$ given by $\mathcal{N}^+ := \{\mathrm{Ad}_k X \ : \ k \in K \text{ and } X \in \Lambda_r^{\llcorner}\}$. In [KrOp08] and [Kro08], Rem.4.12, it was suggested that the domain $\Xi^+$ is homeomorphic to the twisted product $G \times_K \mathcal{N}^+$. For the sake of completeness we give a proof of this fact.

5.1. **Some topological lemmas.** We first need a number of lemmas, which are of topological nature. Our setting is as follows. Let $G$ be a connected Lie group acting properly on a Hausdorff topological space $Z$, and let $K$ be a compact subgroup of $G$. Let $N$ be a Hausdorff topological $K$-space. Assume that there exists a $K$-equivariant continuous map $j : N \to Z$ such that the continuous map

$$\psi : G \times_K N \to Z, \ [g,x] \to g \cdot j(x)$$

is bijective. Denote by $\Sigma$ a closed subset of $N$ such that $K \cdot \Sigma = N$. We are going to discuss necessary and sufficient conditions for $\psi$ to be a homeomorphism.

**Lemma 5.1.** *The following three conditons are equivalent*

(i) *The map $\widetilde{\psi} : G \times \Sigma \to Z, \ (g,x) \to g \cdot j(x)$ is proper,*
(ii) *The map $\widehat{\psi} : G \times N \to Z, \ (g,x) \to g \cdot j(x)$ is proper,*
(iii) *The map $\psi : G \times_K N \to Z, \ [g,x] \to g \cdot j(x)$ is proper.*

*If any of the above conditions is satisfied, then $\psi$ is a homeomorphism, the map $j : N \to j(N)$ is a homeomorphism, and $j(N)$ is closed in $Z$.*

*Proof.* We first show that (i) is equivalent to (ii). Consider the commutative diagram

$$\begin{tikzcd} G \times \Sigma \arrow[d] \arrow[dr, "\widetilde{\psi}"] & \\ G \times N \arrow[r, "\widehat{\psi}"'] & Z\,, \end{tikzcd}$$

where the vertical arrow is the inclusion map. Being $\Sigma$ closed in $N$, such a map is proper. Therefore, if $\widehat{\psi}$ is proper, so is $\widetilde{\psi}$. Conversely, assume that $\widetilde{\psi}$ is proper and let $C$ be a compact subset of $Z$. We claim that the closed subset $\widehat{\psi}^{-1}(C)$ coincides with $K \cdot \widetilde{\psi}^{-1}(C)$, where the $K$-action on $G \times N$ is given by $k \cdot (g,x) := (gk^{-1}, k \cdot x)$. In order to see that $\widehat{\psi}^{-1}(C) \subset K \cdot \widetilde{\psi}^{-1}(C)$, let $(g,x)$ be an element in $\widehat{\psi}^{-1}(C)$ and choose $k \in K$ and $x' \in \Sigma$ such that $x = k \cdot x'$. Then $gk \cdot j(x') = g \cdot j(x) \in C$, implying that $(gk, x') \in \widetilde{\psi}^{-1}(C)$. Thus $(g,x) = k \cdot (gk, x')$ belongs to $K \cdot \widetilde{\psi}^{-1}(C)$. Being the opposite inclusion straightforward, the claim follows.

Since $\widetilde{\psi}^{-1}(C)$ is compact by assumption, it follows that $\widehat{\psi}^{-1}(C) = K \cdot \widetilde{\psi}^{-1}(C)$ is compact (cf. [Bou89], Cor. 1, p. 251). This concludes the proof of the first equivalence.

In order to show that (ii) is equivalent to (iii), consider the commutative diagram

$$\begin{tikzcd} G \times N \arrow[d, "\pi"'] \arrow[dr, "\widehat{\psi}"] & \\ G \times_K N \arrow[r, "\psi"'] & Z\,, \end{tikzcd}$$

where $\pi$ is the natural quotient with respect to the twisted $K$-action. Being $K$ compact, such a map is proper (cf. [Bou89], Prop. 2, p. 252). Therefore, if $\psi$ is proper, so is $\widehat{\psi}$. Conversely, assume that $\widehat{\psi}$ is proper and let $C$ be a compact subset of $Z$. Then the inverse image $\psi^{-1}(C)$ coincides with $\pi(\widehat{\psi}^{-1}(C))$. Thus it is compact, showing that $\psi$ is proper and concluding the proof of the lemma. □

Note that assuming $j : \Sigma \to Z$ proper does not imply $G \times \Sigma \to Z$ proper. For instance, let $G = \mathbb{R}$ act on $\mathbb{R}^2$ by $t \cdot (x, y) = (t + x, y)$, set $N = \Sigma := \{\ s \in \mathbb{R}\ :\ s \leq 0 \text{ or } s > 1\}$ and define $j : \Sigma \to \mathbb{R}^2$ by $j(s) := (0, s)$, for $s \in (-\infty, 0]$, and $j(s) := (\ln(s - 1), s - 1)$, for $s \in (1, +\infty)$. Then $\psi : \mathbb{R} \times \Sigma \to \mathbb{R}^2$ is continuous and bijective but it is not a homeomorphism. In this example $\Sigma \cong j(\Sigma)$ is a non-connected, closed submanifold (with boundary) of $Z$. In higher dimension, e.g. $\dim_{\mathbb{R}} Z = 3$ one can constuct a similar example with $\Sigma$ a contractible, closed submanifold (with boundary) of $Z$.

Now we also assume that $Z$ has the structure of a $G$-equivariant fiber bundle, i.e. that there exists a closed topological $K$-subspace $P$ of $Z$ such that the map

$$G \times_K P \to Z, \quad [g, p] \to g \cdot p$$

is a homeomorphism. Let $\pi : P \to P/K$ be the canonical projection.

**Lemma 5.2.** *If the map $q : \Sigma \to P/K$, given by $x \to \pi\big(G \cdot j(x) \cap P\big)$ is proper, then $\psi : G \times_K N \to Z$, $[g, x] \to g \cdot j(x)$ is a homeomorphism.*

*Proof.* By Lemma 5.1, it is sufficient to show that the map $\widetilde{\psi} : G \times \Sigma \to Z$ is proper. Let $\{(g_n, x_n)\}_n$ be a sequence in $G \times \Sigma$, with $g_n \cdot j(x_n) \to z_0$. Choose $\{(h_n, p_n)\}_n$ in $G \times P$ such that $g_n \cdot j(x_n) = h_n \cdot p_n$. Being the canonical projection $G \times P \to G \times_K P$ proper (cf. [Bou89], Prop. 2, p. 252), the map $G \times P \to Z$, given by $(g, z) \to g \cdot z$, is proper. Thus, by passing to a subsequence if necessary, we may assume that $(h_n, p_n) \to (h_0, p_0)$. In particular, $q(x_n) := \pi(G \cdot j(x_n) \cap P) = \pi(p_n) \to \pi(p_0)$. Since the map $q$ is proper by assumption, by passing to a subsequence if necessary, one has that $x_n \to x_0$, for some $x_0 \in \Sigma$. Thus $j(x_n) \to j(x_0)$. By the properness of the $G$-action, the map $G \times Z \to Z \times Z$, given by $(g, z) \to (z, g \cdot z)$, is proper as well. Therefore, the sequence $\{(g_n, x_n)\}_n$ converges to $(g_0, x_0)$, for some $g_0$ in $G$. As a result the map $\widetilde{\psi} : G \times \Sigma \to Z$ is proper, and the statement follows from Lemma 5.1. □

As a matter of fact, the converse of the above lemma holds true as well. Indeed if $\psi : G \times_K N \to Z$, $[g, x] \to g \cdot j(x)$ is a homeomorphism, then $Z/G$ is homeomorphic to $N/K$, as well as to $P/K$, being $Z$ homeomorphic to $G \times_K P$. Therefore one has a commutative diagram

$$\begin{array}{ccccc} \Sigma & \longrightarrow & G \times_K N & \longrightarrow & Z \\ & \searrow & \downarrow & & \downarrow \\ & & N/K & \longrightarrow & P/K\,, \end{array}$$

where the map $N/K \to P/K$ is a homeomorphism. Being $\Sigma$ closed in $N$, the restriction $\Sigma \to N/K$ of the natural projection $G \times_k N \to N/K$ is proper. Hence the map $q : \Sigma \to P/K$, $x \to \pi(G \cdot j(x) \cap P)$, given in the above diagram as the composition of proper maps, is proper, as claimed.

Also note that, being $Z$ connected by assumption, if $\psi$ is a homeomorphism and $K$ is connected, then $N$ is necessarily connected. Indeed, in this case the principal bundle $G \times N \to G \times_K N$ has connected base and fibers. Thus the total space $G \times N$ is connected, implying that $N$ is connected.

For later use we also give the following corollary.

**Corollary 5.3.** *Assume there exists a continuous, $G$-invariant function $f : Z \to \mathbb{R}$ such that $f \circ j|_\Sigma : \Sigma \to \mathbb{R}$ is proper. Then $\psi$ is a homeomorphism.*

*Proof.* By Lemma 5.1, it is sufficient to show that the map

$$\widetilde{\psi} : G \times \Sigma \to Z,\ (g, x) \to g \cdot j(x)$$

is proper. Let $\{(g_n, x_n)\}_n$ be a sequence in $G \times \Sigma$ such that $\{g_n \cdot j(x_n)\}_n$ converges to an element $z_0$ in $Z$. We need to show that, by replacing it with a subsequence if necessary, the sequence $\{(g_n, x_n)\}_n$ converges in $G \times \Sigma$. Let $U$ be a compact neighborhood of $f(z_0)$ in $\mathbb{R}$. By assumption, the set $V := (f \circ j|_\Sigma)^{-1}(U)$ is a compact subset of $\Sigma$. By the continuity and the $G$-invariance of $f$ one has $f(j(x_n)) = f(g_n \cdot j(x_n)) \to f(z_0)$. Therefore $x_n \in V$ for $n$ large enough. Thus, by passing to a subsequence if necessary, $\{x_n\}_n$ converges to an element $x_0$ of $\Sigma$ and $j(x_n) \to j(x_0)$. Finally, by the properness of the $G$-action, the map $G \times Z \to Z \times Z$, given by $(g, z) \to (z, g \cdot z)$, is proper. Hence, by passing to a subsequence if necessary, $\{(g_n, x_n)\}_n$ converges to $(g_0, x_0)$, for some $g_0$ in $G$. This concludes the proof. □

**Remark 5.4.** The function $f \circ j|_\Sigma$ is proper if and only if $f \circ j$ is proper. Being $\Sigma$ closed in $N$, one implication is clear. For the converse, let $C$ be a compact subset of $\mathbb{R}$. Then

$$(f \circ j)^{-1}(C) = K \cdot (f \circ j|_\Sigma)^{-1}(C)\,,$$

which is compact if $(f \circ j|_\Sigma)^{-1}(C)$ is compact (cf. [Bou89], Cor. I, p. 251).

5.2. **A slice in the anti-holomorphic tangent bundle.** Let $G/K$ be an irreducible Hermitian symmetric space. Resuming the notation of Section 2, denote by $\mathfrak{a}^+$ the open positive Weyl chamber in $\mathfrak{a}$ and by $\overline{\mathfrak{a}^+}$ its topological closure, given by

$$\mathfrak{a}^+ := \{\sum_{j=1}^r x_j A_j \ :\ x_1 > \cdots > x_r > 0\}, \quad \overline{\mathfrak{a}^+} = \{\sum_{j=1}^r x_j A_j \ :\ x_1 \geq \cdots \geq x_r \geq 0\}.$$

Define

$$\mathfrak{a}^{\llcorner} := \{\sum_{j=1}^r x_j A_j \ :\ x_j \geq 0,\ j = 1, \ldots, r\}.$$

The set $\overline{\mathfrak{a}^+}$ is a perfect slice for the Adjoint action of $K$ on $\mathfrak{p}$, and

$$\mathfrak{a}^{\llcorner} = W_K(\mathfrak{a})^+ \cdot \overline{\mathfrak{a}^+}.$$

Similarly, denote by $(\Lambda_r^{\llcorner})^+$ the open positive Weyl chamber in $\Lambda_r^{\llcorner}$, and by $\overline{(\Lambda_r^{\llcorner})^+}$ its topological closure, given by

$$(\Lambda_r^{\llcorner})^+ := \{\sum_{j=1}^r x_j E_j \ :\ x_1 > \cdots > x_r > 0\}, \quad \overline{(\Lambda_r^{\llcorner})^+} = \{\sum_{j=1}^r x_j E_j, \ :\ x_1 \geq \cdots \geq x_r \geq 0\}.$$

By Lemma 4.1 and Corollary 4.2, one has

$$\Lambda_r^{\llcorner} = W_K(\Lambda_r) \cdot \overline{(\Lambda_r^{\llcorner})^+}.$$

Consider the $K$-equivariant map

$$\Psi : \mathfrak{g} \to \mathfrak{p}, \quad X \mapsto [Z_0, X - \theta X] = J_0(X - \theta X)\,, \tag{12}$$

where $Z_0 \in Z(\mathfrak{k})$ is the element defining the complex structure $J_0 = \mathrm{ad}_{Z_0}$. Note that its restriction

$$\Psi|_{\Lambda_r} : \Lambda_r \to \mathfrak{a}$$

is a linear isomorphism.

Consider also the homeomorphism

$$\Phi : \Lambda_r^{\llcorner} \to \mathfrak{a}^{\llcorner}, \quad \sum x_j E_j \to \frac{1}{2} \sum \log(1 + x_j) A_j\,,$$

and the $K$-equivariant isomorphism

$$\tau\colon \mathfrak{p} \to \mathfrak{p}^{0,1}, \quad Y \to -\frac{1}{2}(Y + iJ_0 Y)\,. \tag{13}$$

The isomorphism $\tau$ maps $\mathfrak{a}$, a slice for the $\mathrm{Ad}_K$-action on $\mathfrak{p}$, onto a slice for the $\mathrm{Ad}_K$-action on $\mathfrak{p}^{0,1}$, and induces a homeomorphism between the respective fundamental domains $\overline{\mathfrak{a}^+} \subset \mathfrak{a}$ and $\tau(\overline{\mathfrak{a}^+})$ in $\mathfrak{p}^{0,1}$.

The next lemma is crucial for the main result of this section. It states that in $\Xi^+$ the nilpotent slice $\exp i\Lambda_r^{\llcorner} \cdot x_0$ can be mapped *continuously* onto a slice in $\exp \mathfrak{p}^{0,1} \cdot x_0$, by elements of the abelian group $A = \exp \mathfrak{a}$.

**Lemma 5.5.** *For every $X$ in $\Lambda_r^{\llcorner}$ one has*

$$\exp(iX) = \exp \Phi(X) \exp\left(-\frac{1}{2}(\Psi(X) + iJ_0\Psi(X)\right) \exp i\chi(X),$$

*where $\chi : \Lambda_r^{\llcorner} \to \mathfrak{k}$ is defined by $\sum x_j E_j \to \sum \sinh^{-1}\left(\frac{x_j}{2\sqrt{1+x_j}}\right)(E_j + \theta E_j)$. Thus*

$$\exp(iX) \cdot x_0 = \exp \Phi(X) \exp\left(-\frac{1}{2}(\Psi(X) + iJ_0\Psi(X)\right) \cdot x_0\ .$$

*Proof.* Write $X = \sum x_j E_j$ as a sum of nilpotent elements in the embedded $\mathfrak{sl}(2)$-triples. By Lemma 2.4 (ii), the complex structure $J_0$ of $G/K$ induces the invariant complex structure defined by $\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on each of the rank-one symmetric spaces associated to the $\mathfrak{sl}(2)$-triples. This fact, together with the commutativity of the $\mathfrak{sl}(2)$-triples in $\mathfrak{g}$ and of the corresponding groups in $G^{\mathbb{C}}$, reduces the proof to the case of $G = SL(2,\mathbb{R})$. In this case, the equality to be proved reads as

$$\exp i \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \exp \Phi \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \exp -\frac{1}{2}\left(\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} + i \begin{pmatrix} 0 & -x \\ -x & 0 \end{pmatrix}\right) \mathrm{SO}(2,\mathbb{C})\,.$$

In other words, we are left to check the following matrix identity

$$\begin{pmatrix} 1 & ix \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{1+x} & 0 \\ 0 & \sqrt{1+x}^{-1} \end{pmatrix} \begin{pmatrix} 1 - \frac{x}{2} & i\frac{x}{2} \\ i\frac{x}{2} & 1 + \frac{x}{2} \end{pmatrix} M\,,$$

where $M \in \exp i\mathfrak{so}(2,\mathbb{R}) \subset SO(2,\mathbb{C})$ is the matrix given by

$$M = \exp i \sinh^{-1}\left(\frac{x}{2\sqrt{1+x}}\right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{\sqrt{1+x}} \begin{pmatrix} 1 + \frac{x}{2} & i\frac{x}{2} \\ -i\frac{x}{2} & 1 + \frac{x}{2} \end{pmatrix}\,.$$

□

**Lemma 5.6.**

(i) *Let $X$ be an element in $\overline{(\Lambda_r^{\llcorner})^+}$. Then*

$$Z_K(X) = Z_K(\Psi(X)) = Z_K(\Phi(X))\,.$$

(ii) *Let $X$ and $X'$ be elements in $\overline{(\Lambda_r^{\llcorner})^+}$ such that*

$$\Psi(X') = \mathrm{Ad}_k \Psi(X), \qquad \textit{for some } k \in K.$$

*Then $X' = X$ and $k \in Z_K(X)$.*

*Proof.* (i) We begin by proving that $Z_K(X) = Z_K(\Psi(X))$. Since the map $\Psi(X) = [Z_0, X - \theta X]$ defined in (12) is $K$-equivariant, there is an inclusion

$$Z_K(X) \subset Z_K(\Psi(X)).$$

We prove the opposite one by showing that an element $k \in Z_K(\Psi(X))$ centralizes both $X - \theta X$ and $X + \theta X$. From

$$[Z_0, X - \theta X] = \mathrm{Ad}_k[Z_0, X - \theta X] = [Z_0, \mathrm{Ad}_k(X - \theta X)]$$

and the fact that $\mathrm{ad}_{Z_0}$ is bijective on $\mathfrak{p}$ (it is a complex structure), we obtain that $k \in Z_K(X - \theta X)$. Before showing that $k \in Z_K(X + \theta X)$, we make a small digression.

Given a subset $\Delta$ of $\Delta(\mathfrak{g}, \mathfrak{a})^+$, the associated orbit stratum in the closure of the Weyl chamber $\overline{\mathfrak{a}^+}$ is by definition

$$\overline{\mathfrak{a}^+_\Delta} := \{\, A \in \mathfrak{a}^+ \ : \ \beta(A) = 0 \text{ if } \beta \in \Delta,\ \beta(A) > 0 \text{ if } \beta \in \Delta(\mathfrak{g}, \mathfrak{a})^+ \setminus \Delta, \}\,.$$

Let $H$ be an element in $\mathfrak{a}$. Since $G^{\mathbb{C}}$ is simply connected, the centralizer $Z_{G^{\mathbb{C}}}(H)$ of $H$ in $G^{\mathbb{C}}$ is a connected group (see [Hum95], p.33) with Lie algebra

$$Z_{\mathfrak{g}^{\mathbb{C}}}(H) = Z_{\mathfrak{k}^{\mathbb{C}}}(\mathfrak{a}) \oplus \mathfrak{a}^{\mathbb{C}} \oplus \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}}) \\ \alpha(H) = 0}} \mathfrak{g}^{\alpha}. \tag{14}$$

Moreover, since $\sigma(H) = H$ and $\theta(H) = -H$, the group $Z_{G^{\mathbb{C}}}(H)$ is both $\sigma$ and $\theta$-stable. As a result, if two elements $H_1$ and $H_2$ of $\overline{\mathfrak{a}^+}$ lie in the same orbit stratum, then $Z_{G^{\mathbb{C}}}(H_1) = Z_{G^{\mathbb{C}}}(H_2)$ and likewise $Z_K(H_1) = Z_K(H_2)$.

Write $X = \sum x_j E_j$ and $\Psi(X) = \sum x_j A_j$. Since the elements $\sum x_j A_j$ and $\sum \sqrt{\frac{x_j}{2}} A_j$ lie in the same orbit stratum of $\overline{\mathfrak{a}^+}$, one has $Z_K(\Psi(X)) = Z_K(\sum \sqrt{\frac{x_j}{2}} A_j)$. Moreover, since

$$\sum \sqrt{\frac{x_j}{2}}(E_j - \theta E_j) = [-Z_0, \sum \sqrt{\frac{x_j}{2}} A_j],$$

one also has $Z_K(\Psi(X)) \subset Z_K\big(\sum \sqrt{\frac{x_j}{2}}(E_j - \theta E_j)\big)$. Then the equality

$$Z_K(\Psi(X)) = Z_K(X + \theta X)$$

follows from

$$\mathrm{Ad}_k(X + \theta X) =$$

$$\mathrm{Ad}_k\big(\sum x_j(E_j + \theta E_j)\big) = \mathrm{Ad}_k[\sum \sqrt{\frac{x_j}{2}} A_j, \sum \sqrt{\frac{x_j}{2}}(E_j - \theta E_j)] =$$

$$[\mathrm{Ad}_k(\sum \sqrt{\frac{x_j}{2}} A_j), \mathrm{Ad}_k(\sum \sqrt{\frac{x_j}{2}}(E_j - \theta E_j))] = [\sum \sqrt{\frac{x_j}{2}} A_j, \sum \sqrt{\frac{x_j}{2}}(E_j - \theta E_j)] =$$

$$\sum x_j(E_j + \theta E_j) = X + \theta X\,.$$

Since $X = \frac{1}{2}(X - \theta X) + \frac{1}{2}(X + \theta X)$, we conclude that

$$Z_K(X) = Z_K(\Psi(X)).$$

Next we show that

$$Z_K(\Psi(X)) = Z_K(\Phi(X)).$$

From the definition of the maps $\Psi$ , $\Phi$ and of the roots defining $\mathfrak{a}^+$ (cf. Sect.2) it is clear that $\Psi(X)$ and $\Phi(X)$ lie in the same orbit stratum of $\overline{\mathfrak{a}^+}$. Then the desired equality follows from the above considerations.

(ii) By definition of $\overline{(\Lambda_r^{\llcorner})^+}$, the elements $\Psi(X)$ and $\Psi(X')$ lie in $\overline{\mathfrak{a}^+}$, which is a perfect slice for the $\mathrm{Ad}_K$-action on $\mathfrak{p}$. Then $\Psi(X')=\Psi(X)$ and $k\in Z_K(\Psi(X))=Z_K(X)$. Since the map $\Psi\colon\Lambda_r\to\mathfrak{a}$ is injective, it follows that $X'=X$.

□

**Proposition 5.7.** *Let $G/K$ be an irreducible Hermitian symmetric space. Then the map*

$$\psi : G\times_K\mathcal{N}^+\to\Xi^+,\quad [g,X]\to g\exp(iX)\cdot x_0$$

*is a $G$-equivariant homeomorphism.*

*Proof.* The map $\psi$ is $G$-equivariant by construction. By Lemma 3.1 and Lemma 5.5, it is surjective. Recall that by Corollary 4.2, one has $\mathcal{N}^+=\mathrm{Ad}_K\overline{(\Lambda_r^{\llcorner})^+}$. Hence, in order to prove that $\psi$ is injective it is sufficient to show that the identity

$$g\exp iX\cdot x_0=\exp iX'\cdot x_0, \tag{15}$$

for some $g\in G$ and $X,X'\in\overline{(\Lambda_r^{\llcorner})^+}$, implies

$$g\in K,\quad\text{and}\quad X'=\mathrm{Ad}_gX.$$

By Lemma 5.5, equation (15) is equivalent to

$$g\exp\Phi(X)\exp\left(-\frac{1}{2}(\Psi(X)+iJ_0\Psi(X))\right)\cdot x_0=$$

$$\exp\Phi(X')\exp\left(-\frac{1}{2}(\Psi(X')+iJ_0\Psi(X'))\right)\cdot x_0\ .$$

By Lemma 3.1 it follows that

$$[g\exp\Phi(X),-\frac{1}{2}(\Psi(X)+iJ_0\Psi(X))]=[\exp\Phi(X'),-\frac{1}{2}(\Psi(X')+iJ_0\Psi(X'))]$$

in $G\times_k\mathfrak{p}^{0,1}$, i.e. there exists $k\in K$ such that

$$\exp\Phi(X')=g\exp\Phi(X)k^{-1}\quad\text{and}\quad\Psi(X')=\mathrm{Ad}_k\Psi(X)\,. \tag{16}$$

From the second equality in (16) and Lemma 5.6, one obtains the relations

$$X=X',\quad\text{and}\quad k\in Z_K(\Psi(X))=Z_K(\Phi(X))=Z_K(X),$$

which plugged in the first equality of (16) yield $g=k$. In conclusion, we have obtained

$$g\in Z_K(X),\quad X'=X=\mathrm{Ad}_gX,$$

as desired.

Next we are going to show that $\psi$ is a homeomorphism. Since by Lemma 3.1 the map $G\times_KP\to\Xi^+$, given by $[g,z]\to g\exp Z\cdot z_0$, is a $G$-equivariant diffeomorphism, Lemma 5.2 implies that it is sufficient to show that the following map is proper

$$q:\Lambda_r^{\llcorner}\to(\exp\mathfrak{p}^{0,1}\cdot x_0)/K,\quad X\to\pi(G\exp iX\cdot x_0\cap\exp\mathfrak{p}^{0,1}\cdot x_0),$$

where $\pi:\exp\mathfrak{p}^{0,1}\cdot x_0\to(\exp\mathfrak{p}^{0,1}\cdot x_0)/K$ denotes the canonical projection.

So let $\{X_n\}_n$ be a sequence diverging in $\Lambda_r^{\llcorner}$. Then $\{-\frac{1}{2}(\Psi(X_n)+iJ_0\Psi(X_n))\}_n$ diverges in $\mathfrak{p}^{0,1}$. Thus the sequence $\{\exp-\frac{1}{2}(\Psi(X_n)+iJ_0\Psi(X_n))\cdot x_0\}_n$ diverges in $\exp\mathfrak{p}^{0,1}\cdot x_0$ and, by Lemma 5.5, every element $\exp-\frac{1}{2}(\Psi(X_n)+iJ_0\Psi(X_n))\cdot x_0$ lies in $G\exp iX_n\cdot x_0\cap\exp\mathfrak{p}^{0,1}\cdot x_0$. Since the projection $\pi$ is proper, the sequence $\{\pi(G\exp iX_n\cdot x_0\cap\exp\mathfrak{p}^{0,1}\cdot x_0)=\pi(\exp\left(-\frac{1}{2}(\Psi(X)+iJ_0\Psi(X))\cdot x_0\right)\}_n$ diverges in $\exp\mathfrak{p}^{0,1}\cdot x_0/K$. Thus the map $q$ is proper, as wished. □

From the above proposition we obtain the following consequences.

**Corollary 5.8.** *The restriction of the map (12)*

$$\Psi\colon \mathcal{N}^+ \to \mathfrak{p}, \qquad \Psi(X) = [Z_0, X - \theta X] = J_0(X - \theta X)$$

*is a $K$-equivariant homeomorphism. Likewise, the maps*

$$\mathcal{N}^+ \to \mathfrak{p}, \qquad X \to X - \theta X$$

*and*

$$\Psi^{0,1} : \mathcal{N}^+ \to \mathfrak{p}^{0,1}, \qquad X \to \frac{1}{2}\big(\Psi(X) + iJ_0\Psi(X)\big)$$

*are $K$-equivariant homeomorphisms.*

*Proof.* The map $\Psi$ is $K$-equivariant, since both $\mathrm{ad}_{Z_0}$ and the Cartan involution $\theta$ commute with the Adjoint action of $K$. It is also surjective, since its image contains the closure of the Weyl chamber $\overline{\mathfrak{a}^+}$. In order to show that $\Psi$ is injective, it is enough to consider pairs $X, \mathrm{Ad}_k(X')$, for some $X,\, X' \in \overline{(\Lambda_r^{\llcorner})^+}$ and $k \in K$. Assume that $\Psi(X) = \Psi(\mathrm{Ad}_k(X'))$. Then by Lemma 5.6, one obtains

$$X = X', \quad k \in Z_K(\Psi(X)) = Z_K(X).$$

Hence $X = \mathrm{Ad}_k(X')$, as wished.

It remains to show that $\Psi$ is proper. This follows from the fact that $\Psi(X) \neq 0$, if $X \neq 0$, and $\Psi(tX) = t\Psi(X)$, for all real $t$. This implies that the image of any divergent sequence in $\mathcal{N}^+$ under $\Psi$ is a divergent sequence in $\mathfrak{p}$.

The second part of the statement follows directly from the fact that both $J_0 : \mathfrak{p} \to \mathfrak{p}$ and the map $\mathfrak{p} \to \mathfrak{p}^{0,1}$, given by $Y \to \frac{1}{2}\big(Y + iJ_0(Y)\big)$, are $K$-equivariant linear isomorphisms. □

We conclude this section with another corollary of Proposition 5.7, which will be needed later on.

**Corollary 5.9.** *Let $U$ be an open subset of $\Lambda_r^{\llcorner}$. Then $\mathrm{Ad}_K(U)$ is open in the nilcone $\mathcal{N}^+$.*

*Proof.* As a consequence of Proposition 5.7, the map $\mathcal{N}^+ \to \exp i\mathcal{N}^+ \cdot x_0 \subset \Xi^+$, given by $X \to \exp iX \cdot x_0$, is a homeomorphism onto its (closed) image. Moreover, it follows that

$$\exp i\mathrm{Ad}_K U \cdot x_0 = G \exp iU \cdot x_0 \cap \exp i\mathcal{N}^+ \cdot x_0\,.$$

Thus, in order to prove the statement, it is sufficient to show that $G \exp iU \cdot x_0$ is open in $\Xi^+$.

For this note that $\Psi(U)$ is an open subset in the union $W_K(\mathfrak{a})^+ \cdot \overline{\mathfrak{a}}^+$ of closures of Weyl chambers of $\mathfrak{a}$. Thus $\mathrm{Ad}_K\Psi(U)$ is open in $\mathfrak{p}$ and consequently the set

$$\Big\{\mathrm{Ad}_K\big(-\frac{1}{2}(\Psi(U) + iJ_0\Psi(X))\big) :\ X \in U\Big\}$$

is open in $\mathfrak{p}^{0,1}$. Since the bundle map $G \times_K \mathfrak{p}^{0,1} \to \Xi^+$, given by $[g, Z] \to g \exp Z \cdot x_0$, a diffeomorphism, the set

$$V := \{G \exp\big(-\frac{1}{2}(\Psi(U) + iJ_0\Psi(X))\big) \cdot x_0 :\ X \in U\}$$

is open as well in $\Xi^+$. Finally, by Lemma 5.5 the set $G \exp iU \cdot x_0$ coincides with $V$. Hence it is open, as wished. □

## 6. An example.

In this section, we give a different proof of Proposition 5.7 in the case of $G = Sp(2,\mathbb{R})$ and $G = Sp(1,\mathbb{R}) \cong SL(2,\mathbb{R})$. This proof uses Corollary 5.3 and a global $G$-invariant function $f : \Xi^+ \to \mathbb{R}$, with the property that the map

$$\Lambda_2^{\llcorner} \to \mathbb{R}, \quad X \to f(\exp iX \cdot x_0)$$

is proper. As a matter of fact, the function $f$ is the restriction of a $G$-invariant function defined on all of $G^{\mathbb{C}}/K^{\mathbb{C}}$.

Consider the real symplectic group

$$G = Sp(r,\mathbb{R}) = \left\{ Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M^{2r\times 2r}(\mathbb{R}) \ : \ {}^tZJZ = J \right\}, \qquad J := \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$$

and its complexification $G^{\mathbb{C}} = Sp(r,\mathbb{C})$. By Witt's theorem, $G^{\mathbb{C}}$ acts transitively on the Grasmannian of $J$-isotropic complex $r$-planes in $\mathbb{C}^{2r}$

$$Y = \{ \mathbf{x} \text{ complex } r\text{-plane in } \mathbb{C}^{2r} \ : \ J|\mathbf{x} \times \mathbf{x} = 0 \}.$$

By considering all possible bases of $\mathbf{x}$, given as $r$-tuples of column vectors in $\mathbb{C}^{2r}$, we view $Y$ as the quotient of

$$\widehat{Y} := \left\{ \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \ : \ R_1,\, R_2 \in M^{r\times r}(\mathbb{C}),\ \operatorname{rank}\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = r,\ \begin{pmatrix} {}^tR_1 & {}^tR_2 \end{pmatrix} J \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = 0 \right\}$$

by the right action of $GL(r,\mathbb{C})$ defined by

$$M \cdot \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} := \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} M^{-1}, \qquad M \in GL(r,\mathbb{C}).$$

Note that $G^{\mathbb{C}}$ acts on $\widehat{Y}$ by left multiplication and that the canonical projection

$$\widehat{Y} \to Y, \qquad \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \to \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

is $G^{\mathbb{C}}$-equivariant.

Fix the base point $\mathbf{x}_+ = \begin{bmatrix} iI_r \\ I_r \end{bmatrix} \in Y$. Then $G \cdot \mathbf{x}_+ \cong G/K$, where

$$K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \ : \ A + iB \in U(n) \right\}.$$

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the associated Cartan decomposition of $\mathfrak{g}$, where

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \ : \ {}^tA = -A,\ {}^tB = B \right\}, \quad \mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \ : \ {}^tA = A,\ {}^tB = B \right\}.$$

The complex structure of $\mathfrak{p}$ is given by $J_0 := \operatorname{ad}_{Z_0}$, where $Z_0 = \frac{1}{2}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Under the action of $J_0$, the complexification $\mathfrak{p}^{\mathbb{C}}$ of $\mathfrak{p}$ decomposes as the direct sum of the $\pm i$-eigenspaces $\mathfrak{p}^{1,0} \oplus \mathfrak{p}^{0,1}$, namely

$$\mathfrak{p}^{1,0} = \left\{ \begin{pmatrix} Z & iZ \\ iZ & -Z \end{pmatrix} \ : \ {}^tZ = Z, \right\}, \quad \mathfrak{p}^{0,1} = \left\{ \begin{pmatrix} Z & -iZ \\ -iZ & -Z \end{pmatrix} \ : \ {}^tZ = Z, \right\}.$$

The flag manifold

$$Y = G^{\mathbb{C}} \cdot \mathbf{x}_+ \cong G^{\mathbb{C}}/Q, \quad \text{where} \quad Q = K^{\mathbb{C}} \exp \mathfrak{p}^{0,1},$$

is the compact dual symmetric space of $G/K$, and the complexification $G^{\mathbb{C}}/K^{\mathbb{C}}$ of $G/K$ can be realized as a dense open orbit in the product $Y \times \overline{Y}$

$$G^{\mathbb{C}}/K^{\mathbb{C}} \cong G^{\mathbb{C}} \cdot x_0 = \left\{ \left( \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \right) \in Y \times \overline{Y} \ : \begin{vmatrix} R_1 & \bar{S}_1 \\ R_2 & \bar{S}_2 \end{vmatrix} \neq 0 \right\},$$

where $x_0 = (\mathbf{x}_+, \mathbf{x}_+)$ (see [FHW05], p. 68).

Define two real $G$-invariant functions on $G^{\mathbb{C}}/K^{\mathbb{C}}$ as follows

$$f_1\left(\begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}\right) = \left\| \frac{\left| \begin{pmatrix} {}^tR_1 & {}^tR_2 \end{pmatrix} J \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \right|}{\begin{vmatrix} R_1 & \bar{S}_1 \\ R_2 & \bar{S}_2 \end{vmatrix}} \right\|^2$$

$$f_2\left(\begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}\right) = \frac{\left| \begin{pmatrix} {}^tR_1 & {}^tR_2 \end{pmatrix} J \begin{pmatrix} \bar{R}_1 \\ \bar{R}_2 \end{pmatrix} \right| \left| \begin{pmatrix} {}^tS_1 & {}^tS_2 \end{pmatrix} J \begin{pmatrix} \bar{S}_1 \\ \bar{S}_2 \end{pmatrix} \right|}{\left\| \begin{vmatrix} R_1 & \bar{S}_1 \\ R_2 & \bar{S}_2 \end{vmatrix} \right\|^2}.$$

A simple computation shows that for

$$X = \begin{pmatrix} 0 & \cdots & 0 & x_1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & x_r \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \in \Lambda_r,$$

one has

$$f_1(\exp iX \cdot x_0) = (1 - x_1^2) \ldots (1 - x_r^2) \qquad \text{and} \qquad f_2(\exp iX \cdot x_0) = x_1^2 \ldots x_r^2.$$

For $r = 2$, define the $G$-invariant function $f := 1 - f_1 + f_2$ on $G^{\mathbb{C}}/K^{\mathbb{C}}$. Then, by restricting it to $\exp i\Lambda_2 \cdot x_0$, one obtains a map

$$\Lambda_2 \to \mathbb{R}, \qquad X = x_1 E_1 + x_2 E_2 \to f(\exp iX \cdot x_0) = x_1^2 + x_2^2.$$

which is an exhaustion function on $\Lambda_r^{\llcorner}$. This fact, together with Corollary 5.3, yields a different proof of Proposition 5.7 for $G = Sp(2, \mathbb{R})$.

A similar proof works for $G = SL(2, \mathbb{R}) = Sp(1, \mathbb{R})$, using the global $G$-invariant function $f_2$.

It would be interesting to obtain a similar global $G$-invariant function on $G^{\mathbb{C}}/K^{\mathbb{C}}$ in the higher rank case and in general for all Hermitian symmetric spaces. In the case of $Sp(r, \mathbb{R})$, for $r \geq 3$, we know no global $G$-invariant functions whose restrictions to $\exp(i\Lambda_r) \cdot x_0$ define other linearly independent symmetric functions in the ring $\mathbb{R}[x_1^2, \ldots, x_r^2]$. Note that, as a consequence of Proposition 5.7, every symmetric function in $\mathbb{R}[x_1^2, \ldots, x_r^2]$ extends continuosly and $G$-equivariantly at least to $\Xi^+ \cup \Xi^-$.

## 7. Orbit structure of $\Xi^+$.

By the results of the previous section, the map

$$\psi \colon G \times_K \mathcal{N}^+ \to \Xi^+, \qquad [g, X] \to g \exp iX \cdot x_0$$

is a $G$-equivariant homeomorphism. Hence, every $G$-orbit in $\Xi^+$ meets $\exp i\mathcal{N}^+ \cdot x_0$ in a $K$-orbit and the $G$-orbit structure of $\Xi^+$ is completely determined by the $K$-orbit structure of the nilpotent cone $\mathcal{N}^+ = \mathrm{Ad}_K \Lambda_r^{\llcorner}$. In this section we give further details.

**Corollary 7.1.** *Let $X$ be an element in $\Lambda_r^{\llcorner}$, and let $\exp iX \cdot x_0$ be the corresponding point in $\Xi^+$. Then*

$$G_{\exp iX\cdot x_0} = Z_K(X) = Z_K([\theta X, X])\,.$$

*Proof.* Since $\exp iX \cdot x_0 = \psi([e,X])$, by Proposition 5.7 one has

$$G_{\exp iX\cdot x_0} = G_{[e,X]} = Z_K(X)\,,$$

which proves the first equality.

To prove the second equality, write $X = \sum x_j E_j$, with $x_j \geq 0$, for all $j$. It is clear that

$$\Psi(X) := \sum_j x_j A_j \quad \text{and} \quad [\theta X, X] = \sum_j x_j^2 A_j$$

belong to the same orbit stratum in $\mathfrak{a}^{\llcorner}$. In particular, $Z_K(\Psi(X)) = Z_K([\theta X, X])$. Since $Z_K(X) = Z_K(\Psi(X))$ (by (i) of Lemma 5.6), the rest of the statement follows. $\square$

The abelian subspace $\mathfrak{a}$ is a slice for the Adjoint action of $K$ on $\mathfrak{p}$. The generic elements in $\mathfrak{a}$ are those lying on maximal dimensional $\mathrm{Ad}_K$-orbits, i.e.

$$\mathfrak{a}_{gen} = \{H \in \mathfrak{a} \;:\; Z_K(H) = Z_K(\mathfrak{a})\}.$$

At Lie algebra level, one has

$$Z_{\mathfrak{k}}(H) = \mathfrak{a} \oplus Z_{\mathfrak{k}}(\mathfrak{a}) \oplus \bigoplus_{\alpha(H)=0} \mathfrak{g}[\alpha]_{\mathfrak{k}},$$

where $\mathfrak{g}[\alpha]_{\mathfrak{k}}$ is the $\mathfrak{k}$-component of the $\theta$-stable subspace $\mathfrak{g}[\alpha] = \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{-\alpha}$ of $\mathfrak{g}$. The fact that $\Delta(\mathfrak{g}, \mathfrak{a})$ is either of type $C_r$ or $BC_r$, implies that

$$\mathfrak{a}_{gen} = \Big\{ \sum_j a_j A_j \;:\; a_j \neq 0 \text{ and } a_j \neq \pm a_l, \text{ for } j, l = 1, \ldots, r \text{ and } j \neq l \Big\}\,. \qquad (17)$$

Since $\Lambda_r^{\llcorner}$ is a slice for the $\mathrm{Ad}_K$-action on $\mathcal{N}^+$, we define generic elements in $\Lambda_r^{\llcorner}$ in a similar way.

**Definition 7.2.** *An element $X \in \Lambda_r^{\llcorner}$ is generic if $Z_K(X) = Z_K(\Lambda_r^{\llcorner})$. The set of generic elements in $\Lambda_r^{\llcorner}$ is denoted by $(\Lambda_r^{\llcorner})_{gen}$.*

**Lemma 7.3.** *An element $X$ in $\Lambda_r^{\llcorner}$ is generic if and only if $\Psi(X) = [Z_0, X - \theta X]$ (resp. $[\theta X, X]$) is generic in $\mathfrak{a}$. In particular the set $(\Lambda_r^{\llcorner})_{gen}$ is given by*

$$(\Lambda_r^{\llcorner})_{gen} = \{\sum_j x_j E_j \;:\; x_j \neq 0 \text{ and } x_j \neq x_l, \text{ for } j = 1, \ldots, r \text{ and } j \neq l\},$$

*and is dense in $\Lambda_r^{\llcorner}$.*

*Proof.* Write $X = \sum_j x_j E_j$, with $x_j \geq 0$, for all $j$. We already observed that $\Psi(X)$ and $[\theta X, X]$ lie in the same orbit stratum in $\mathfrak{a}$. Moreover, $Z_K(X) = Z_K(\Psi(X))$, by (i) of Lemma 5.6, and $Z_K(\Lambda_r) = Z_K(\Lambda_r^{\llcorner}) = Z_K(\mathfrak{a})$, by Lemma 4.1. From (17) it follows that $X$ is generic if and only if $x_j \neq 0$ and $x_j \neq x_l$, for $j, l = 1, \ldots, r$ and $j \neq l$, as claimed $\square$

**Lemma 7.4.** *Let $X \in \Lambda_r^{\llcorner}$ and $k \in K$ be elements such that $\mathrm{Ad}_k X \in \Lambda_r$. Then*
(i) $\mathrm{Ad}_k X$ *lies in $\Lambda_r^{\llcorner}$, implying that $\mathcal{N}^+ \cap \Lambda_r = \Lambda_r^{\llcorner}$,*

(ii) *there exists* $n \in N_K(\Lambda_r)$ *such that* $\mathrm{Ad}_k X = \mathrm{Ad}_n X$.
*In other words, the intersection* $\mathrm{Ad}_K X \cap \Lambda_r$*, of the* $\mathrm{Ad}_K$*-orbit of* $X$ *with* $\Lambda_r$*, is given by the* $W_K(\Lambda_r)$*-orbit of* $X$ *in* $\Lambda_r^{\llcorner}$.

*Proof.* (i) We first consider the case when $k$ is an element of $N_K(\mathfrak{a})$ and we set $n := k$. Then $\mathrm{Ad}_n$ acts on $\mathfrak{a}$ by signed permutations of the $A_j$.

*Claim.* If for some indices $i, h \in \{1, \dots, r\}$ one has $\mathrm{Ad}_n(A_i) = A_h$, then $\mathrm{Ad}_n(E_i) \in \mathfrak{g}^{\lambda_h}$; if $\mathrm{Ad}_n(A_i) = -A_h$, then $\mathrm{Ad}_n(E_i) \in \mathfrak{g}^{-\lambda_h}$.
*Proof of the claim.* From $[A_i, E_i] = 2E_i$, by applying $\mathrm{Ad}_n$ to both terms of the equation we obtain

$$[\mathrm{Ad}_n A_i, \mathrm{Ad}_n E_i] = [A_h, \mathrm{Ad}_n E_i] = 2\mathrm{Ad}_n E_i.$$

Then, in order to show that $\mathrm{Ad}_n E_i \in \mathfrak{g}^{\lambda_h}$, we need to show that $[A_l, \mathrm{Ad}_n E_i] = 0$, for all $l \neq h$. Write

$$[A_l, \mathrm{Ad}_n E_i] = \mathrm{Ad}_n[\mathrm{Ad}_{n^{-1}} A_l, E_i]$$

and observe that $\mathrm{Ad}_{n^{-1}} A_l \in \{\pm A_m\}$, for some $m \neq i$. Then

$$\mathrm{Ad}_n[\mathrm{Ad}_{n^{-1}} A_l, E_i] = \mathrm{Ad}_n[\pm A_m, E_i] = 0,$$

as desired. A similar argument shows the second statement, and concludes the proof of the claim.

Write $X = \sum x_j E_j$, with $x_j \geq 0$, and $\mathrm{Ad}_n X = \sum y_j E_j$, with $y_j \in \mathbb{R}$. Then $\Psi(X) = \sum x_j A_j$ and, since $\Psi$ is $\mathrm{Ad}_K$-equivariant, one has

$$\mathrm{Ad}_n(\Psi(X)) = \sum x_j \mathrm{Ad}_n A_j = \Psi(\mathrm{Ad}_n X) = \sum y_j A_j\,.$$

Thus, given $i \in \{1, \dots, r\}$, one has $y_h = x_i \geq 0$, if $\mathrm{Ad}_n A_i = A_h$, and $y_h = -x_i \leq 0$, if $\mathrm{Ad}_n A_i = -A_h$. In order to show that $\mathrm{Ad}_n X = \sum y_j E_j$ lies in $\Lambda_r^{\llcorner}$, we prove that $x_i = 0$ whenever $\mathrm{Ad}_n A_i = -A_h$.

Assume by contradiction that this is not the case. By the above claim, each $\mathrm{Ad}_n E_j$ lies in one of the root spaces of the direct sum $\Lambda_r \oplus \theta\Lambda_r = \bigoplus_j \mathfrak{g}^{\lambda_j} \oplus \mathfrak{g}^{-\lambda_j}$. Moreover, $\mathrm{Ad}_n X = \sum x_j \mathrm{Ad}_n E_j$ has a non-zero component in $\mathfrak{g}^{-\lambda_h}$. This contradicts the fact that $\mathrm{Ad}_n X$ lies in $\Lambda_r$ and concludes the case when $k = n$ is an element of $N_K(\mathfrak{a})$.

Next, the general case. Both elements $\Psi(X)$ and $\Psi(\mathrm{Ad}_k X) = \mathrm{Ad}_k(\Psi(X))$ belong to $\mathfrak{a}$ and, by [Kna04], Lemma 7.38, p.459, there exists an element $n \in N_K(\mathfrak{a})$ such that

$$\mathrm{Ad}_k(\Psi(X)) = \mathrm{Ad}_n(\Psi(X)).$$

Thus $n^{-1}k$ lies in $Z_K(\Psi(X))$ and also in $Z_K(X)$, by (i) of Lemma 5.6. Therefore

$$\mathrm{Ad}_k X = \mathrm{Ad}_n X.$$

Since we already showed that $\mathrm{Ad}_n X$ belongs to $\Lambda_r^{\llcorner}$, the proof of (i) is now complete.
(ii) We first consider the case of a generic element $X$ in $\Lambda_r^{\llcorner}$. By Lemma 7.3, both $\Psi(X) = \sum x_j A_j$ and $\mathrm{Ad}_k(\Psi(X))$ are generic in $\mathfrak{a}$, implying that $k \in N_K(\mathfrak{a})$. We need to show that $k \in N_K(\Lambda_r)$.

Assume by contradiction that this is not the case. Then, by (iii) of Lemma 4.1, there exist $i$ and $h$ in $\{1, \dots, r\}$ such that $\mathrm{Ad}_k A_i = -A_h$. By the claim contained in the proof if part (i), each $\mathrm{Ad}_k E_j$ lies in one of the root spaces of $\Lambda_r \oplus \theta\Lambda_r$ and $\mathrm{Ad}_k E_i \in \mathfrak{g}^{-\lambda_h}$. Since Lemma 7.3 implies that all $x_j$ are strictly positive, $\mathrm{Ad}_k X = \sum x_j \mathrm{Ad}_k E_j$ has a non-zero component in $\mathfrak{g}^{-\lambda_h}$. This contradicts the fact that $\mathrm{Ad}_k X$ lies in $\Lambda_r$. Therefore $k \in N_K(\Lambda_r)$, as wished.

Now let $X$ be an arbitrary element in $\Lambda_r^{\llcorner}$. By (i) we know that $\mathrm{Ad}_k X \in \Lambda_r^{\llcorner}$. Choose fundamental systems of open neighborhoods $\{U_X^m\}_{m\in\mathbb{N}}$ and $\{U_{\mathrm{Ad}_k X}^m\}_{m\in\mathbb{N}}$ of $X$ and $\mathrm{Ad}_k X$ in $\Lambda_r^{\llcorner}$, respectively. By Corollary 5.9, the sets $\mathrm{Ad}_K U_X^m$ and $\mathrm{Ad}_K U_{\mathrm{Ad}_k X}^m$

are open in $\mathcal{N}^+$. By considering intersections if necessary, we may assume that $\mathrm{Ad}_K U_X^m = \mathrm{Ad}_K U^m_{\mathrm{Ad}_k X}$, for all $m \in \mathbb{N}$.

For each $m \in \mathbb{N}$ choose an element $X_m$ in $(\Lambda_r^\llcorner)^{gen} \cap U_X^m$. Then there exists $k_m \in K$ such that $\mathrm{Ad}_{k_m} X_m \in U^m_{\mathrm{Ad}_k X}$. By construction $X_m \to X$ and $\mathrm{Ad}_{k_m} X_m \to \mathrm{Ad}_k X$. Moreover, by the first part of the proof of (ii), there exists elements $n_m \in N_K(\Lambda_r)$ such that $\mathrm{Ad}_{k_m} X_m = \mathrm{Ad}_{n_m} X_m$. Being $N_K(\Lambda_r)$ compact, we may assume that $n_m \to n \in N_K(\Lambda_r)$. Thus

$$\mathrm{Ad}_k X = \lim_m \mathrm{Ad}_{k_m} X_m = \lim_m \mathrm{Ad}_{n_m} X_m = \mathrm{Ad}_n X\,,$$

with $n \in N_K(\Lambda_r)$, as wished. □

By Lemma 4.1 the closure $\overline{(\Lambda_r^\llcorner)}^+$ of the open chamber

$$(\Lambda_r^\llcorner)^+ := \{x_1 E_1 + \cdots + x_r E_r \ : \ x_1 > x_2 > \cdots > x_r > 0\}$$

is a perfect slice for the $W_K(\Lambda_r)$-action on $\Lambda_r^\llcorner$.

**Corollary 7.5.**
(i) *The closure* $\overline{(\Lambda_r^\llcorner)}^+$ *of the open chamber* $(\Lambda_r^\llcorner)^+$ *is a perfect slice for the* $\mathrm{Ad}_K$*-action on* $\mathcal{N}^+$.
(ii) *For* $X \in \Lambda_r^\llcorner$ *one has*

$$G \exp iX \cdot x_0 \bigcap \exp i\Lambda_r^\llcorner \cdot x_0 = \exp i(W_K(\Lambda_r) \cdot X) \cdot x_0$$

(iii) *There are homeomorphisms of orbit spaces*

$$\Xi^+/G \cong \Lambda_r^\llcorner / W_K(\Lambda_r) \cong \overline{(\Lambda_r^\llcorner)}^+ .$$

*Proof.* Part (i) follows from (ii) of Lemma 7.4. For parts (ii) and (iii), Lemma 7.4 implies that every $G$-orbit in $G \times_K \mathcal{N}^+$ intersects $\Lambda_r^\llcorner \cong \{[e, X] \in G \times_K \mathcal{N}^+ \ : \ X \in \Lambda_r^\llcorner\}$ in a $W_K(\Lambda_r)$ orbit. Since by Proposition 5.7, the map $G \times_K \mathcal{N}^+ \to \Xi^+$, given by $[g, X] \to g \exp iX$, is a $G$-equivariant homeomorphism, the statements follow. □

**Remark.** Observe that inside $\Xi^+$ there is a proper inclusion

$$\exp i\Lambda_r^\llcorner \cdot x_0 \ \subset \ \Xi^+ \cap \exp i\Lambda_r \cdot x_0,$$

and that

$$\{X \in \Lambda_r \ : \ \exp iX \cdot x_0 \in \Xi^+\} = \bigoplus_{j=1}^{r} (-1, \infty) E_j$$

(cf. [Kro08], p. 286). In fact, there exist elements $X \in \Lambda_r^\llcorner$, $Y \in \Lambda_r \setminus \Lambda_r^\llcorner$ and $g \in G \setminus K$ such that

$$g \exp iX \cdot x_0 = \exp iY \cdot x_0.$$

For example, for $G/K = SL(2,\mathbb{R})/SO(2,\mathbb{R})$, take $-1 < x < 1$ and $b := \sqrt{1 - x^2}$. Then $\begin{pmatrix} 0 & b \\ -1/b & 0 \end{pmatrix} \in G$ and $\begin{pmatrix} -ix/b & 1/b \\ -1/b & -ix/b \end{pmatrix} \in SO(2,\mathbb{C})$; moreover the following relation holds

$$\begin{pmatrix} 0 & b \\ -1/b & 0 \end{pmatrix} \begin{pmatrix} 1 & ix \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -ix \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -ix/b & 1/b \\ -1/b & -ix/b \end{pmatrix} .$$

This shows that the elements

$$\exp i \begin{pmatrix} 0 & -x \\ 0 & 0 \end{pmatrix} \cdot x_0 \quad \text{and} \quad \exp i \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \cdot x_0$$

lie on the same $G$-orbit in $\Xi^+$, even though not on the same $K$-orbit.

On the subdomains

$$(-1,\infty)E_1 \oplus \cdots \oplus (-1,1)E_{\bar{j}} \oplus \cdots \oplus (-1,\infty)E_r\,,$$

which are defined for $\bar{j} \in \{1,\ldots,r\}$, one has additional symmetries which identify different elements on the same $G$-orbit in $\Xi^+$. Namely, for $-1 < x_{\bar{j}} < 1$, let $g_{\bar{j}}$ be the image of the element

$$\begin{pmatrix} 0 & \sqrt{1-x_{\bar{j}}^2} \\ -1/\sqrt{1-x_{\bar{j}}^2} & 0 \end{pmatrix}$$

in the $SL(2,\mathbb{R})$-subgroup of $G$ generated by the $\mathfrak{sl}(2)$-triple $\{E_{\bar{j}}, \theta E_{\bar{j}}, A_{\bar{j}}\}$. Then

$$g_{\bar{j}} \exp i(x_1E_1+\cdots+x_{\bar{j}}E_{\bar{j}}+\cdots+x_rE_r)\cdot x_0 = \exp i(x_1E_1+\cdots-x_{\bar{j}}E_{\bar{j}}+\cdots+x_rE_r)\cdot x_0\,.$$

Thus inside the $\bar{j}^{th}$ subdomain of $\Lambda_r$ defined as above, the elements $X$ and $r_{\bar{j}}(X)$, with $r_{\bar{j}}$ the reflection with respect to the $\bar{j}^{th}$ coordinate plane, are mapped into each other by $g_{\bar{j}}$. Therefore they lie on the same $G$-orbit, even though not on the same $K$-orbit.

## 8. The domain $\Xi^+$ and its distinguished Stein subdomains.

Let $G/K$ be a rank-one Hermitian symmetric space. In [GeIa08] it was shown that, beside the crown $\Xi$, the domain $\Xi^+$ contains another distinguished $G$-invariant subdomain with the peculiarity that its boundary contains no principal orbits of $G^{\mathbb{C}}/K^{\mathbb{C}}$ (i.e. closed $G$-orbits of maximal dimension).

In the tube case $SL(2,\mathbb{R})/SO(2,\mathbb{R})$, such a subdomain $S^+$ arises from the compactly causal structure of a symmetric $G$-orbit in the semisimple boundary $\partial_s\Xi$ of the crown and it is Stein. It also turns out that every Stein, invariant, proper subdomain of $\Xi^+$ is either contained in $\Xi$ or in $S^+$. In the non-tube case $SU(n,1)/U(n)$, for $n>1$, such a subdomain $\Omega^+$ arises from the compactly causal structure of the orbit of a proper subgroup of $G$ in $\partial_s\Xi$. The domain $\Omega^+$ is not Stein and contains no invariant Stein subdomains. In this case, every Stein, invariant, proper subdomain of $\Xi^+$ is contained in $\Xi$.

The purpose of this section is to prove that the domains $S^+$ and $\Omega^+$ have higher rank analogues, which are contained in $\Xi^+$. Since the proofs rely on the rank-one reduction, we recall the rank-one case in detail.

8.1. **The rank-one case.** We begin with the *tube-case* $G/K = SL(2,\mathbb{R})/SO(2,\mathbb{R})$. Fix the $\mathfrak{sl}(2,\mathbb{R})$-triple

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \theta E = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{18}$$

normalized as in (5), and the complex structure $J_0 = \mathrm{ad}_{Z_0}$ determined by the element $Z_0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in Z(\mathfrak{k})$. In [Kro08] and [KrOp08] the crown $\Xi$ and the domain $\Xi^+$ were described as follows

$$\Xi = G\exp i(-1,1)E\cdot x_0 = G\exp i[0,1)E\cdot x_0,$$
$$\Xi^+ = G\exp i(-1,\infty)E\cdot x_0 = G\exp i[0,\infty)E\cdot x_0,$$

where $x_0 = (eQ, eQ)$ (see Section 3). Set $\mathfrak{a} = \mathbb{R}A$ and define

$$g_1 := \exp(i\frac{\pi}{2}\frac{A}{2}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} \in \exp i\mathfrak{a}, \tag{19}$$

where $\frac{1}{2}A$ is the dual root of $\alpha$ in $\mathfrak{a}$. Since $\alpha(\frac{\pi}{2}\frac{A}{2}) = \frac{\pi}{2}$, the point $x_1 := g_1 \cdot x_0$ lies on the semisimple boundary of $\Xi$. The orbit $G \cdot x_1$ is diffeomorphic to the symmetric space of Cayley type $G/H = SL(2,\mathbb{R})/SO(1,1)$ (both compactly and non-compactly causal), with involution $\tau = \mathrm{Ad}_{g_1^2}\theta$ (see [GeIa08], Lemma 4.3). The associated symmetric algebra is given by

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \qquad \mathfrak{h} = \mathbb{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathfrak{q} = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The abelian subspace $\mathfrak{a}$ lies in $\mathfrak{q} \cap \mathfrak{p}$, and the triple $\{E, \theta E, A\}$ satisfies the further condition $\theta E = -\tau E$. Set $T := E + \theta E$. Then

$$Z_0 = \frac{1}{2}T$$

and $\mathfrak{c} = \mathbb{R}T$ is a compact Cartan subspace in $\mathfrak{q} \cap \mathfrak{k}$. Since $G/H$ is a compactly causal symmetric space of *rank-one*, there exist precisely two proper, open, convex, $\mathrm{Ad}_H$-invariant, elliptic cones $W^\pm$ in $\mathfrak{q}$, intersecting $\mathfrak{c}$ in the open halflines $\pm(0,\infty)T$, and satisfying $\overline{W^\pm_{min}} = \pm\overline{conv\,(\mathrm{Ad}_H(\mathbb{R}^+ Z_0))}$. Define

$$S^+ := G \exp iW^+ \cdot x_1 = G \exp i(0,\infty)T \cdot x_1.$$

Since the isotropy subgroup of $x_1$ in $G^{\mathbb{C}}$ is given by $H^{\mathbb{C}} := g_1 K^{\mathbb{C}} {g_1}^{-1}$, the map

$$G^{\mathbb{C}}/H^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}, \quad gH^{\mathbb{C}} \to gg_1K^{\mathbb{C}},$$

is a $G^{\mathbb{C}}$-equivariant biholomorphism. Moreover $G \exp iW^+ H^{\mathbb{C}}/H^{\mathbb{C}}$ is a Stein domain in $G^{\mathbb{C}}/H^{\mathbb{C}}$ ([Nee99], Thm. 3.5, p. 205). Consequently $S^+$ is a Stein, $G$-invariant domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$ with the orbit $G \cdot x_1$ in its boundary.

In the next lemma we show that $\Xi^+$ contains both the crown $\Xi$ and the domain $S^+$. An analogous computation was carried out in [KrOp08], Sect. 3.2, for the crown domain using the hyperbolic model $SO_0(1,2,\mathbb{C})/SO(2,\mathbb{C})$.

**Lemma 8.1.** *Set* $k_0 = \exp\frac{\pi}{4}T$.

(i) *For* $t \in (-\pi/4, \pi/4)$ *define* $a_1(t) = \exp\frac{1}{\sqrt{\cos 2t}}A$. *One has*

$$\exp itA \cdot x_0 = k_0 a_1(t) \exp i \sin 2tE \cdot x_0\,. \tag{20}$$

*In particular* $\exp itA \cdot x_0 \in G \exp i \sin 2tE \cdot x_0$ *and*

$$\Xi = G \exp i[0,1)E \cdot x_0.$$

(ii) *For* $t \in (0,\infty)$ *define* $a_2(t) = \exp\frac{1}{\sqrt{\sinh 2t}}A$. *One has*

$$\exp itT\, g_1 \cdot x_0 = k_0 a_2(t) \exp i \cosh 2tE \cdot x_0\,. \tag{21}$$

*In particular* $\exp itT\, g_1 \cdot x_0 \in G \exp i \cosh 2tE \cdot x_0$ *and*

$$S^+ = G \exp i(1,\infty)E \cdot x_0.$$

*Proof.* Part $(i)$ follows by showing that

$$\exp itA = k_0 a_1(t) \exp i \sin 2tE\, k,$$

for some $k \in SO(2,\mathbb{C})$. The proof is a simple matrix computation with

$$\exp itA = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, \quad k_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad a_1(t) = \begin{pmatrix} \frac{1}{\sqrt{\cos 2t}} & 0 \\ 0 & \sqrt{\cos 2t} \end{pmatrix}$$

$$\exp i \sin 2tE = \begin{pmatrix} 1 & i\sin 2t \\ 0 & 1 \end{pmatrix}, \quad k = \frac{1}{\sqrt{2\cos 2t}}\begin{pmatrix} e^{-it} & -e^{it} \\ e^{it} & e^{-it} \end{pmatrix}.$$

The second equality follows directly from equation (20) and the definition of $\Xi$.

Similarly, part $(ii)$ follows by showing that

$$k = g_1^{-1}\,(\exp itT)^{-1}\,k_0\,a_2(t)\,\exp i\cosh 2tE$$

is an element of $SO(2,\mathbb{C})$. The proof is a simple matrix computation with

$$g_1^{-1} = \begin{pmatrix} \frac{1-i}{\sqrt{2}} & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{pmatrix}, \quad (\exp itT)^{-1} = \begin{pmatrix} \cosh t & -i\sinh t \\ i\sinh t & \cosh t \end{pmatrix}, \quad k_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$a_2(t) = \begin{pmatrix} \frac{1}{\sqrt{\sinh 2t}} & 0 \\ 0 & \sqrt{\sinh 2t} \end{pmatrix}, \quad \exp i\cosh 2tE = \begin{pmatrix} 1 & i\cosh 2t \\ 0 & 1 \end{pmatrix}.$$

The final part of the statement follows from equation (21) and the definition of $S^+$. □

In Example 6.3 in [GeIa08] it is shown that the orbit $G\cdot w$ of the point $w := \exp iE\cdot x_0$ is a real hypersurface in $\Xi^+$, lying in the common boundary of $\Xi$ and $S^+$ inside $\Xi^+$ and having $G\cdot x_1$ in its closure. This fact together with Lemma 8.1 yields the following description of $\Xi^+$.

**Proposition 8.2.** *The domain $\Xi^+$ in $SL(2,\mathbb{C})/SO(2,\mathbb{C})$ is given by*

$$\Xi^+ = G\exp i[0,\infty)E\cdot x_0 = \Xi\cup G\cdot w\cup S^+,$$

*where $G\cdot w$ is a hypersurface orbit lying in the common boundary of $\Xi$ and $S^+$.*

In the *non-tube case* $SU(n,1)/U(n)$, for $n>1$, an analogue of Proposition 8.2 holds true. Define $x_1 = g_1\cdot x_0$, where $g_1 = \exp(i\frac{\pi}{2}\frac{A}{2})$ and $\alpha(A)=1$. Since $\alpha(\frac{\pi}{2}\frac{A}{2}) = \frac{\pi}{4}$ and $2\alpha(\frac{\pi}{2}\frac{A}{2}) = \frac{\pi}{2}$, the point $x_1$ lies on the semisimple boundary of the crown. In Example 6.3 in [GeIa08], one can see that the orbit $G\cdot x_1$ is a homogeneous space of dimension $\dim_{\mathbb{R}} G\cdot x_1 = 2(2n-1)$ and that it is not a $G$-symmetric space. The group $\widehat{G} := Z_G(g_1^4)$ is a proper subgroup of $G$ and the orbit $\widehat{G}\cdot x_1 \subset G\cdot x_1$ is a symmetric space diffeomorphic to $SU(1,1)/SO(1,1)\cong SL(2,\mathbb{R})/SO(1,1)$, embedded in $G^{\mathbb{C}}/K^{\mathbb{C}}$ as a totally real submanifold. The isotropy subgroups of $x_1$ in $G$ and in $\widehat{G}$ coincide and the slice representation at $x_1$ is equivalent to the isotropy representation of $\widehat{G}\cdot x_1$. This can be seen most clearly at Lie algebra level. Consider the restricted root decomposition of $\mathfrak{g} = \mathfrak{su}(n,1)$

$$\mathfrak{g} = Z_{\mathfrak{k}}(\mathfrak{a})\oplus\mathfrak{a}\oplus\mathfrak{g}^{\alpha}\oplus\mathfrak{g}^{-\alpha}\oplus\mathfrak{g}^{2\alpha}\oplus\mathfrak{g}^{-2\alpha},$$

and denote by $\mathfrak{su}(1,1)_{2\alpha}$ the 3-dimensional Lie subalgebra spanned by the vectors $A\in\mathfrak{a}$, $E\in\mathfrak{g}^{2\alpha}$ and $\theta E\in\mathfrak{g}^{-2\alpha}$, normalized as in (5). Then the Lie algebra of $\widehat{G}$ and the isotropy subalgebra at $x_1$ are given by

$$\widehat{\mathfrak{g}} = Z_{\mathfrak{k}}(\mathfrak{a})\oplus\mathfrak{su}(1,1)_{2\alpha} \quad\text{and}\quad \mathfrak{g}_{x_1} = \widehat{\mathfrak{g}}_{x_1} = Z_{\mathfrak{k}}(\mathfrak{a})\oplus\mathbb{R}(E-\theta E),$$

respectively. The tangent space to the orbit $G\cdot x_1$

$$T_{x_1}(G\cdot x_1)\cong\mathfrak{g}^{\alpha}\oplus\mathfrak{g}^{-\alpha}\oplus\mathbb{R}\mathfrak{a}\oplus\mathbb{R}(E+\theta E)$$

contains the $\mathrm{Ad}_{G_{x_1}}$-invariant subspace

$$T_{x_1}(\widehat{G}\cdot x_1)\cong\mathbb{R}\mathfrak{a}\oplus\mathbb{R}(E+\theta E),$$

which is isomorphic to the tangent space of the Cayley type symmetric space $SL(2,\mathbb{R})/SO(1,1)$ endowed with the isotropy action. Moreover multiplication by $i$ defines an equivariant isomorphism onto the slice representation at $x_1$. Recall that by Lemma 2.4 the element $Z_0\in Z(\mathfrak{k})$ defining the complex structure of $G/K$ can be written as $Z_0 = S+T_0$, where $S\in Z_K(\mathfrak{a})$ and $T_0 = \frac{1}{2}(E+\theta E)$. Denote by $W^+$

the maximal proper, open, convex, $\mathrm{Ad}_{G_{x_1}}$-invariant, elliptic cone in $T_{x_1}(\widehat{G}\cdot x_1)$, satisfying $\overline{W^+} = \overline{conv\left(\mathrm{Ad}_{G_{x_1}}(\mathbb{R}^+T_0)\right)}$. Then

$$\Omega^+ = G\exp iW^+\cdot x_1 = G\,(\exp i(0,\infty)T_0)\,g_1\cdot x_0$$

is an open $G$-invariant domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$.

In Example 4.7 and Example 6.3 in [GeIa08] it was shown that the orbit $G\cdot w$ of the point $w := \exp iE\cdot x_0$, is a real hypersurface in $\Xi^+$, lying in the common boundary of $\Xi$ and $\Omega^+$ and having $G\cdot x_1$ in its closure.

**Proposition 8.3.** *The domain $\Xi^+$ in $SL(n+1,\mathbb{C})/GL(n,\mathbb{C})$ is given by*

$$\Xi^+ = G\exp i[0,\infty)E\cdot x_0 = \Xi\cup G\cdot w\cup\Omega^+,$$

*where $G\cdot w$ is a hypersurface orbit lying in the common boundary of $\Xi$ and $\Omega^+$.*

Like the domain $S^+$ in the $SL(2,\mathbb{R})$-case, the domain $\Omega^+$ has the peculiarity that its boundary $\partial\Omega^+$ consists of non-principal $G$-orbits in $G^{\mathbb{C}}/K^{\mathbb{C}}$. But unlike $S^+$, the domain $\Omega^+$ is not Stein and contains no $G$-invariant Stein subdomains (see [GeIa08], Ex. 6.3).

8.2. **The higher rank case.** Let $G/K$ be a Hermitian symmetric space of rank $r>1$. Denote by $\{\omega_1,\ldots,\omega_r\}$ the dual basis of the simple roots $\{\alpha_1,\ldots,\alpha_r\}$. Define

$$g_1 := \exp(i\frac{\pi}{2}\frac{\omega_r}{k_r}) \in \exp i\mathfrak{a}\,, \tag{22}$$

where $k_r$ is the coefficient of the $r$-th simple restricted root $\alpha_r$ in the highest root $\alpha_h\in\Delta(\mathfrak{g},\mathfrak{a})^+$. If $G/K$ is of tube type, then $\Delta(\mathfrak{g},\mathfrak{a})$ is of type $C_r$ and the highest root is given by $\alpha_h = 2\alpha_1+\ldots+2\alpha_{r-1}+\alpha_r$. Hence $k_r=1$ and $g_1=\exp(i\frac{\pi}{2}\omega_r)$. If $G/K$ is not of tube type, then $\Delta(\mathfrak{g},\mathfrak{a})$ is of type $BC_r$ and $\alpha_h = 2\alpha_1+\ldots+2\alpha_r$. Hence $k_r=2$ and $g_1=\exp(i\frac{\pi}{2}\frac{\omega_r}{2})$.

In both cases $|\alpha(\frac{\pi}{2}\frac{\omega_r}{k_r})|\le\frac{\pi}{2}$, for all restricted roots $\alpha$, and $|\lambda_r(\frac{\pi}{2}\frac{\omega_r}{k_r})| = \frac{\pi}{2}$, where $\lambda_r$ is as in (3). This shows that $x_1 = g_1\cdot x_0$ is a point on the semisimple boundary of the crown domain. For $j=1,\ldots,r$, define

$$g_{1,j} := \exp(i\frac{\pi}{2}\frac{A_j}{2}),$$

where $A_j$ is as in (4). The element $g_{1,j}$ lies in the $SL(2,\mathbb{C})$-subgroup of $G^{\mathbb{C}}$ corresponding to the $j^{\text{th}}$ triple defined in (4).

**Lemma 8.4.** *One has*

$$\omega_r = \frac{1}{2}(A_1+A_2+\ldots+A_r), \qquad \text{in the tube case,}$$

$$\omega_r = A_1+A_2+\ldots+A_r, \qquad \text{in the non-tube case.}$$

*As a consequence, the following identity holds*

$$g_1 = \prod_{j=1}^{r} g_{1,j}.$$

*Proof.* In the tube case, (1) and the relations $\lambda_i(\frac{1}{2}A_j) = \delta_{ij}$, imply that $\alpha_j(\frac{1}{2}(A_1 + A_2 + \ldots + A_r)) = \delta_{jr}$, for $j = 1, \ldots, r$. Therefore $\omega_r = \frac{1}{2}(A_1 + A_2 + \ldots + A_r)$.
In the non-tube case, (2) and the relations $\lambda_i(\frac{1}{2}A_j) = \delta_{ij}$ imply that $\alpha_j(A_1 + A_2 + \ldots + A_r) = \delta_{jr}$, for $j = 1, \ldots, r$. Thus $\omega_r = A_1 + A_2 + \ldots + A_r$, proving the first part of the statement. Since the $\mathfrak{sl}(2,\mathbb{R})$-triples defined in (4) commute, one has

$$g_{1,1} \cdot \ldots \cdot g_{1,r} = \exp(i\frac{\pi}{2}\frac{A_1}{2}) \cdot \ldots \cdot \exp(i\frac{\pi}{2}\frac{A_r}{2}) =$$
$$= \exp(i\frac{\pi}{2}(\frac{1}{2}(A_1 + A_2 + \ldots + A_r))) = g_1\,,$$

as claimed. □

8.2.1. **The tube case.** Let $G/K$ be an irreducible Hermitian symmetric space of tube type. We begin by showing that the semisimple boundary of the crown domain $\Xi$ contains a point $x_1$ whose $G$-orbit is an irreducible symmetric space $G/H$ of Cayley type. As a consequence, $x_1$ also lies on the boundary of two $G$-invariant Stein domains $S^\pm \subset G^\mathbb{C}/K^\mathbb{C}$, arising from the compactly causal structure of $G/H$. Such domains appear in a larger class of Stein domains studied by Neeb in [Nee99]. The main purpose of this subsection is to show that the domain $\Xi^+$ contains both $\Xi$ and the domain $S^+$, as well as part of their boundaries.

**Lemma 8.5.** *Let $G/K$ be an irreducible Hermitian symmetric space of tube type. Then the $G$-orbit of the point $x_1 = g_1 \cdot x_0$ in $G^\mathbb{C}/K^\mathbb{C}$ is a totally real semisimple symmetric space $G/H$ of Cayley type, with involution $\tau = \mathrm{Ad}_{g_1^2}\theta$ and $H = G^\tau$. The space $G/H$ has the same rank, real rank and dimension as $G/K$.*

*Proof.* In the tube case $\omega_r = \frac{1}{2}(A_1 + A_2 + \ldots + A_r)$. It is easy to check that $|\alpha(\frac{\pi}{2}\omega_r)| \leq \frac{\pi}{2}$, for every root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$ and that $\alpha_r(\frac{\pi}{2}\omega_r) = \frac{\pi}{2}$. This shows that $x_1$ lies on the semisimple boundary $\partial_s\Xi$ of the crown domain $\Xi$. More precisely, one has $\alpha(\frac{\pi}{2}\omega_r) \in \mathbb{Z}\frac{\pi}{2}$, for every $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$. Then the orbit $G \cdot x_1$, with the involution $\tau = \mathrm{Ad}_{g_1}\theta\mathrm{Ad}_{g_1^{-1}} = \mathrm{Ad}_{g_1^2}\theta$, is a pseudo-Riemannian symmetric space, say $G/H$, of the same rank, real rank and dimension as $G/K$ (see [Gea12], Lemma 2.2). Since $x_1 \in \partial_s\Xi$, by [GiKr02], Thm. B, the space $G/H$ is a non-compactly causal symmetric space.

From the definition of $\tau$ and Lemma 8.4, one can check that the further conditions $\theta E_j = -\tau E_j$, for $j = 1, \ldots, r$, are satisfied. Consequently, all the vectors $T_j := E_j + \theta E_j$, and in particular the element $Z_0 = \frac{1}{2}\sum_j T_j$ in the center of $\mathfrak{k}$ (see Prop. 2.6), are contained in $\mathfrak{q} \cap \mathfrak{k}$. By Thm. 1.3.8 and Rem. 1.3.9 in [HiOl97], the space $G/H$ is also compactly causal, and therefore of Cayley type, as claimed. □

Let $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \tau)$ be the symmetric algebra associated to the Cayley type symmetric space $G/H$ and let $W^\pm$ denote the *maximal* proper, open, convex, $\mathrm{Ad}_H$-invariant, elliptic cones in $\mathfrak{q}$. Set $H^\mathbb{C} = g_1 K^\mathbb{C} g_1^{-1}$. Then the two domains $G \exp iW^\pm H^\mathbb{C}/H^\mathbb{C}$ in $G^\mathbb{C}/H^\mathbb{C}$ are Stein (cf. [Nee99], Thm. 3.5, p. 205), and likewise

$$S^\pm := G \exp iW^\pm \cdot x_1 = G \exp iW^\pm\, g_1 \cdot x_0$$

are $G$-invariant, Stein domains in $G^\mathbb{C}/K^\mathbb{C}$.

It is important to observe that for the Cayley type symmetric space $G/H$, the *maximal* and the *minimal* proper, open, convex, $\mathrm{Ad}_H$-invariant, elliptic cones in $\mathfrak{q}$ coincide: under the Adjoint action of $H$, the space $\mathfrak{q}$ decomposes as the direct sum of irreducibles subspaces $\mathfrak{q}^+ \oplus \mathfrak{q}^-$, with the property that $\mathfrak{q}^- = -\theta\mathfrak{q}^+$. Each summand contains closed, convex, $\mathrm{Ad}_H$-invariant cones $\pm C_+ \subset \mathfrak{q}^+$ and $\pm C_- \subset \mathfrak{q}^-$, with the property that the minimal elliptic and hyperbolic closed cones in $\mathfrak{q}$ are given by $\pm(C_+ - C_-)$ and $\pm(C_+ + C_-)$, respectively (cf. [HiOl97], p.53). In particular, for the minimal closed, $\mathrm{Ad}_H$-invariant elliptic cone $\overline{W^+_{min}}$, there is an isomorphism $\overline{W^+_{min}} \cong C_+ + C_+$.

Denote by $C^0_+$ the interior of $C_+$. Since the symmetric space $G/K$ is biholomorphic to the tube domain $\mathfrak{q}^+ + iC^0_+$ (see [HiOl97], Rem.2.6.9, p.55), the cone $C_+$ is selfadjoint (i.e. it coincides with its dual cone). As a consequence, the minimal proper, closed, convex, $\mathrm{Ad}_H$-invariant, elliptic cone in $\mathfrak{q}$ is selfadjoint and coincides with the maximal one, which by definition is its dual cone $\left(\overline{W^+_{min}}\right)^*$. The same is true for the respective interior parts.

Let $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \tau)$ be the symmetric algebra associated to the Cayley type symmetric space $G/H$. Since the involutions $\theta$ and $\tau$ commute, $\mathfrak{g}$ has a joint eigenspace decomposition $\mathfrak{g} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}) \oplus (\mathfrak{q} \cap \mathfrak{k}) \oplus (\mathfrak{q} \cap \mathfrak{p})$. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{q} \cap \mathfrak{p}$. Then $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ and in $\mathfrak{q}$ (see [HiOl97], Prop. 3.1.11, p.77).

Fix a set of commuting $\mathfrak{sl}(2,\mathbb{R})$-triples $\{E_j, \theta E_j, A_j\}$ as in (4). As we remarked in the proof of Lemma 8.5, each $T_j := E_j + \theta E_j$ is contained in $\mathfrak{q} \cap \mathfrak{k}$ and $\mathfrak{c} := \mathrm{span}_{\mathbb{R}}\{T_1, \ldots, T_r\}$ is a compact Cartan subspace in $\mathfrak{q}$. In particular, $\mathfrak{c}$ contains the element $Z_0 = \frac{1}{2}(T_1 + \ldots + T_r) \in Z(\mathfrak{k})$ (see Prop. 2.6).

**Lemma 8.6.** *Let $G/K$ be an irreducible Hermitian symmetric space of tube type. Then*

$$S^+ = G\left(\exp i \bigoplus_{j=1}^{r} (0,\infty) T_j\right) g_1 \cdot x_0.$$

*Proof.* A proper, closed, convex, $\mathrm{Ad}_H$-invariant, elliptic cone in $\mathfrak{q}$ intersects the compact Cartan subspace $\mathfrak{c}$ in a proper, closed, convex, $W_H(\mathfrak{c})$-invariant, elliptic cone. Since the cone $\overline{W^+}$ is selfadjoint (i.e. maximal and minimal), we can identify the intersection $\overline{W^+_{\mathfrak{c}}} := \overline{W^+} \cap \mathfrak{c}$ with a minimal proper, closed, convex, $W_H(\mathfrak{c})$-invariant, elliptic cone in $\mathfrak{c}$. We prove the lemma by showing that

$$\overline{W^+_{\mathfrak{c}}} = \bigoplus_{j=1}^{r} [0,\infty) T_j.$$

In order to do this we first observe that

$$W_H(\mathfrak{c}) \cong W_{H\cap K}(\mathfrak{c}) \cong W_{H^0\cap K}(\mathfrak{c}),$$

where the second isomorphism follows from the fact that the space $G^c/H$ is non-compactly causal, with $i\mathfrak{c}$ hyperbolic maximal abelian in $i\mathfrak{q}$. Then, by [HiOl97], Thm. 3.1.18 and Thm. 3.1.20, the group $H$ is essentially connected, i.e. $H = H^0 Z_{H\cap K}(i\mathfrak{c})$ (see [HiOl97], Def. 3.1.16).

Next we need to recall the characterization of the minimal proper, closed, convex, $W_{H^0}(\mathfrak{c})$-invariant, elliptic cones in $\mathfrak{c}$ (see [KrNe96]). Consider the restricted root system $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$ of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{c}^{\mathbb{C}}$. Define the Lie subalgebra $\mathfrak{r} = \mathfrak{q} \cap \mathfrak{k} \oplus [\mathfrak{q} \cap \mathfrak{k}, \mathfrak{q} \cap \mathfrak{k}] \subset \mathfrak{k}$. A root $\alpha \in \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$ is called compact if $\mathfrak{g}^{\alpha} \cap \mathfrak{r}^{\mathbb{C}} \neq \{0\}$, and non-compact otherwise. Denote by $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})_c$ and $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})_n$ the compact and non-compact roots in $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$, respectively. The root system $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$ is called split if $\mathfrak{g}^{\alpha} \subset \mathfrak{k}^{\mathbb{C}}$, for all compact roots $\alpha$. The Weyl group $W_{H^0\cap K}(\mathfrak{c})$ is isomorphic to the group $W_c$ generated by the reflections in the compact roots ([KrNe96], Def.III.9 and Prop. V.2.i). If the positive non-compact roots $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})_n$ are stable under the group $W_c$, the system $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})^+$ is called $\mathfrak{r}$-adapted.

If the symmetric algebra $(\mathfrak{g}, \tau)$ is compactly causal then the restricted root system $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$ is split and admits an $\mathfrak{r}$-adapted positive system. Moreover the minimal proper, closed, convex, $W_{H^0\cap K}(\mathfrak{c})$-invariant, elliptic cones in $\mathfrak{c}$ have the following characterization

$$\overline{iW_{\mathfrak{c}}^{\pm}} := \pm\mathrm{cone}(\{h_\alpha\}_{\alpha\in\Delta(\mathfrak{g}^{\mathbb{C}},\mathfrak{c}^{\mathbb{C}})_n}),$$

where $h_\alpha \in i\mathfrak{c}$ is defined by $\alpha(H) = B(H, h_\alpha)$.

Now we come to our situation: since $\mathfrak{c}$ is the image of $\mathfrak{a}$ under a Cayley transform, the root system $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$ is isomorphic to the ordinary restricted root system $\Delta(\mathfrak{g}, \mathfrak{a})$, and is of type $C_r$. For simplicity, identify $\mathfrak{c}_{\mathbb{R}} = i\mathfrak{c}$ with $\mathfrak{c}_{\mathbb{R}}^*$ using the Killing form. Since the restrictions of the roots $\tilde\lambda_1, \dots, \tilde\lambda_r$ defined in Lemma 2.1 are non-compact in $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$, one has that

$$\mathrm{cone}(\{2e_j\}_j) \subset \overline{iW_{\mathfrak{c}}^{+}}.$$

The fact that the image of $\mathrm{cone}(\{2e_j\}_{j=1,\dots,r})$ under the reflections with respect to roots of the form $\pm(e_i + e_j)$, for $1 \le i < j \le r$, is not contained in any regular cone in $i\mathfrak{c}$, implies that such roots are necessarily non-compact. It follows that

$$\mathrm{cone}(\{2e_j\}_j) = \mathrm{cone}(\{2e_j, (e_i + e_k)\}_{j,\, i\neq k}) \subset \overline{iW_{\mathfrak{c}}^{+}}.$$

We claim that all roots of the form $\pm(e_i - e_j)$, for $1 \le i < j \le r$ are compact. In order to see this, first observe that the compact roots are a non-empty proper subset of $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$. Then assume by contradiction that there is a non-compact root of the form $e_i - e_k$, for some $i < k$. Without loss of generality, we may also assume that either $e_i - e_j$, for some $i < j$, or $e_j - e_k$, for some $j < k$, is compact. From the $W_c$-invariance of the cone $\overline{iW_{\mathfrak{c}}^{+}}$ and

$$r_{e_i-e_j}(e_i - e_k) = e_j - e_k \quad \text{and} \quad r_{e_j-e_k}(e_i - e_k) = e_i - e_j,$$

we deduce that either $e_j - e_k$ or $e_i - e_j$ is a non-compact roots and lies in $\overline{iW_{\mathfrak{c}}^{+}}$ as well. From $(e_i - e_j) + (e_j - e_k) = (e_i + e_j) - 2e_k$, we obtain that $\mathbb{R}2e_k \subset \overline{iW_{\mathfrak{c}}^{+}}$; similarly, from $(e_i - e_k) + (e_i - e_j) = 2e_i - (e_k + e_j)$, we obtain that $\mathbb{R}(e_k + e_j) \subset iW_{\mathfrak{c}}^{+}$. In both cases the assumption that $\overline{iW_{\mathfrak{c}}^{+}}$ is a proper cone is contradicted. Hence

$$\mathrm{cone}(\{2e_j\}_j) = \overline{iW_{\mathfrak{c}}^{+}},$$

as desired. □

Now we can prove that the domain $\Xi^+$ contains both the crown domain $\Xi$ and the domain $S^+$.

**Proposition 8.7.** *Let $G/K$ be an irreducible Hermitian symmetric space of tube type. Then the domain $\Xi^+$ contains the crown*

$$\Xi = G \exp i \bigoplus_{j=1}^{r} [0, 1)E_j \cdot x_0\,,$$

*and the domain*

$$S^+ = G \exp i \bigoplus_{j=1}^{r} (1, \infty)E_j \cdot x_0.$$

*Proof.* The first equality was proved in [KrOp08]. The second one follows from $G$-invariance, and rank-1 reduction. Indeed by Lemma 8.6 and Lemma 8.1, we have

$$S^+ = G \left( \prod_{j=1}^{r} \exp i(0, \infty)T_j \right) g_1 \cdot x_0 =$$

$$= G\left(\prod_{j=1}^{r}\exp i(0,\infty)T_j\right)\prod_{j=1}^{r}g_{1,j}\cdot x_0 = G\left(\prod_{j=1}^{r}\exp i(0,\infty)T_j g_{1,j}\right)\cdot x_0 =$$
$$= G\prod_{j=1}^{r}\exp i(1,\infty)E_j\cdot x_0,$$

as claimed. □

8.2.2. **The non-tube case**. Assume now that $G/K$ is not of tube type. Consider the point $x_1 = g_1\cdot x_0$, where $g_1 = \exp(i\frac{\pi}{2}\frac{\omega_r}{2})$ is as in (22). Since $|\alpha(\frac{\pi}{2}\frac{\omega_r}{2})| \le \frac{\pi}{2}$, for all $\alpha\in\Delta(\mathfrak{g},\mathfrak{a})$, and $2\alpha_r(\frac{\pi}{2}\frac{\omega_r}{2}) = \frac{\pi}{2}$, the point $x_1$ lies on the boundary of the crown domain. More precisely, $\alpha(\frac{\pi}{2}\frac{\omega_r}{2})\in\mathbb{Z}\frac{\pi}{4}$, for all $\alpha\in\Delta(\mathfrak{g},\mathfrak{a})$, and $\alpha_r(\frac{\pi}{2}\frac{\omega_r}{2}) = \frac{\pi}{4}$.

Then, by [Gea12], Lemma 2.1, the following facts hold: the $G$-orbit of $x_1$ is not a $G$-symmetric space; the group $\widehat{G} := Z_G(g_1^4)$ is a reductive proper subgroup of $G$; the orbit $\widehat{G}\cdot x_1\subset G\cdot x_1$ is a reductive symmetric space with involution $\tau = \mathrm{Ad}_{g_1^2}\theta$, of the same rank and real rank as $G/K$, but of strictly smaller dimension. The isotropy subgroups of $x_1$ in $G$ and in $\widehat{G}$ coincide, and the slice representation at $x_1$ is equivalent to the isotropy representation of $\widehat{G}\cdot x_1$.

**Lemma 8.8.** *The orbit $\widehat{G}\cdot x_1$ is diffeomorphic to the Cayley symmetric space associated to the tube type Hermitian symmetric space contained in $G/K$.*

*Proof.* One easily verifies that $\mathrm{Ad}_{g_1^4}$ is an involution of $G^{\mathbb{C}}$, commuting both with the Cartan involution $\Theta$ of $G^{\mathbb{C}}$ and with the conjugation $\sigma$ relative to $G$. Since $G^{\mathbb{C}}$ is simply connected, $\widehat{G}^{\mathbb{C}} = Z_{G^{\mathbb{C}}}(g_1^4) = \mathrm{Fix}(G^{\mathbb{C}},\mathrm{Ad}_{g_1^4})$ is a connected reductive group. Moreover, it is the complexification of $\widehat{U}$, the fixed point subgroup of $\mathrm{Ad}_{g_1^4}$ on the simply connected compact real form $U$ of $G^{\mathbb{C}}$.

From the classification of simply connected, compact symmetric spaces one sees that the following three cases occur:

$G = SU(r,s),\ (r<s),\quad G^{\mathbb{C}} = SL(r+s,\mathbb{C}),\quad \widehat{G}^{\mathbb{C}} = S(GL(s-r,\mathbb{C})\times GL(2r,\mathbb{C})),$
$G = Spin^*(2r),\quad G^{\mathbb{C}} = Spin^*(2r,\mathbb{C})\quad \widehat{G}^{\mathbb{C}} = \mathbb{C}^*Spin^*(2(r-1),\mathbb{C}),$
$G = E_{6(-14)},\ (r=2),\quad G^{\mathbb{C}} = E_6,\quad \widehat{G}^{\mathbb{C}} = \mathbb{C}^*Spin(10,\mathbb{C}).$

From the above table one sees that $\widehat{G}^{\mathbb{C}}$ can be written as the commuting product
$$\widehat{G}^{\mathbb{C}} = M^{\mathbb{C}}G^{\mathbb{C}}_{tube}, \tag{23}$$
where $M^{\mathbb{C}}$ is a subgroup of $Z_{K^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})$ and $G^{\mathbb{C}}_{tube}$ denotes the simply connected complexification of the connected, Hermitian, simple group acting on the tube-type symmetric space contained in $G/K$. By [Gea12], Lemma 2.1(iv), the isotropy subgroup of $x_1$ in $\widehat{G}^{\mathbb{C}}$ is given by $(\widehat{G}^{\mathbb{C}})^\tau := \mathrm{Fix}(\widehat{G}^{\mathbb{C}},\tau)$. Since the involution $\tau$ preserves the subgroups $M^{\mathbb{C}}$ and $G^{\mathbb{C}}_{tube}$ and $\tau|_{M^{\mathbb{C}}} = Id|_{M^{\mathbb{C}}}$, there is an isomorphism of coset spaces
$$\widehat{G}^{\mathbb{C}}/(\widehat{G}^{\mathbb{C}})^\tau \cong G^{\mathbb{C}}_{tube}/(G^{\mathbb{C}}_{tube})^\tau.$$
Moreover, since the involutions $\sigma$ and $\tau$ commute on $\widehat{G}^{\mathbb{C}}$, there is also an isomorphism
$$\widehat{G}/\widehat{G}^\tau \cong G_{tube}/(G_{tube})^\tau.$$
This last fact can be seen most clearly at Lie algebra level:

$$\mathfrak{g} = \mathfrak{su}(r,s),\ (r<s),\quad \widehat{\mathfrak{g}} = \mathfrak{u}(s-r)\oplus\mathfrak{su}(r,r),\quad \widehat{\mathfrak{g}}_{x_1} = \mathfrak{g}_{x_1} = \mathfrak{u}(s-r)\oplus\mathfrak{sl}(r,\mathbb{C})\oplus\mathbb{R};$$
$$\mathfrak{g} = \mathfrak{so}^*(2r),\ (r \text{ odd}),\quad \widehat{\mathfrak{g}} = \mathbb{R}\oplus\mathfrak{so}^*(2(r-1)),\quad \widehat{\mathfrak{g}}_{x_1} = \mathfrak{g}_{x_1} = \mathbb{R}\oplus\mathfrak{sl}(r-1,\mathbb{H})\oplus\mathbb{R};$$
$$\mathfrak{g} = \mathfrak{e}_{6(-14)},\ (r=2),\quad \widehat{\mathfrak{g}} = \mathbb{R}\oplus\mathfrak{so}(2,8),\quad \widehat{\mathfrak{g}}_{x_1} = \mathfrak{g}_{x_1} = \mathbb{R}\oplus\mathfrak{so}(1,1)\oplus\mathfrak{so}(1,7).$$
□

As a result of the above discussion, we have reduced ourselves to the case of a Hermitian symmetric space of tube type $G_{tube}/(G_{tube})^\tau$, with $G_{tube}^{\mathbb{C}}$ simply-connected. Recall that by Lemma 2.4, the element $Z_0 \in Z(\mathfrak{k})$ determining the complex structure of $G/K$ can be written as

$$Z_0 = S + T_0,$$

where $S \in Z_K(\mathfrak{a})$ and $T_0 = \frac{1}{2}\sum T_j$, with $T_j = E_j + \theta E_j$. Observe that $Z_0$ lies in $\widehat{\mathfrak{g}}$ and $T_0$ lies in $\widehat{\mathfrak{g}}_{tube}$. Denote then by $W^+$ the maximal proper, open, convex, $\mathrm{Ad}_{(G_{tube})^\tau}$-invariant elliptic cone in $T_{x_1}(\widehat{G}_{tube} \cdot x_1)$, which satisfies $\overline{W^+} = \overline{conv\left(\mathrm{Ad}_{(G_{tube})^\tau}(\mathbb{R}^+T_0)\right)}$. Then

$$\Omega^+ = G \exp iW^+ \cdot x_1 = G \exp iW^+ g_1 \cdot x_0$$

is an open $G$-invariant domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$. By similar considerations as in the previous section one obtains that

$$\Omega^+ = G \exp i \bigoplus_{j=1}^{r} (0,\infty) T_j \, g_1 \cdot x_0.$$

and an analogue of Proposition 8.7 holds true.

**Proposition 8.9.** *Let $G/K$ be an irreducible Hermitian symmetric space which is not of tube-type. The domain $\Xi^+$ contains two distinguished invariant subdomains, namely the crown domain*

$$\Xi = G \exp i \bigoplus_{j=1}^{r} [0,1) E_j \cdot x_0 \,,$$

*and the domain*

$$\Omega^+ = G \exp i \bigoplus_{j=1}^{r} (1,\infty) E_j \cdot x_0.$$

We will see in a forthcoming paper that like in the rank-one case of non-tube type, the domain $\Omega^+$ is not Stein and contains no $G$-invariant Stein subdomains.

## 9. Final remarks.

Recall that the domain $\Xi^+$ is $G$-equivariantly diffeomorphic to the anti-holomorphic tangent bundle of $G/K$. From Lemma 5.5 and Lemma 3.1, we obtain another natural description of the crown $\Xi$ and of the domains $S^+$ (resp. $\Omega^+$) inside $\Xi^+$, by means of their intersections with the image of the slice $\mathfrak{a}$ under the map (13).

**Corollary 9.1.** *One has*

$$\Xi = G \exp i \bigoplus_{j=1}^{r} [0,1) \frac{1}{2}(A_j + iJ_0A_j) \cdot x_0 = G \exp i \bigoplus_{j=1}^{r} (-1,1) \frac{1}{2}(A_j + iJ_0A_j) \cdot x_0$$

*and*

$$S^+ = G \exp i \bigoplus_{j=1}^{r} (1,\infty) \frac{1}{2}(A_j + iJ_0A_j) \cdot x_0 =$$

$$G \exp i \bigoplus_{j=1}^{r} \big((-\infty,-1) \cup (1,\infty)\big) \frac{1}{2}(A_j + iJ_0A_j) \cdot x_0 \,.$$

*A similar description holds true for $\Omega^+$.*

*Proof.* Recall that by Lemma 8.7 and Lemma 8.9 one has

$$S^+ = G \exp i \bigoplus_{j=1}^{r} (1,\infty) E_j \cdot x_0 \quad \text{and} \quad \Omega^+ = G \exp i \bigoplus_{j=1}^{r} (1,\infty) E_j \cdot x_0$$

inside $\Xi^+ = G \exp i \bigoplus_{j=1}^{r} [0,\infty) E_j \cdot x_0$. Then the result follows from Lemma 5.5 and the fact that the Weyl group $W_K(\mathfrak{a})$ acts by signed permutations of $A_1, \ldots, A_r$ on $\mathfrak{a}$ and by signed permutations of $\{A_1 + iJ_0A_1, \ldots, A_r + iJ_0A_r\}$ in $\{A + iJ_0A \ : \ A \in \mathfrak{a}\}$, which is a slice for the $K$-action on $\mathfrak{p}^{0,1}$.

□

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