ORBIT STRUCTURE OF A DISTINGUISHED STEIN INVARIANT DOMAIN IN THE COMPLEXIFICATION OF A HERMITIAN SYMMETRIC SPACE

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Abstract. We carry out a detailed study of Ξ^+, a distinguished G-invariant Stein domain in the complexification of an irreducible Hermitian symmetric space G/K. The domain Ξ^+ contains the crown domain Ξ and is naturally diffeomorphic to the anti-holomorphic tangent bundle of G/K. The unipotent parametrization of Ξ^+ introduced in [KrOp08] and [Krö08] suggests that Ξ^+ also admits the structure of a twisted bundle $G \times_K \mathcal{N}^+$, with fiber a nilpotent cone $\mathcal{N}^+$. Here we give a complete proof of this fact and use it to describe the G-orbit structure of Ξ^+ via the K-orbit structure of $\mathcal{N}^+$. In the tube case, we also single out a Stein, G-invariant domain contained in $\Xi^+ \setminus \Xi$ which is relevant in the classification of envelopes of holomorphy of invariant subdomains of Ξ^+.

1. Introduction

Let G/K be a non-compact, irreducible, Riemannian symmetric space. Its Lie group complexification $G^{\mathbb{C}}/K^{\mathbb{C}}$ is a Stein manifold and left translations by elements of G are holomorphic transformations of $G^{\mathbb{C}}/K^{\mathbb{C}}$. In [AkGi90], Akhiezer and Gindikin introduced the crown domain Ξ in $G^{\mathbb{C}}/K^{\mathbb{C}}$, with the aim of determining a complex G-manifold whose analytic properties would reflect the harmonic analysis of G/K and the representation theory of G. Since then its complex analytic properties have been extensively studied by several authors.

In the Hermitian case, Krötz and Opdam recently introduced two Stein G-invariant domains Ξ^+ and Ξ^- in $G^{\mathbb{C}}/K^{\mathbb{C}}$, with $\Xi^+ \cap \Xi^- = \Xi$, which are maximal with respect to properness of the G-action on $G^{\mathbb{C}}/K^{\mathbb{C}}$. The relevance of Ξ and of the domains Ξ^+ and Ξ^- for the representation theory of G was underlined in Theorem 1.1 in [Krö08]. Here we carry out a detailed analysis of the G-orbit structure of Ξ^+. Since Ξ^+ and Ξ^- are G-equivariantly anti-biholomorphic, the same analysis applies to Ξ^- as well.

Let G/K be an irreducible Hermitian symmetric space and let $G^{\mathbb{C}}/Q$ be its compact dual symmetric space, which is denoted by $\overline{G^{\mathbb{C}}/Q}$ when endowed with the opposite complex structure. The space $G^{\mathbb{C}}/K^{\mathbb{C}}$ admits an equivariant holomorphic embedding

$$G^{\mathbb{C}}/K^{\mathbb{C}} \cong G^{\mathbb{C}} \cdot x_0 \subset G^{\mathbb{C}}/Q \times \overline{G^{\mathbb{C}}/Q}$$

as the open dense orbit through $x_0 := (eQ, eQ) \in G^{\mathbb{C}}/Q \times \overline{G^{\mathbb{C}}/Q}$, under the $G^{\mathbb{C}}$-action defined by

$$g \cdot (x, y) := (g \cdot x, \sigma(g) \cdot y)\,.$$

Here σ denotes the conjugation of $G^{\mathbb{C}}$ with respect to G. Let $\pi_1 : G^{\mathbb{C}}/Q \times \overline{G^{\mathbb{C}}/Q} \to G^{\mathbb{C}}/Q$ be the projection onto the first factor. The G-invariant domain Ξ^+ is defined by

$$\Xi^+ := (\pi_1)^{-1}(D) \cap G^{\mathbb{C}} \cdot x_0,$$

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where $D := G \cdot eQ$ is the Borel embedding of G/K in $G^{\mathbb{C}}/Q$. The domain Ξ^+ contains the crown Ξ as the subset $D \times \overline{D}$ and the G-action on Ξ^+ is proper.

The above definition leads to a natural G-equivariant diffeomorphism between the anti-holomorphic tangent bundle of G/K and Ξ^+, via the map

$$G \times_K \mathfrak{p}^{0,1} \to \Xi^+, \qquad [g, Z] \mapsto g \exp Z \cdot x_0. \tag{1}$$

The anti-holomorphic G-equivariant involution on $G^{\mathbb{C}}/K^{\mathbb{C}}$ induced by σ maps Ξ^+ diffeomorphically onto $\Xi^- := \pi_2^{-1}(\overline{D}) \cap G^{\mathbb{C}} \cdot x_0$.

An alternative construction of the domain Ξ^+ was given in [Krö08] and [KrOp08], via its unipotent parametrization. In the notation of Section 2, let $\lambda_1, \dots, \lambda_r$ be a maximal set of long strongly orthogonal real restricted roots, and let $E_j \in \mathfrak{g}^{\lambda_j}$, for $j = 1, \dots, r$, be root vectors normalized as in Definition 2.1. Consider the closed hyperoctant

$$\Lambda_r^{\llcorner} := \mathrm{span}_{\mathbb{R}_{\geq 0}}\{E_1, \dots, E_r\}$$

and the subcone $\mathcal{N}^+ := \mathrm{Ad}_K \Lambda_r^{\llcorner}$ of the nilpotent cone of $\mathfrak{g}$. Then

$$\Xi^+ = G \exp i \bigoplus_j (-1, \infty) E_j \cdot x_0 = G \exp i \Lambda_r^{\llcorner} \cdot x_0.$$

It was also suggested in [KrOp08] and [Krö08] that the map

$$\psi \colon G \times_K \mathcal{N}^+ \to \Xi^+, \quad [g, X] \mapsto g \exp iX \cdot x_0$$

is a G-equivariant homeomorphism.

The first goal of the paper is to give a complete and selfcontained proof of this fact. The main difficulty is to show that the map ψ is open. This is not a priori obvious because at every point of the slice $\exp i\Lambda_r^{\llcorner} \cdot x_0 \subset \Xi^+$, lying on a singular G-orbit, the tangent spaces to the orbit and to the slice itself do not span the whole tangent space to Ξ^+.

Let $P := \exp \mathfrak{p}^{0,1} \cdot x_0$ be the K-invariant fiber in the domain $\Xi^+ \cong G \times_K \mathfrak{p}^{0,1}$. We first use a topological argument (Lemma 4.2) to show that our goal is equivalent to proving that the projection

$$\Lambda_r^{\llcorner} \to P/K, \quad X \mapsto G \exp iX \cdot x_0 \cap P\,,$$

is proper. Then we check that such a projection is proper by using a novel decomposition inside $G^{\mathbb{C}}$, relating a unipotent element $\exp iX$, with $X \in \Lambda_r^{\llcorner}$, to an element in $\exp Z\, K^{\mathbb{C}}$, with $Z \in \mathfrak{p}^{0,1}$, lying on the same G-orbit (see Lemma 4.5 and Thm. 4.7). Possibly, a similar argument leads to a characterization of smooth twisted bundles in the context of proper G-actions on differentiable manifolds, as considered by R. S. Palais and C.-L. Terng in [PaTe87].

In view of the bundle structure defined by ψ, the G-orbit structure of Ξ^+ is completely determined by the Ad_K-orbit structure of the cone $\mathcal{N}^+$. We show that a fundamental domain for the action of the Weyl group $W_K(\Lambda_r^{\llcorner})$ on the hyperoctant $\Lambda_r^{\llcorner}$ is a perfect slice for the K-action on $\mathcal{N}^+$ and hence it determines a perfect slice for the G-action on Ξ^+. Moreover, there is a one-to-one correspondence between the orbit strata of the $W_K(\Lambda_r^{\llcorner})$-action on the closed hyperoctant $\Lambda_r^{\llcorner}$ and the orbit strata of the G-action on Ξ^+.

The second goal of the paper is to prove that, in the tube-case, Ξ^+ contains a distinguished Stein, G-invariant subdomain S^+, which arises from the compactly causal structure of a semisimple symmetric orbit G/H in the boundary of Ξ. A first evidence of this fact comes from the rank-one case $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ studied in [GeIa08], where it is also shown that every proper, Stein, invariant subdomain of Ξ^+ is either contained in Ξ or in S^+.

The domain S^+ is G-equivariantly biholomorphic to an invariant domain in the Lie group complexification of the symmetric space G/H and its Steiness follows from a result of K. H. Neeb in [Nee99]. Here we show that it is contained in Ξ^+ by proving the following identity (Prop. 7.5)

$$S^+ = G \exp i \bigoplus_{j=1}^{r} (1, \infty) E_j \cdot x_0.$$

From the classification of envelopes of holomorphy of invariant domains in Ξ^+ (see [GeIa13]), it follows that, like in the rank-one case, every proper, Stein, invariant domain of Ξ^+ is contained either in Ξ or in S^+. In the non-tube case, there is no Stein analogue of S^+. At the end of the paper we give some details on the non-tube case.

The paper is organized as follows. In Section 2 we set up the notation and collect some basic facts about Hermitian symmetric spaces. In Section 3 we study the action of the Weyl group $W_K(\Lambda_r^\llcorner)$ of the hyperoctant $\Lambda_r^\llcorner$. In Section 4 we recall the unipotent model of Ξ^+ and prove that the map

$$\psi \colon G \times_K \mathcal{N}^+ \to \Xi^+, \quad [g, X] \mapsto g \exp iX \cdot x_0$$

is a G-equivariant homeomorphism. In Section 5 we give an alternative proof of the above fact for the symmetric spaces $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ and $Sp(2,\mathbb{R})/U(2)$, by using global G-invariant functions on concrete models of $G^{\mathbb{C}}/K^{\mathbb{C}}$. In Section 6 we study the G-orbit structure of Ξ^+ by means of the Ad_K-orbit structure of $\mathcal{N}^+$. Finally, in Section 7 we show that the domain S^+ is contained in Ξ^+ by expressing it in the unipotent parametrization of Ξ^+.

2. Preliminaries

Let G/K be an irreducible Hermitian symmetric space of the non-compact type. We may assume G to be a connected, non-compact, real simple Lie group contained in its simple, simply connected universal complexification $G^{\mathbb{C}}$, and K to be a maximal compact subgroup of G. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. Denote by θ both the Cartan involution of G with respect to K and the derived involution of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$. The *rank* of G/K is by definition $r = \dim \mathfrak{a}$. The adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$ determines the restricted root decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus Z_{\mathfrak{k}}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g},\mathfrak{a})} \mathfrak{g}^\alpha,$$

where $\Delta(\mathfrak{g},\mathfrak{a}) = \{\alpha \in \mathfrak{a}^* \setminus \{0\} \ : \ \mathfrak{g}^\alpha \neq \{0\}\}$ is the restricted root system, $\mathfrak{g}^\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X, \ H \in \mathfrak{a}\}$ is the α-restricted root space, and $Z_{\mathfrak{k}}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. A set of simple roots $\Pi_{\mathfrak{a}}$ in $\Delta(\mathfrak{g},\mathfrak{a})$ uniquely determines a set of positive restricted roots $\Delta^+(\mathfrak{g},\mathfrak{a})$ and an Iwasawa decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \qquad \text{where} \quad \mathfrak{n} = \bigoplus_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}^\alpha.$$

The restricted root system of a Lie algebra $\mathfrak{g}$ of Hermitian type is either of type C_r (if G/K is of tube type) or of type BC_r (if G/K is not of tube type) (cf. [Moo64]), i.e. there exists a basis $\{e_1, \dots, e_r\}$ of $\mathfrak{a}^*$ for which

$$\Delta(\mathfrak{g},\mathfrak{a}) = \{\pm 2e_j, \ 1 \le j \le r, \ \pm e_j \pm e_k, \ 1 \le j \neq k \le r\}, \quad \text{for type } C_r,$$

$$\Delta(\mathfrak{g},\mathfrak{a}) = \{\pm e_j, \ \pm 2e_j, \ 1 \le j \le r, \ \pm e_j \pm e_k, \ 1 \le j \neq k \le r\}, \quad \text{for type } BC_r.$$

Since $\mathfrak{g}$ admits a compact Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g}$, there exists a set of r long strongly orthogonal restricted roots $\{\lambda_1, \dots, \lambda_r\}$ (i.e. such that $\lambda_j \pm \lambda_k \notin \Delta(\mathfrak{g},\mathfrak{a})$, for $j \neq k$), which are restrictions of *real* roots with respect to a maximally split θ-stable Cartan subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ extending $\mathfrak{a}$. Choosing as simple roots

$$\Pi_{\mathfrak{a}} = \{e_1 - e_2, \dots, e_{r-1} - e_r, 2e_r\}, \quad \text{for type } C_r, \tag{2}$$

$$\Pi_{\mathfrak{a}} = \{e_1 - e_2, \dots, e_{r-1} - e_r, e_r\}, \quad \text{for type } BC_r, \tag{3}$$

the roots $\{\lambda_1, \dots, \lambda_r\}$ are given by

$$\lambda_1 = 2e_1, \dots, \lambda_r = 2e_r. \tag{4}$$

In both cases, the Weyl group $W_K(\mathfrak{a}) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is isomorphic to the group of signed permutations of $\{e_1, \dots, e_r\}$, and therefore of $\{\lambda_1, \dots, \lambda_r\}$. Denote by $W_K(\mathfrak{a})^+$ the subgroup of $W_K(\mathfrak{a})$ isomorphic to the the group of ordinary permutations of $\{e_1, \dots, e_r\}$. This subgroup is generated by the reflections in the first $r-1$ simple restricted roots.

For $j = 1, \dots, r$, choose $E_j \in \mathfrak{g}^{\lambda_j}$ such that the $\mathfrak{sl}(2)$-triple

$$\{E_j,\ \theta E_j,\ A_j := [\theta E_j, E_j]\} \tag{5}$$

is normalized as follows

$$[A_j, E_j] = 2E_j, \quad [A_j, \theta E_j] = -2\theta E_j. \tag{6}$$

The vectors $\{A_1, \dots, A_r\}$ form a basis of $\mathfrak{a}$ which is orthogonal with respect to the restriction of the Killing form and one has

$$[E_j, E_k] = [E_j, \theta E_k] = 0, \quad [A_j, E_k] = \lambda_k(A_j)E_k = 0, \quad \text{for } j \neq k. \tag{7}$$

In particular the above $\mathfrak{sl}(2)$-triples commute with each other and $\{A_1, \dots, A_r\}$ is the dual basis of $\{e_1, \dots, e_r\}$. As a consequence, the action of $W_K(\mathfrak{a})$ and of $W_K(\mathfrak{a})^+$ on $\mathfrak{a}$ is by signed permutations and by ordinary permutations of $\{A_1, \dots, A_r\}$, respectively.

Observe that relations (6) and (5) determine the vectors E_j only up to sign. Fix an invariant complex structure J_0 of G/K. We are going to define the unique choice of the vectors E_j which is compatible with J_0, in the sense that the r-dimensional polydisk, associated with the r commuting $\mathfrak{sl}(2)$-triples in $\mathfrak{g}$, is holomorphically embedded in G/K.

Identify $\mathfrak{p}$ with the tangent space to G/K at the base point eK. The complex structure J_0 is uniquely determined by its restriction to $\mathfrak{p}$ and it is given by $J_0 = ad_{Z_0}|\mathfrak{p}$, for some $Z_0 \in Z(\mathfrak{k})$. More precisely, consider a compact Cartan subalgebra of $\mathfrak{g}$ of the form $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{c}$, where $\mathfrak{s}$ is a Cartan subalgebra of $Z_{\mathfrak{k}}(\mathfrak{a})$, $\mathfrak{c} := \mathrm{span}\{T_1, \dots, T_r\}$, and $T_j := E_j + \theta E_j$, for $j = 1, \dots, r$. Then $Z_0 \in \mathfrak{t}$ and can be written as $Z_0 = S + \sum_j a_j T_j$, for some $S \in \mathfrak{s}$ and $a_j \in \mathbb{R}$. Since $J_0^2 = -Id$ and the algebra $Z_{\mathfrak{k}}(\mathfrak{a})$ acts trivially on the 1-dimensional root spaces $\mathfrak{g}^{\lambda_j}$ and $\mathfrak{g}^{-\lambda_j}$, one has

$$J_0(E_j - \theta E_j) = [Z_0, E_j - \theta E_j] = 2a_j A_j, \quad \text{with} \quad a_j = \pm\frac{1}{2}.$$

Definition 2.1. *The choice of the E_j is compatible with the complex structure J_0 if, for all $j = 1, \dots, r$, one has*

$$J_0(E_j - \theta E_j) = A_j.$$

Equivalently, $a_j = \frac{1}{2}$, for all $j = 1, \dots, r$.

Consider the Lie algebra homomorphism $\mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}$ mapping the triple

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \theta E = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8}$$

to $\{E_j, \theta E_j, A_j\}$, for some j. Endow $SL(2, \mathbb{R})/SO(2, \mathbb{R})$ with the unique invariant complex structure defined by $\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then the induced embedding

$$SL(2, \mathbb{R})/SO(2, \mathbb{R}) \to G/K$$

is holomorphic if and only if the choice of the vector E_j agrees with Definition 2.1. Otherwise it is anti-holomorphic.

Remark 2.2. Fix the vectors E_j as in Definition 2.1 and set

$$W_j := \frac{1}{2}\left((E_j - \theta E_j) - iA_j\right), \qquad W_{-j} := \overline{W}_j\,. \tag{9}$$

Then the vectors W_j in $\mathfrak{g}^{\mathbb{C}}$ span the root spaces $\mathfrak{g}^{\widetilde{\lambda}_j}$ of a set of strongly orthogonal, non-compact, imaginary roots $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_r$ in $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, satisfying $\widetilde{\lambda}_j(-iZ_0) = 1$. Moreover $[W_j, W_{-j}] = -iT_j$, for $j = 1, \ldots, r$. Then, by the discussion on p. 254 and Koranyi-Wolf's Theorem A.3.5 in [HiOl97], the following conditions are equivalent
(a) G/K is of tube type, i.e. $\Delta(\mathfrak{g}, \mathfrak{a})$ is reduced of type C_r,
(b) $Z_0 = \frac{1}{2}\sum_j T_j$.

3. The Weyl group $W_K(\Lambda_r)$

Resume the notation of Section 2. For $j = 1, \ldots, r$, let E_j be the unique vector in $\mathfrak{g}^{\lambda_j}$ which is compatible with the complex structure J_0 of G/K in the sense of Definition 2.1. Define

$$\Lambda_r := \operatorname{span}_{\mathbb{R}}\{E_1, \ldots, E_r\} \quad \text{and} \quad \Lambda_r^{\llcorner} := \operatorname{span}_{\mathbb{R}_{\geq 0}}\{E_1, \ldots, E_r\}. \tag{10}$$

Consider the Adjoint action of K on $\mathfrak{g}$ and define the groups

$$Z_K(\Lambda_r) := \{k \in K \,:\, \mathrm{Ad}_k X = X,\ \forall\, X \in \Lambda_r\},\ \ N_K(\Lambda_r) := \{k \in K \,:\, \mathrm{Ad}_k \Lambda_r = \Lambda_r\},$$
$$W_K(\Lambda_r) := N_K(\Lambda_r)/Z_K(\Lambda_r)\,.$$

Consider the K-equivariant map

$$\Psi : \mathfrak{g} \to \mathfrak{p}, \quad X \mapsto [Z_0, X - \theta X] = J_0(X - \theta X)\,, \tag{11}$$

where $Z_0 \in Z(\mathfrak{k})$ is the element defining the complex structure $J_0 = \mathrm{ad}_{Z_0}$ of G/K. Note that its restriction $\Psi|_{\Lambda_r} : \Lambda_r \to \mathfrak{a}$ is a linear isomorphism (cf. Def. 2.1).

Lemma 3.1.
(i) $Z_K(\Lambda_r) = Z_K(\mathfrak{a})$.
(ii) *$N_K(\Lambda_r)$ is a subgroup of $N_K(\mathfrak{a})$, implying that $W_K(\Lambda_r)$ is a subgroup of $W_K(\mathfrak{a})$.*
(iii) *The group $W_K(\Lambda_r)$ coincides with the subgroup $W_K(\mathfrak{a})^+$ of $W_K(\mathfrak{a})$, acting on $\mathfrak{a}$ by permutations of $\{A_1, \ldots, A_r\}$. Moreover, $W_K(\Lambda_r)$ acts on Λ_r by permutations of $\{E_1, \ldots, E_r\}$.*

Proof. Since the map Ψ defined in (11) is K-equivariant and $\Psi|_{\Lambda_r} : \Lambda_r \to \mathfrak{a}$ is an isomorphism, there are inclusions $N_K(\Lambda_r) \subset N_K(\mathfrak{a})$ and $Z_K(\Lambda_r) \subset Z_K(\mathfrak{a})$. In order to show that $Z_K(\mathfrak{a}) \subset Z_K(\Lambda_r)$, recall that every restricted root space is invariant under the Adjont action of $Z_K(\mathfrak{a})$ on $\mathfrak{g}$. Since Λ_r is the direct sum of the one-dimensional restricted root spaces $\mathfrak{g}^{\lambda_j}$, for $j = 1, \ldots, r$, it follows that $Z_K(\mathfrak{a})$ is a subgroup of $N_K(\Lambda_r)$. The injectivity of the $N_K(\Lambda_r)$-equivariant isomorphism $\Psi|_{\Lambda_r}$ implies that $Z_K(\mathfrak{a}) \subset Z_K(\Lambda_r)$, proving (i) and (ii).
(iii) We already showed that $W_K(\Lambda_r) \subset W_K(\mathfrak{a})$. Next we show that $W_K(\Lambda_r)$ contains the subgroup $W_K(\mathfrak{a})^+$. Recall that the subgroup $W_K(\mathfrak{a})^+$ acts on $\mathfrak{a}$ by permutations of $A_1, \ldots, A_r$ and on $\mathfrak{a}^*$ by permutations of the basis vectors $e_1, \ldots, e_r$ defined in Section 2. As a result, the corresponding elements in K permute the root spaces $\mathfrak{g}^{\lambda_1}, \ldots, \mathfrak{g}^{\lambda_r}$ and thus normalize Λ_r. This proves the inclusion

$$W_K(\mathfrak{a})^+ \subset W_K(\Lambda_r).$$

In order to prove equality, assume by contradiction that there exists $k \in N_K(\Lambda_r)$ lying in $W_K(\mathfrak{a}) \setminus W_K(\mathfrak{a})^+$. Since $W_K(\mathfrak{a})$ acts on $\mathfrak{a}$ by signed permutations of $A_1, \ldots, A_r$, there exist indices $j, h \in \{1, \ldots, r\}$ for which $\mathrm{Ad}_k(A_j) = -A_h$. By applying Ad_k to both terms of the relation $[A_j, E_j] = 2E_j$, we obtain

$$[A_h, \mathrm{Ad}_k E_j] = -2\mathrm{Ad}_k E_j.$$

We claim that $[A_l, \mathrm{Ad}_k E_j] = 0$, for all $l \neq h$. From the identity

$$[A_l, \mathrm{Ad}_k E_j] = \mathrm{Ad}_k[\mathrm{Ad}_{k^{-1}} A_l, E_j]$$

and the fact that k normalizes $\mathfrak{a}$, we have that $\mathrm{Ad}_{k^{-1}} A_l \in \{\pm A_m\}$, for some $m \neq j$. Thus

$$\mathrm{Ad}_k[\mathrm{Ad}_{k^{-1}} A_l, E_j] = \mathrm{Ad}_k[\pm A_m, E_j] = 0,$$

as claimed. It follows that $\mathrm{Ad}_k E_j \in \mathfrak{g}^{-\lambda_h}$, contradicting the assumption that k normalizes Λ_r. So $W_K(\mathfrak{a})^+ = W_K(\Lambda_r)$, proving the first part of (iii).

Finally, since $\Psi|_{\Lambda_r}(E_j) = A_j$ and $W_K(\mathfrak{a})^+$ acts on $\mathfrak{a}$ by permutations of $A_1, \ldots, A_r$, the equivariance of the isomorphism $\Psi|_{\Lambda_r}$ implies that $W_K(\Lambda_r) = W_K(\mathfrak{a})^+$ acts on Λ_r by permutations of $E_1, \ldots, E_r$. This concludes the proof of (iii) and of the lemma. $\square$

Corollary 3.2. *The group $W_K(\Lambda_r)$ preserves the closed hyperoctant $\Lambda_r^{\llcorner}$. Hence*

$$W_K(\Lambda_r) := N_K(\Lambda_r)/Z_K(\Lambda_r) = N_K(\Lambda_r^{\llcorner})/Z_K(\Lambda_r^{\llcorner}).$$

4. The domain Ξ^+ as a nilpotent cone bundle

As it was mentioned in the introduction, an alternative description of the domain Ξ^+ was given in [Krö08], p.286, and [KrOp08], Sect. 8, via its unipotent parametrization. For $j = 1, \ldots, r$, fix the unique vectors $E_j \in \mathfrak{g}^{\lambda_j}$ compatible with the complex structure J_0 of G/K (see Def. 2.1). Define Λ_r and $\Lambda_r^{\llcorner}$ as in (10) and consider the subcone $\mathcal{N}^+ := \mathrm{Ad}_K \Lambda_r^{\llcorner}$ of the nilpotent cone of $\mathfrak{g}$. In [Krö08] it was shown that

$$\Xi^+ = G \exp i \bigoplus_{j=1}^{r} (-1, \infty) E_j \cdot x_0 = G \exp i \Lambda_r^{\llcorner} \cdot x_0,$$

and it was suggested that the map

$$\psi \colon G \times_K \mathcal{N}^+ \to \Xi^+, \quad [g, X] \mapsto g \exp iX \cdot x_0$$

is a G-equivariant homeomorphism. The main result of this section is a complete self-contained proof of this fact. It is obtained by combining a topological approach with a novel decomposition in $G^{\mathbb{C}}$ relating a unipotent element $\exp iX$, with $X \in \Lambda_r^{\llcorner}$, to an element $\exp Z\, K^{\mathbb{C}}$, with $Z \in \mathfrak{p}^{0,1}$, lying on the same G-orbit (see Lemma 4.5 and Thm. 4.7).

4.1. **Some topological lemmas.** This subsection contains some preliminary results, which are of topological nature. Our setting is as follows. Let G be a connected Lie group acting properly on a connected Hausdorff topological space Z, and let K be a compact subgroup of G. Let N be a Hausdorff topological K-space. Assume that there exists a K-equivariant continuous map $j : N \to Z$ such that the continuous map

$$\psi : G \times_K N \to Z,\ [g, x] \to g \cdot j(x)$$

is bijective. Denote by Σ a closed subset of N such that $K \cdot \Sigma = N$. We discuss necessary and sufficient conditions for ψ to be a homeomorphism.

Lemma 4.1. *The following three conditons are equivalent:*

(i) *the map* $\widetilde{\psi} : G \times \Sigma \to Z,\ (g, x) \to g \cdot j(x)$ *is proper,*
(ii) *the map* $\widehat{\psi} : G \times N \to Z,\ (g, x) \to g \cdot j(x)$ *is proper,*
(iii) *the map* $\psi : G \times_K N \to Z,\ [g, x] \to g \cdot j(x)$ *is proper.*

If any of the above conditions is satisfied, then ψ *is a homeomorphism, the map* $j : N \to j(N)$ *is a homeomorphism, and* $j(N)$ *is closed in* Z.

Proof. We first show that (i) is equivalent to (ii). Consider the commutative diagram

$$\begin{tikzcd} G \times \Sigma \arrow[d] \arrow[dr, "\widetilde{\psi}"] & \\ G \times N \arrow[r, "\widehat{\psi}"'] & Z , \end{tikzcd}$$

where the vertical arrow is the inclusion map. Since Σ is closed in N, such a map is proper. Therefore, if $\widehat{\psi}$ is proper, so is $\widetilde{\psi}$. Conversely, assume that $\widetilde{\psi}$ is proper and let C be a compact subset of Z. We claim that the closed subset $\widehat{\psi}^{-1}(C)$ coincides with $K \cdot \widetilde{\psi}^{-1}(C)$, where the K-action on $G \times N$ is given by $k \cdot (g, x) := (gk^{-1}, k \cdot x)$. In order to see that $\widehat{\psi}^{-1}(C) \subset K \cdot \widetilde{\psi}^{-1}(C)$, let (g, x) be an element in $\widehat{\psi}^{-1}(C)$ and choose $k \in K$ and $x' \in \Sigma$ such that $x = k \cdot x'$. Then $gk \cdot j(x') = g \cdot j(x) \in C$, implying that $(gk, x') \in \widetilde{\psi}^{-1}(C)$. Thus $(g, x) = k \cdot (gk, x')$ belongs to $K \cdot \widetilde{\psi}^{-1}(C)$. The opposite inclusion is straightforward, and the claim follows.

Since $\widetilde{\psi}^{-1}(C)$ is compact by assumption, it follows that $\widehat{\psi}^{-1}(C) = K \cdot \widetilde{\psi}^{-1}(C)$ is compact (cf. [Bou89], Cor. 1, p. 251). This concludes the proof of the first equivalence. In order to show that (ii) is equivalent to (iii), consider the commutative diagram

$$\begin{tikzcd} G \times N \arrow[d, "\pi"'] \arrow[dr, "\widehat{\psi}"] & \\ G \times_K N \arrow[r, "\psi"'] & Z , \end{tikzcd}$$

where π is the natural quotient map with respect to the twisted K-action. Since K is compact, such a map is proper (cf. [Bou89], Prop. 2, p. 252). Therefore, if ψ is proper, so is $\widehat{\psi}$. Conversely, assume that $\widehat{\psi}$ is proper and let C be a compact subset of Z. Then the inverse image $\psi^{-1}(C)$ coincides with $\pi(\widehat{\psi}^{-1}(C))$. Thus it is compact, implying that ψ is proper and concluding the proof of the lemma. $\square$

Note that assuming $j : \Sigma \to Z$ proper does not imply $G \times \Sigma \to Z$ proper. For instance, let $G = \mathbb{R}$ act on $\mathbb{R}^2$ by $t \cdot (x, y) = (t + x, y)$. Set $N = \Sigma := \{\ s \in \mathbb{R}\ :\ s \leq 0 \text{ or } s > 1\}$ and define $j : \Sigma \to \mathbb{R}^2$ by $j(s) := (0, s)$, for $s \in (-\infty, 0]$, and $j(s) := (\ln(s - 1), s - 1)$, for $s \in (1, +\infty)$. Then $\psi : \mathbb{R} \times \Sigma \to \mathbb{R}^2$ is continuous and bijective but it is not a homeomorphism. In this example $\Sigma \cong j(\Sigma)$ is a disconnected, closed submanifold (with boundary) of Z. In higher dimension, e.g. $\dim_{\mathbb{R}} Z = 3$, one can constuct a similar example with Σ a contractible, closed submanifold (with boundary) of Z.

Now we assume that in addition Z has the structure of a G-equivariant fiber bundle, i.e. that there exists a closed topological K-subspace P of Z such that the map

$$G \times_K P \to Z, \quad [g,p] \to g \cdot p$$

is a homeomorphism.

Lemma 4.2. *If the map $q : \Sigma \to P/K$, given by $x \to P \cap G \cdot j(x)$ is proper, then $\psi : G \times_K N \to Z,\ [g,x] \to g \cdot j(x)$ is a homeomorphism.*

Proof. By Lemma 4.1, it is sufficient to show that the map $\widetilde{\psi} : G \times \Sigma \to Z$ is proper. Let $\{(g_n, x_n)\}_n$ be a sequence in $G \times \Sigma$, with $g_n \cdot j(x_n) \to z_0$. Choose $\{(h_n, p_n)\}_n$ in $G \times P$ such that $g_n \cdot j(x_n) = h_n \cdot p_n$. Since the canonical projection $G \times P \to G \times_K P$ is proper (cf. [Bou89], Prop. 2, p. 252), the map $G \times P \to Z$, given by $(g,z) \to g \cdot z$, is proper. Thus, by passing to a subsequence if necessary, we may assume that $(h_n, p_n) \to (h_0, p_0)$. In particular, $q(x_n) := P \cap G \cdot j(x_n) = K \cdot p_n \to K \cdot p_0$. Since the map q is proper by assumption, by passing to a subsequence if necessary, one has that $x_n \to x_0$, for some $x_0 \in \Sigma$. Thus $j(x_n) \to j(x_0)$. By the properness of the G-action, the map $G \times Z \to Z \times Z$, given by $(g,z) \to (z, g \cdot z)$, is proper as well. Therefore, the sequence $\{(g_n, x_n)\}_n$ converges to (g_0, x_0), for some g_0 in G. As a result the map $\widetilde{\psi} : G \times \Sigma \to Z$ is proper, and the statement follows from Lemma 4.1. $\square$

As a matter of fact, the converse of the above lemma holds true as well. Indeed if $\psi : G \times_K N \to Z,\ [g,x] \to g \cdot j(x)$ is a homeomorphism, then Z/G is homeomorphic to N/K, as well as to P/K, being Z homeomorphic to $G \times_K P$. Therefore there is a commutative diagram

$$\begin{tikzcd} \Sigma \arrow[r] \arrow[dr] & G \times_K N \arrow[r, "\psi"] \arrow[d] & Z \arrow[d] \\ & N/K \arrow[r] & P/K\,, \end{tikzcd}$$

where the map $N/K \to P/K$ is a homeomorphism. As Σ is closed in N, the restriction $\Sigma \to N/K$ of the natural projection $G \times_K N \to N/K$ is proper. Hence the map $q : \Sigma \to P/K,\ x \to P \cap G \cdot j(x)$, given in the above diagram as the composition of proper maps, is proper, as claimed.

Note that, being Z connected by assumption, if ψ is a homeomorphism and K is connected, then N is necessarily connected. Indeed, in this case the principal bundle $G \times N \to G \times_K N$ has connected base and fibers. Thus the total space $G \times N$ is connected, implying that N is connected.

For later use we also mention the following corollary.

Corollary 4.3. *Assume that there exists a continuous, G-invariant function $f \colon Z \to \mathbb{R}$ such that $f \circ j|_\Sigma : \Sigma \to \mathbb{R}$ is proper. Then ψ is a homeomorphism.*

Proof. By Lemma 4.1, it is sufficient to show that the map

$$\widetilde{\psi} : G \times \Sigma \to Z,\ (g,x) \to g \cdot j(x)$$

is proper. Let $\{(g_n, x_n)\}_n$ be a sequence in $G \times \Sigma$ such that $\{g_n \cdot j(x_n)\}_n$ converges to an element z_0 in Z. We need to show that, by passing to a subsequence if necessary, the sequence $\{(g_n, x_n)\}_n$ converges in $G \times \Sigma$. Let U be a compact neighborhood of $f(z_0)$ in $\mathbb{R}$. By assumption, the set $V := (f \circ j|_\Sigma)^{-1}(U)$ is a compact subset of Σ. By the continuity and the G-invariance of f one has $f(j(x_n)) = f(g_n \cdot j(x_n)) \to f(z_0)$. Therefore $x_n \in V$ for n large enough. Thus, by passing to a subsequence if necessary, $\{x_n\}_n$ converges to an element x_0 of Σ and $j(x_n) \to j(x_0)$. Finally, by the properness of the G-action, the map $G \times Z \to Z \times Z$, given by $(g, z) \to (z, g \cdot z)$, is proper. Hence, by passing to a subsequence if necessary, $\{(g_n, x_n)\}_n$ converges to (g_0, x_0), for some g_0 in G. This concludes the proof of the corollary. □

Remark 4.4. The function $f \circ j|_\Sigma$ is proper if and only if $f \circ j$ is proper. As Σ is closed in N, one implication is clear. For the converse, let C be a compact subset of $\mathbb{R}$. Then

$$(f \circ j)^{-1}(C) = K \cdot (f \circ j|_\Sigma)^{-1}(C)\,,$$

which is compact if $(f \circ j|_\Sigma)^{-1}(C)$ is compact (cf. [Bou89], Cor. I, p. 251).

4.2. **A slice in the anti-holomorphic tangent bundle.** Let G/K be an irreducible Hermitian symmetric space. Resuming the notation of Section 2, denote by $\mathfrak{a}^+$ the open positive Weyl chamber in $\mathfrak{a}$ and by $\overline{\mathfrak{a}^+}$ its topological closure, given by

$$\mathfrak{a}^+ := \{\sum_{j=1}^r x_j A_j \ :\ x_1 > \cdots > x_r > 0\}, \quad \overline{\mathfrak{a}^+} = \{\sum_{j=1}^r x_j A_j \ :\ x_1 \geq \cdots \geq x_r \geq 0\}.$$

Define the closed hyperoctant

$$\mathfrak{a}^{\llcorner} := \{\sum_{j=1}^r x_j A_j \ :\ x_j \geq 0,\ j = 1, \dots, r\}.$$

The set $\overline{\mathfrak{a}^+}$ is a perfect slice for the adjoint action of K on $\mathfrak{p}$, and

$$\mathfrak{a}^{\llcorner} = W_K(\mathfrak{a})^+ \cdot \overline{\mathfrak{a}^+}.$$

Similarly, denote by $(\Lambda_r^{\llcorner})^+$ the open positive Weyl chamber in $\Lambda_r^{\llcorner}$, and by $\overline{(\Lambda_r^{\llcorner})^+}$ its topological closure, given by

$$(\Lambda_r^{\llcorner})^+ := \{\sum_{j=1}^r x_j E_j \ :\ x_1 > \cdots > x_r > 0\}, \quad \overline{(\Lambda_r^{\llcorner})^+} = \{\sum_{j=1}^r x_j E_j, \ :\ x_1 \geq \cdots \geq x_r \geq 0\},$$

respectively. By Lemma 3.1 and Corollary 3.2, one has

$$\Lambda_r^{\llcorner} = W_K(\Lambda_r) \cdot \overline{(\Lambda_r^{\llcorner})^+}.$$

Consider the homeomorphism

$$\Phi : \Lambda_r^{\llcorner} \to \mathfrak{a}^{\llcorner}, \quad \sum x_j E_j \to \frac{1}{2} \sum \log(1 + x_j) A_j\,,$$

and the K-equivariant linear isomorphism

$$\tau \colon \mathfrak{p} \to \mathfrak{p}^{0,1}, \quad Y \to -\frac{1}{2}(Y + iJ_0 Y)\,. \tag{12}$$

The isomorphism τ maps $\mathfrak{a}$, a slice for the Ad_K-action on $\mathfrak{p}$, onto a slice for the Ad_K-action on $\mathfrak{p}^{0,1}$, and induces a homeomorphism between the respective fundamental domains $\overline{\mathfrak{a}^+} \subset \mathfrak{a}$ and $\tau(\overline{\mathfrak{a}^+})$ in $\mathfrak{p}^{0,1}$.

The next lemma is crucial for the main result of this section. It states that inside Ξ^+ the nilpotent slice $\exp i\Lambda_r^{\llcorner} \cdot x_0$ can be mapped *continuously* onto a slice in $\exp \mathfrak{p}^{0,1} \cdot x_0$, by elements of the abelian group $A = \exp \mathfrak{a}$.

Lemma 4.5. *For every X in $\Lambda_r^{\llcorner}$ one has*

$$\exp(iX) = \exp\Phi(X)\exp\left(-\frac{1}{2}(\Psi(X)+iJ_0\Psi(X)\right)\exp i\chi(X),$$

where $\chi : \Lambda_r^{\llcorner} \to \mathfrak{k}$ is defined by $\sum x_j E_j \to \sum \sinh^{-1}\left(\frac{x_j}{2\sqrt{1+x_j}}\right)(E_j + \theta E_j)$. Thus

$$\exp(iX)\cdot x_0 = \exp\Phi(X)\exp\left(-\frac{1}{2}(\Psi(X)+iJ_0\Psi(X)\right)\cdot x_0\ .$$

Proof. Write $X = \sum x_j E_j$ as a sum of nilpotent elements in the embedded $\mathfrak{sl}(2)$-triples defined in (5). By Definition 2.1, the complex structure J_0 of G/K induces the invariant complex structure defined by $\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on each of the associated rank-one symmetric spaces. This fact, together with the commutativity of such triples in $\mathfrak{g}$ and of the corresponding groups in $G^{\mathbb{C}}$, reduces the proof to the case of $G = SL(2,\mathbb{R})$. Then the equality to be proved reads as

$$\exp i\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \exp\Phi\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\exp -\frac{1}{2}\left(\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} + i\begin{pmatrix} 0 & -x \\ -x & 0 \end{pmatrix}\right)\mathrm{SO}(2,\mathbb{C})\,.$$

One can easily check that the matrix

$$M = \exp i\sinh^{-1}\left(\frac{x}{2\sqrt{1+x}}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{\sqrt{1+x}}\begin{pmatrix} 1+\frac{x}{2} & i\frac{x}{2} \\ -i\frac{x}{2} & 1+\frac{x}{2} \end{pmatrix}$$

belongs to $\exp i\mathfrak{so}(2,\mathbb{R}) \subset SO(2,\mathbb{C})$, and that the following identity holds

$$\begin{pmatrix} 1 & ix \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{1+x} & 0 \\ 0 & \sqrt{1+x}^{-1} \end{pmatrix}\begin{pmatrix} 1-\frac{x}{2} & i\frac{x}{2} \\ i\frac{x}{2} & 1+\frac{x}{2} \end{pmatrix} M\,.$$

This concludes the proof of the lemma. □

Lemma 4.6.
(i) *Let X be an element in $\overline{(\Lambda_r^{\llcorner})^+}$. Then*

$$Z_K(X) = Z_K(\Psi(X)) = Z_K(\Phi(X))\,.$$

(ii) *Let X and X' be elements in $\overline{(\Lambda_r^{\llcorner})^+}$ such that*

$$\Psi(X') = \mathrm{Ad}_k\Psi(X), \qquad \textit{for some } k \in K.$$

Then $X' = X$ and $k \in Z_K(X)$.

Proof. (i) We begin by proving that $Z_K(X) = Z_K(\Psi(X))$. Since the map $\Psi(X) = [Z_0, X - \theta X]$ defined in (11) is K-equivariant, there is an inclusion

$$Z_K(X) \subset Z_K(\Psi(X)).$$

We prove the opposite one by showing that an element $k \in Z_K(\Psi(X))$ centralizes both $X - \theta X$ and $X + \theta X$. From

$$[Z_0, X-\theta X] = \mathrm{Ad}_k[Z_0, X-\theta X] = [Z_0, \mathrm{Ad}_k(X-\theta X)]$$

and the fact that ad_{Z_0} is bijective on $\mathfrak{p}$ (it is a complex structure), we obtain that $k \in Z_K(X-\theta X)$. Before showing that $k \in Z_K(X+\theta X)$, we make a small digression.

Given a subset Δ of $\Delta(\mathfrak{g},\mathfrak{a})^+$, the associated orbit stratum in the closure of the Weyl chamber $\overline{\mathfrak{a}^+}$ is by definition

$$\overline{\mathfrak{a}^+_\Delta} := \{\, A \in \mathfrak{a}^+ \ :\ \beta(A) = 0 \text{ if } \beta \in \Delta,\ \beta(A) > 0 \text{ if } \beta \in \Delta(\mathfrak{g},\mathfrak{a})^+ \setminus \Delta\}\,.$$

Let H be an element in $\mathfrak{a}$. Since $G^{\mathbb{C}}$ is simply connected, the centralizer $Z_{G^{\mathbb{C}}}(H)$ of H in $G^{\mathbb{C}}$ is a connected group (see [Hum95], p.33) with Lie algebra

$$Z_{\mathfrak{g}^{\mathbb{C}}}(H) = Z_{\mathfrak{k}^{\mathbb{C}}}(\mathfrak{a}) \oplus \mathfrak{a}^{\mathbb{C}} \oplus \bigoplus_{\substack{\alpha\in\Delta(\mathfrak{g}^{\mathbb{C}},\mathfrak{a}^{\mathbb{C}})\\ \alpha(H)=0}} \mathfrak{g}^{\alpha}. \tag{13}$$

Moreover, since $\sigma(H) = H$ and $\theta(H) = -H$, the group $Z_{G^{\mathbb{C}}}(H)$ is both σ and θ-stable. As a result, if two elements H_1 and H_2 of $\overline{\mathfrak{a}^+}$ lie in the same orbit stratum, then $Z_{G^{\mathbb{C}}}(H_1) = Z_{G^{\mathbb{C}}}(H_2)$ and likewise $Z_K(H_1) = Z_K(H_2)$.

Write $X = \sum x_j E_j$ and $\Psi(X) = \sum x_j A_j$. Since the elements $\sum x_j A_j$ and $\sum \sqrt{\frac{x_j}{2}} A_j$ lie in the same orbit stratum of $\overline{\mathfrak{a}^+}$, one has $Z_K(\Psi(X)) = Z_K(\sum \sqrt{\frac{x_j}{2}} A_j)$. Moreover, since

$$\sum_j \sqrt{\frac{x_j}{2}}(E_j - \theta E_j) = [-Z_0, \sum_j \sqrt{\frac{x_j}{2}} A_j],$$

one also has $Z_K(\Psi(X)) \subset Z_K\big(\sum \sqrt{\frac{x_j}{2}}(E_j - \theta E_j)\big)$. Then the equality

$$Z_K(\Psi(X)) = Z_K(X + \theta X)$$

follows from

$$\mathrm{Ad}_k(X + \theta X) =$$

$$\mathrm{Ad}_k\big(\sum_j x_j(E_j + \theta E_j)\big) = \mathrm{Ad}_k[\sum_j \sqrt{\frac{x_j}{2}} A_j, \sum_j \sqrt{\frac{x_j}{2}}(E_j - \theta E_j)] =$$

$$[\mathrm{Ad}_k(\sum_j \sqrt{\frac{x_j}{2}} A_j), \mathrm{Ad}_k(\sum_j \sqrt{\frac{x_j}{2}}(E_j - \theta E_j))] = [\sum_j \sqrt{\frac{x_j}{2}} A_j, \sum_j \sqrt{\frac{x_j}{2}}(E_j - \theta E_j)] =$$

$$\sum_j x_j(E_j + \theta E_j) = X + \theta X\,.$$

Since $X = \frac{1}{2}(X - \theta X) + \frac{1}{2}(X + \theta X)$, we conclude that

$$Z_K(X) = Z_K(\Psi(X)).$$

Next we show that

$$Z_K(\Psi(X)) = Z_K(\Phi(X)).$$

From the definition of the maps Ψ, Φ and of the roots defining $\mathfrak{a}^+$ (cf. Sect. 2) it is clear that $\Psi(X)$ and $\Phi(X)$ lie in the same orbit stratum of $\overline{\mathfrak{a}^+}$. Then the desired equality follows from the above considerations.

(ii) By definition of $\overline{(\Lambda_r^{\llcorner})^+}$, the elements $\Psi(X)$ and $\Psi(X')$ lie in $\overline{\mathfrak{a}^+}$, which is a perfect slice for the Ad_K-action on $\mathfrak{p}$. Then $\Psi(X') = \Psi(X)$ and $k \in Z_K(\Psi(X)) = Z_K(X)$. Since the map $\Psi\colon \Lambda_r \to \mathfrak{a}$ is injective, it follows that $X' = X$. $\square$

Theorem 4.7. *Let G/K be an irreducible Hermitian symmetric space. Then the map*

$$\psi : G \times_K \mathcal{N}^+ \to \Xi^+, \quad [g, X] \to g \exp iX \cdot x_0$$

is a G-equivariant homeomorphism.

Proof. The map ψ is G-equivariant by construction. Since $\Xi^+ = G \exp \mathfrak{p}^{0,1} \cdot x_0$ (see (1)), Lemma 4.5 implies that ψ is surjective. Recall that by Corollary 3.2, one has $\mathcal{N}^+ = \mathrm{Ad}_K \overline{(\Lambda_r^{\llcorner})^+}$. Hence, in order to prove that ψ is injective, it is sufficient to show that if the identity

$$g \exp iX \cdot x_0 = \exp iX' \cdot x_0, \tag{14}$$

holds true for some $g \in G$ and $X, X' \in \overline{(\Lambda_r^{\llcorner})^+}$, then

$$g \in K, \quad \text{and} \quad X' = \mathrm{Ad}_g X.$$

By Lemma 4.5, equation (14) is equivalent to

$$g \exp \Phi(X) \exp \left(-\frac{1}{2}(\Psi(X) + iJ_0\Psi(X)) \right) \cdot x_0 =$$

$$\exp \Phi(X') \exp \left(-\frac{1}{2}(\Psi(X') + iJ_0\Psi(X')) \right) \cdot x_0 \ .$$

Then, by identifying Ξ^+ with $G \times_K \mathfrak{p}^{0,1}$ under the G-equivariant diffeomorphism (1), the above identity becomes

$$[g \exp \Phi(X), -\frac{1}{2}(\Psi(X) + iJ_0\Psi(X))] = [\exp \Phi(X'), -\frac{1}{2}(\Psi(X') + iJ_0\Psi(X'))] \,.$$

In other words, there exists $k \in K$ such that

$$\exp \Phi(X') = g \exp \Phi(X) k^{-1} \quad \text{and} \quad \Psi(X') = \mathrm{Ad}_k \Psi(X) \,. \tag{15}$$

From the second equality in (15) and Lemma 4.6, one obtains the relations

$$X = X' \quad \text{and} \quad k \in Z_K(\Psi(X)) = Z_K(\Phi(X)) = Z_K(X),$$

which plugged in the first equality of (15) yield $g = k$. In conclusion, we have obtained

$$g \in Z_K(X) \quad \text{and} \quad X' = X = \mathrm{Ad}_g X,$$

as desired.

Next we are going to show that ψ is a homeomorphism. Consider the K-invariant fiber $P := \exp \mathfrak{p}^{0,1} \cdot x_0$ in $\Xi^+ \cong G \times_K \mathfrak{p}^{0,1}$. Since the map $G \times_K P \to \Xi^+$, given by $[g, z] \to g \cdot z$, is a G-equivariant diffeomorphism, by Lemma 4.2 it is sufficient to show that the following map is proper

$$q : \Lambda_r^{\llcorner} \to P/K, \quad X \to P \cap G \exp iX \cdot x_0 \,.$$

So let $\{X_n\}_n$ be a sequence diverging in $\Lambda_r^{\llcorner}$. Then $\{-\frac{1}{2}(\Psi(X_n) + iJ_0\Psi(X_n))\}_n$ diverges in $\mathfrak{p}^{0,1}$. Consequently, the sequence $\{\exp -\frac{1}{2}(\Psi(X_n) + iJ_0\Psi(X_n)) \cdot x_0\}_n$ diverges in $\exp \mathfrak{p}^{0,1} \cdot x_0$ and, by Lemma 4.5, every element $\exp -\frac{1}{2}(\Psi(X_n) + iJ_0\Psi(X_n)) \cdot x_0$ lies in $G \exp iX_n \cdot x_0 \cap \exp \mathfrak{p}^{0,1} \cdot x_0$. Since the canonical projection $\exp \mathfrak{p}^{0,1} \cdot x_0 \to \exp \mathfrak{p}^{0,1} \cdot x_0/K$ is proper, the sequence $\{\exp \mathfrak{p}^{0,1} \cdot x_0 \cap G \exp iX_n \cdot x_0 = \exp\left(-\frac{1}{2}(\Psi(X) + iJ_0\Psi(X) \cdot x_0\right)\}_n$ diverges in $\exp \mathfrak{p}^{0,1} \cdot x_0/K$. Thus the map q is proper, as wished. □

From the above proposition we obtain the following consequences.

Corollary 4.8. *The restriction of the map* (11)

$$\Psi \colon \mathcal{N}^+ \to \mathfrak{p}, \qquad \Psi(X) = [Z_0, X - \theta X] = J_0(X - \theta X)$$

is a K-equivariant homeomorphism. Likewise, the maps

$$\mathcal{N}^+ \to \mathfrak{p}, \qquad X \to X - \theta X$$

and

$$\Psi^{0,1} : \mathcal{N}^+ \to \mathfrak{p}^{0,1}, \qquad X \to \frac{1}{2}\big(\Psi(X) + iJ_0\Psi(X)\big)$$

are K-equivariant homeomorphisms.

Proof. The map Ψ is K-equivariant, since both ad_{Z_0} and the Cartan involution θ commute with the Adjoint action of K. It is also surjective, since its image contains the closure of the Weyl chamber $\overline{\mathfrak{a}^+}$. In order to show that Ψ is injective, it is enough to consider pairs of elements X and $\mathrm{Ad}_k(X')$, with $X,\, X' \in \overline{(\Lambda_r^{\llcorner})^+}$ and $k \in K$. Assume that $\Psi(X) = \Psi(\mathrm{Ad}_k(X'))$. Then by Lemma 4.6, one obtains

$$X = X', \quad k \in Z_K(\Psi(X)) = Z_K(X).$$

In particular $X = \mathrm{Ad}_k(X')$, as wished.

It remains to show that Ψ is proper. This follows from the fact that $\Psi(X) \neq 0$, if $X \neq 0$, and $\Psi(tX) = t\Psi(X)$, for all $t \in \mathbb{R}$. As a consequence, the image of any divergent sequence in $\mathcal{N}^+$ under Ψ is a divergent sequence in $\mathfrak{p}$.

The second part of the statement follows directly from the fact that both $J_0 : \mathfrak{p} \to \mathfrak{p}$ and the map $\mathfrak{p} \to \mathfrak{p}^{0,1}$, given by $Y \to \frac{1}{2}\big(Y + iJ_0(Y)\big)$, are K-equivariant linear isomorphisms. □

5. An example.

In this section, we give a different proof of Theorem 4.7 in the cases of $G = Sp(2,\mathbb{R})$ and $G = Sp(1,\mathbb{R}) \cong SL(2,\mathbb{R})$. This proof uses Corollary 4.3 and a global G-invariant function $f : \Xi^+ \to \mathbb{R}$, with the property that the map

$$\Lambda_r^{\llcorner} \to \mathbb{R}, \quad X \to f(\exp iX \cdot x_0)$$

is proper. As a matter of fact, the function f is the restriction of a G-invariant function defined on $G^{\mathbb{C}}/K^{\mathbb{C}}$.

Consider the real symplectic group

$$G = Sp(r,\mathbb{R}) = \left\{ Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M^{2r\times 2r}(\mathbb{R}) \ :\ {}^tZJZ = J \right\}, \qquad J := \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$$

and its complexification $G^{\mathbb{C}} = Sp(r,\mathbb{C})$. By Witt's theorem, $G^{\mathbb{C}}$ acts transitively on the Grassmannian of J-isotropic complex r-planes in $\mathbb{C}^{2r}$

$$Y = \{\, \mathbf{x} \text{ complex } r\text{-plane in } \mathbb{C}^{2r} \ :\ J|_{\mathbf{x}\times\mathbf{x}} = 0 \,\}.$$

By considering all possible bases of $\mathbf{x}$, given as r-tuples of column vectors in $\mathbb{C}^{2r}$, we view Y as the quotient of

$$\widehat{Y} := \left\{ \mathcal{R} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \ :\ R_1,\, R_2 \in M^{r\times r}(\mathbb{C}),\ \mathrm{rank}\mathcal{R} = r,\ {}^t\mathcal{R}J\mathcal{R} = 0 \right\}$$

by the right action of $GL(r,\mathbb{C})$ defined by

$$M \cdot \mathcal{R} := \mathcal{R}M^{-1}, \qquad M \in GL(r,\mathbb{C}).$$

Note that $G^{\mathbb{C}}$ acts on $\widehat{Y}$ by left multiplication and that the canonical projection

$$\widehat{Y} \to Y, \qquad \mathcal{R} \to [\mathcal{R}]$$

is $G^{\mathbb{C}}$-equivariant. Fix the base point $\mathbf{x}_+ = \begin{bmatrix} iI_r \\ I_r \end{bmatrix} \in Y$. Then the complexification $G^{\mathbb{C}}/K^{\mathbb{C}}$ of G/K can be realized in the product $Y \times \overline{Y}$ as the open dense orbit

$$G^{\mathbb{C}}/K^{\mathbb{C}} \cong G^{\mathbb{C}} \cdot x_0 = \left\{ ([\mathcal{R}], [\mathcal{S}]) \in Y \times \overline{Y} \ :\ \left|\mathcal{R}\overline{\mathcal{S}}\right| \neq 0 \right\},$$

where $x_0 = (\mathbf{x}_+, \mathbf{x}_+)$ and $\left|\mathcal{R}\overline{\mathcal{S}}\right|$ denotes the determinant of the matrix formed by $\mathcal{R}$ and $\overline{\mathcal{S}}$ (see [FHW05], p. 68). Define two real G-invariant functions on $G^{\mathbb{C}}/K^{\mathbb{C}}$ as follows

$$f_1([\mathcal{R}], [\mathcal{S}]) = \left\| \frac{\left|{}^t\mathcal{R}J\mathcal{S}\right|}{\left|\mathcal{R}\overline{\mathcal{S}}\right|} \right\|^2, \qquad f_2([\mathcal{R}], [\mathcal{S}]) = \frac{\left|{}^t\mathcal{R}J\overline{\mathcal{R}}\right| \left|{}^t\mathcal{S}J\overline{\mathcal{S}}\right|}{\left\|\left|\mathcal{R}\overline{\mathcal{S}}\right|\right\|^2}.$$

A simple computation shows that for

$$X = \begin{pmatrix} O & D \\ O & O \end{pmatrix} \in \Lambda_r, \quad \text{with} \quad D = diag(x_1, \ldots, x_r),$$

one has

$$f_1(\exp iX \cdot x_0) = (1 - x_1^2) \ldots (1 - x_r^2) \qquad \text{and} \qquad f_2(\exp iX \cdot x_0) = x_1^2 \ldots x_r^2.$$

For $r = 2$, define the G-invariant function $f := 1 - f_1 + f_2$ on $G^{\mathbb{C}}/K^{\mathbb{C}}$. By composing f with the embedding $\Lambda_2^{\llcorner} \to \exp i\Lambda_2^{\llcorner} \cdot x_0$, given by $X \to \exp iX \cdot x_0$, one obtains an exhaustion function of $\Lambda_2^{\llcorner}$

$$\Lambda_2^{\llcorner} \to \mathbb{R}, \qquad X = x_1 E_1 + x_2 E_2 \to f(\exp iX \cdot x_0) = x_1^2 + x_2^2.$$

This fact, together with Corollary 4.3, yields an alternative proof of Theorem 4.7 for $G = Sp(2, \mathbb{R})$. A similar proof works for $G = SL(2, \mathbb{R}) = Sp(1, \mathbb{R})$, using the global G-invariant function f_2.

It would be interesting to obtain similar global smooth G-invariant functions on $G^{\mathbb{C}}/K^{\mathbb{C}}$ in the higher rank case and in general for all Hermitian symmetric spaces. For instance, in the case of $G = Sp(r, \mathbb{R})$, for $r \geq 3$, we know no global G-invariant function whose restriction to $\exp i\Lambda_r^{\llcorner} \cdot x_0$ determines a non-constant symmetric polynomial on Λ_r other than $(1 - x_1^2) \ldots (1 - x_r^2)$ or $x_1^2 \ldots x_r^2$.

Note that as a consequence of Theorem 4.7, every function h on $\exp i\Lambda_r \cdot x_0$, arising from a symmetric polynomial in the ring $\mathbb{R}[x_1^2, \ldots, x_r^2]$, extends continuously and G-equivariantly at least to $\Xi^+ \cup \Xi^-$. It would be interesting to know whether such an extension is smooth and if a further extension to a G-invariant, smooth function defined on $G^{\mathbb{C}}/K^{\mathbb{C}}$ exists. If so, one could look for an explicit global realization of h, e.g. in terms of the coordinates of $G^{\mathbb{C}}/K^{\mathbb{C}}$ in $Y \times \overline{Y}$.

6. G-orbit structure of Ξ^+.

By Theorem 4.7, the map

$$\psi \colon G \times_K \mathcal{N}^+ \to \Xi^+, \qquad [g, X] \to g \exp iX \cdot x_0$$

is a G-equivariant homeomorphism. Hence, every G-orbit in Ξ^+ meets $\exp i\mathcal{N}^+ \cdot x_0$ in a K-orbit and the G-orbit structure of Ξ^+ is completely determined by the K-orbit structure of the nilpotent cone $\mathcal{N}^+ = \mathrm{Ad}_K \Lambda_r^{\llcorner}$. Moreover, by Corollary 4.8, the cone $\mathcal{N}^+$ is K-equivariantly homeomorphic to $\mathfrak{p}$. In this section we give further details.

Corollary 6.1. *Let X be an element in $\Lambda_r^{\llcorner}$, and let $\exp iX \cdot x_0$ be the corresponding point in Ξ^+. Then the isotropy subgroup of $\exp iX \cdot x_0$ in G is given by*

$$G_{\exp iX \cdot x_0} = Z_K(X) = Z_K(\Psi(X)).$$

Proof. Since $\exp iX \cdot x_0 = \psi([e, X])$, by Theorem 4.7 one has

$$G_{\exp iX \cdot x_0} = G_{[e,X]} = Z_K(X),$$

which proves the first equality. The second equality follows from Corollary 4.8. □

Definition 6.2. *An element $X \in \Lambda_r^{\llcorner}$ is generic if $\exp iX \cdot x_0$ lies on a maximal dimensional G-orbit in Ξ^+. Equivalently, if $Z_K(X) = Z_K(\Lambda_r^{\llcorner})$. The set of generic elements in $\Lambda_r^{\llcorner}$ is denoted by $(\Lambda_r^{\llcorner})_{gen}$.*

Lemma 6.3. *An element X in $\Lambda_r^\llcorner$ is generic if and only if $\Psi(X) = [Z_0, X - \theta X]$ is generic in $\mathfrak{a}$. In particular the set $(\Lambda_r^\llcorner)_{gen}$ is given by*

$$(\Lambda_r^\llcorner)_{gen} = \{\sum_j x_j E_j \ : \ x_j \neq 0 \text{ and } x_j \neq x_l, \text{ for } j, l = 1, \dots, r \text{ and } j \neq l\},$$

and is dense in $\Lambda_r^\llcorner$.

Proof. By Corollary 6.1 one has $Z_K(X) = Z_K(\Psi(X))$. Moreover $\Psi(\Lambda_r^\llcorner) = \mathfrak{a}^\llcorner$ and $Z_K(\Lambda_r^\llcorner) = Z_K(\Lambda_r) = Z_K(\mathfrak{a})$ (see Lemma 3.1). Hence X is generic if and only if $Z_K(\Psi(X)) = Z_K(\mathfrak{a})$, i.e. if and only if $\Psi(X)$ is a generic element of $\mathfrak{a}$.

For $H \in \mathfrak{a}$ the Lie algebra of $Z_K(H)$ is given by

$$Z_{\mathfrak{k}}(H) = \mathfrak{a} \oplus Z_{\mathfrak{k}}(\mathfrak{a}) \oplus \bigoplus_{\alpha(H)=0} \mathfrak{g}[\alpha]_{\mathfrak{k}},$$

where $\mathfrak{g}[\alpha]_{\mathfrak{k}}$ is the $\mathfrak{k}$-component of the θ-stable subspace $\mathfrak{g}[\alpha] = \mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}$ of $\mathfrak{g}$. From this and the fact that $\Delta(\mathfrak{g}, \mathfrak{a})$ is either of type C_r or BC_r, one has

$$\mathfrak{a}_{gen} = \Big\{ \sum_j a_j A_j \ : \ a_j \neq 0 \text{ and } a_j \neq \pm a_l, \text{ for } j, l = 1, \dots, r \text{ and } j \neq l \Big\}.$$

Given an element $X = \sum x_j E_j \in \Lambda_r^\llcorner$, one has $\Psi(X) = \sum x_j A_j$. Thus X is generic if and only if $x_j \neq 0$ and $x_j \neq x_l$, for $j, l = 1, \dots, r$ and $j \neq l$, as claimed. □

Proposition 6.4. *Let $X \in \Lambda_r^\llcorner$ and $k \in K$ be elements such that $\mathrm{Ad}_k X \in \Lambda_r$. Then*
(i) *$\mathrm{Ad}_k X$ lies in $\Lambda_r^\llcorner$, implying that $\mathcal{N}^+ \cap \Lambda_r = \Lambda_r^\llcorner$,*
(ii) *there exists $n \in N_K(\Lambda_r)$ such that $\mathrm{Ad}_k X = \mathrm{Ad}_n X$.*
In particular $\Lambda_r^\llcorner$ is closed in $\mathcal{N}^+$ and the intersection $\mathrm{Ad}_K X \cap \Lambda_r$, of the Ad_K-orbit of X with Λ_r, is given by the $W_K(\Lambda_r)$-orbit of X in $\Lambda_r^\llcorner$.

Proof. (i) We first consider the case when k is an element of $N_K(\mathfrak{a})$ and we set $n := k$. Then Ad_n acts on $\mathfrak{a}$ by signed permutations of the A_j.

Claim. If for some indices $i, h \in \{1, \dots, r\}$ one has $\mathrm{Ad}_n(A_i) = A_h$, then $\mathrm{Ad}_n(E_i) \in \mathfrak{g}^{\lambda_h}$; if $\mathrm{Ad}_n(A_i) = -A_h$, then $\mathrm{Ad}_n(E_i) \in \mathfrak{g}^{-\lambda_h}$.

Proof of the claim. From $[A_i, E_i] = 2E_i$, by applying Ad_n to both terms of the equation, we obtain

$$[\mathrm{Ad}_n A_i, \mathrm{Ad}_n E_i] = [A_h, \mathrm{Ad}_n E_i] = 2\mathrm{Ad}_n E_i.$$

Then, in order to show that $\mathrm{Ad}_n E_i \in \mathfrak{g}^{\lambda_h}$, we need to show that $[A_l, \mathrm{Ad}_n E_i] = 0$, for all $l \neq h$. Write $[A_l, \mathrm{Ad}_n E_i] = \mathrm{Ad}_n[\mathrm{Ad}_{n^{-1}} A_l, E_i]$, and observe that $\mathrm{Ad}_{n^{-1}} A_l \in \{\pm A_m\}$, for some $m \neq i$. Then

$$\mathrm{Ad}_n[\mathrm{Ad}_{n^{-1}} A_l, E_i] = \mathrm{Ad}_n[\pm A_m, E_i] = 0,$$

as desired. A similar argument shows the second statement, and concludes the proof of the claim.

Write $X = \sum x_j E_j$, with $x_j \geq 0$, and $\mathrm{Ad}_n X = \sum y_j E_j$, with $y_j \in \mathbb{R}$. Then $\Psi(X) = \sum x_j A_j$ and, since Ψ is Ad_K-equivariant, one has

$$\mathrm{Ad}_n(\Psi(X)) = \sum x_j \mathrm{Ad}_n A_j = \Psi(\mathrm{Ad}_n X) = \sum y_j A_j \,.$$

Thus, given $i \in \{1, \dots, r\}$, one has $y_h = x_i \geq 0$, if $\mathrm{Ad}_n A_i = A_h$, and $y_h = -x_i \leq 0$, if $\mathrm{Ad}_n A_i = -A_h$. In order to show that $\mathrm{Ad}_n X = \sum y_j E_j$ lies in $\Lambda_r^\llcorner$, we prove that $x_i = 0$ whenever $\mathrm{Ad}_n A_i = -A_h$.

Assume by contradiction that this is not the case. By the above claim, each $\mathrm{Ad}_n E_j$ lies in one of the root spaces of the direct sum $\Lambda_r \oplus \theta\Lambda_r = \bigoplus_j \mathfrak{g}^{\lambda_j} \oplus$

$\mathfrak{g}^{-\lambda_j}$. Moreover, $\mathrm{Ad}_n X = \sum x_j \mathrm{Ad}_n E_j$ has a non-zero component in $\mathfrak{g}^{-\lambda_h}$. This contradicts the fact that $\mathrm{Ad}_n X$ lies in Λ_r and concludes the proof in the case when $k = n$ is an element of $N_K(\mathfrak{a})$.

Next, the general case. Both elements $\Psi(X)$ and $\Psi(\mathrm{Ad}_k X) = \mathrm{Ad}_k(\Psi(X))$ belong to $\mathfrak{a}$ and, by [Kna04], Lemma 7.38, p.459, there exists an element $n \in N_K(\mathfrak{a})$ such that

$$\mathrm{Ad}_k(\Psi(X)) = \mathrm{Ad}_n(\Psi(X)).$$

Thus $n^{-1}k$ lies in $Z_K(\Psi(X))$ and also in $Z_K(X)$, by (i) of Lemma 4.6. Therefore

$$\mathrm{Ad}_k X = \mathrm{Ad}_n X.$$

Since we already showed that $\mathrm{Ad}_n X$ belongs to $\Lambda_r^{\llcorner}$, the proof of (i) is now complete.
(ii) By (i), both X and $\mathrm{Ad}_k X$ lie in $\Lambda_r^{\llcorner}$. Since $\Psi \colon \mathcal{N}^+ \to \mathfrak{p}$ is a K-equivariant homeomorphism (Cor. 4.8) and $\Psi(\Lambda_r^{\llcorner}) = \mathfrak{a}^{\llcorner}$, both $\Psi(X)$ and $\mathrm{Ad}_k \Psi(X)$ belong to $\mathfrak{a}^{\llcorner}$. Of course they lie on the same $W_K(\mathfrak{a})$-orbit. Recall that $W_K(\mathfrak{a})$ acts on $\mathfrak{a}$ by signed permutations and that, by definition, $\mathfrak{a}^{\llcorner} := \{\sum_{j=1}^r x_j A_j \ : \ x_j \geq 0,\ j = 1, \dots, r\}$. Thus there exists $\gamma \in W_K(\mathfrak{a})^+$ such that

$$\mathrm{Ad}_k \Psi(X) = \gamma \cdot \Psi(X)\,.$$

Furthermore, $W_K(\mathfrak{a})^+ = W_K(\Lambda_r^{\llcorner})$ by Lemma 3.1, implying that there exists $n \in N_K(\Lambda_r^{\llcorner})$ such that $\gamma = nZ_K(\mathfrak{a})$ and

$$\mathrm{Ad}_k \Psi(X) = \mathrm{Ad}_n \Psi(X)\,.$$

Now, by applying $\Psi^{-1} \colon \mathfrak{p} \to \mathcal{N}^+$ to both sides of the above equality, one obtains $\mathrm{Ad}_k X = \mathrm{Ad}_n X$, as wished. □

By Lemma 3.1 the closure $\overline{(\Lambda_r^{\llcorner})}^+$ of the open chamber

$$(\Lambda_r^{\llcorner})^+ := \{x_1 E_1 + \cdots + x_r E_r \ : \ x_1 > x_2 > \cdots > x_r > 0\}$$

is a perfect slice for the $W_K(\Lambda_r)$-action on $\Lambda_r^{\llcorner}$.

Corollary 6.5.
(i) *The closure* $\overline{(\Lambda_r^{\llcorner})}^+$ *of the open chamber* $(\Lambda_r^{\llcorner})^+$ *is a perfect slice for the* Ad_K*-action on* $\mathcal{N}^+$*.*
(ii) *For* $X \in \Lambda_r^{\llcorner}$ *one has*

$$G \exp iX \cdot x_0 \bigcap \exp i\Lambda_r^{\llcorner} \cdot x_0 = \exp i(W_K(\Lambda_r) \cdot X) \cdot x_0.$$

(iii) *There are homeomorphisms of orbit spaces*

$$\Xi^+/G \cong \Lambda_r^{\llcorner}/W_K(\Lambda_r) \cong \overline{(\Lambda_r^{\llcorner})}^+\,.$$

Proof. Part (i) follows from Proposition 6.4. For parts (ii) and (iii), Proposition 6.4(ii) implies that every G-orbit in $G \times_K \mathcal{N}^+$ intersects the closed subset $\Lambda_r^{\llcorner} \cong \{[e, X] \in G \times_K \mathcal{N}^+ \ : X \in \Lambda_r^{\llcorner}\}$ of $\mathcal{N}^+$ in a $W_K(\Lambda_r)$ orbit. Then the statements follow from the G-equivariance of the homeomorphism $\psi : G \times_K \mathcal{N}^+ \to \Xi^+$ (see Thm. 4.7). □

Remark 6.6. Observe that inside Ξ^+ there is a proper inclusion

$$\exp i\Lambda_r^{\llcorner} \cdot x_0 \ \subset \ \Xi^+ \cap \exp i\Lambda_r \cdot x_0,$$

and that the sets $\{X \in \Lambda_r \ : \ \exp iX \cdot x_0 \in \Xi^+\}$ and $\bigoplus_{j=1}^r (-1, \infty) E_j$ coincide (see [Krö08], p. 286). In fact, there exist elements $X \in \Lambda_r^{\llcorner}$, $Y \in \Lambda_r \setminus \Lambda_r^{\llcorner}$ and $g \in G \setminus K$ such that

$$g \exp iX \cdot x_0 = \exp iY \cdot x_0.$$

For example, for $G/K = SL(2,\mathbb{R})/SO(2,\mathbb{R})$, take $-1 < x < 1$ and $b := \sqrt{1-x^2}$. Then $\begin{pmatrix} 0 & b \\ -1/b & 0 \end{pmatrix} \in G$ and $\begin{pmatrix} -ix/b & 1/b \\ -1/b & -ix/b \end{pmatrix} \in SO(2,\mathbb{C})$. The relation

$$\begin{pmatrix} 0 & b \\ -1/b & 0 \end{pmatrix} \begin{pmatrix} 1 & ix \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -ix \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -ix/b & 1/b \\ -1/b & -ix/b \end{pmatrix}$$

shows that the elements $\exp i \begin{pmatrix} 0 & -x \\ 0 & 0 \end{pmatrix} \cdot x_0$ and $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \cdot x_0$ lie on the same G-orbit in Ξ^+, even though not on the same K-orbit.

In the higher rank case, for $\bar{\jmath} \in \{1, \ldots, r\}$, consider the subdomains

$$(-1,\infty)E_1 \oplus \cdots \oplus (-1,1)E_{\bar{\jmath}} \oplus \cdots \oplus (-1,\infty)E_r \tag{16}$$

of $\bigoplus_{j=1}^r (-1,\infty)E_j \subset \Lambda_r$. On each of them there are additional symmetries (induced by the G-action on Ξ^+) which identify elements which do not lie on the same Ad_K-orbit in $\mathfrak{g}$ (cf. Prop. 6.4). Namely, for $-1 < x < 1$, let $g_{\bar{\jmath}}$ be the image of the element

$$\begin{pmatrix} 0 & \sqrt{1-x^2} \\ -1/\sqrt{1-x^2} & 0 \end{pmatrix}$$

in the $SL(2,\mathbb{R})$-subgroup of G generated by the $\mathfrak{sl}(2)$-triple $\{E_{\bar{\jmath}}, \theta E_{\bar{\jmath}}, A_{\bar{\jmath}}\}$. Then

$$g_{\bar{\jmath}} \exp i(x_1E_1 + \cdots + x_{\bar{\jmath}}E_{\bar{\jmath}} + \cdots + x_rE_r) \cdot x_0 = \exp i(x_1E_1 + \cdots - x_{\bar{\jmath}}E_{\bar{\jmath}} + \cdots + x_rE_r) \cdot x_0 \,.$$

This shows that inside the $\bar{\jmath}^{th}$ subdomain of Λ_r defined in (16), the element $g_{\bar{\jmath}}$ induces the reflection with respect to the $\bar{\jmath}^{th}$-coordinate plane.

7. A distinguished Stein subdomain of Ξ^+.

Let G/K be an irreducible Hermitian symmetric space. The boundary of the crown domain Ξ contains a point whose G-orbit has locally minimal dimension. In the tube case, such an orbit is a Cayley type symmetric space G/H. From the compactly causal structure of G/H two distinguished G-invariant Stein domains $S^\pm$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$ arise, whose boundary contains G/H. The purpose of this section is to prove that one of these domains, namely S^+, is contained in Ξ^+. In the non-tube case, there is no Stein analogue of the domains $S^\pm$ (see Rem. 7.7).

Denote by $\{\omega_1, \ldots, \omega_r\}$ the dual basis of the simple roots $\{\alpha_1, \ldots, \alpha_r\}$, where $r = \mathrm{rank}(G/K)$. Define

$$g_1 := \exp\left(i\frac{\pi}{2}\frac{\omega_r}{k_r}\right) \in \exp i\mathfrak{a}\,, \tag{17}$$

where k_r is the coefficient of the r-th simple restricted root α_r in the highest root $\alpha_h \in \Delta(\mathfrak{g},\mathfrak{a})^+$. If G/K is of tube type, then the restricted root system is of type C_r and the highest root is given by $\alpha_h = 2\alpha_1 + \ldots + 2\alpha_{r-1} + \alpha_r$. Hence $k_r = 1$ and $g_1 = \exp(i\frac{\pi}{2}\omega_r)$. If G/K is not of tube type, then the restricted root system is of type BC_r and $\alpha_h = 2\alpha_1 + \ldots + 2\alpha_r$. Hence $k_r = 2$ and $g_1 = \exp(i\frac{\pi}{2}\frac{\omega_r}{2})$.

In both cases $|\alpha(\frac{\pi}{2}\frac{\omega_r}{k_r})| \le \frac{\pi}{2}$, for all restricted roots α, and $|\lambda_r(\frac{\pi}{2}\frac{\omega_r}{k_r})| = \frac{\pi}{2}$, where λ_r is as in (4). This shows that $x_1 = g_1 \cdot x_0$ is a point on the boundary of the crown domain. For $j = 1, \ldots, r$, define

$$g_{1,j} := \exp\left(i\frac{\pi}{2}\frac{A_j}{2}\right),$$

where A_j is as in (5). The element $g_{1,j}$ lies in the $SL(2,\mathbb{C})$-subgroup of $G^{\mathbb{C}}$ corresponding to the j^{th} triple defined in (5).

Lemma 7.1. *One has*

$$g_1 = \prod_{j=1}^{r} g_{1,j}.$$

Proof. In the tube case, (2) and the relations $\lambda_i(\frac{1}{2}A_j) = \delta_{ij}$, imply that $\alpha_j(\frac{1}{2}(A_1 + A_2 + \ldots + A_r)) = \delta_{jr}$, for $j = 1, \ldots, r$. Therefore $\omega_r = \frac{1}{2}(A_1 + A_2 + \ldots + A_r)$.
In the non-tube case, (3) and the relations $\lambda_i(\frac{1}{2}A_j) = \delta_{ij}$ imply that $\alpha_j(A_1 + A_2 + \ldots + A_r) = \delta_{jr}$, for $j = 1, \ldots, r$. Thus $\omega_r = A_1 + A_2 + \ldots + A_r$. Since $\mathfrak{a}$ is abelian, the identity

$$g_{1,1} \cdot \ldots \cdot g_{1,r} = \exp\Big(i\frac{\pi}{2}\frac{A_1}{2}\Big) \cdot \ldots \cdot \exp\Big(i\frac{\pi}{2}\frac{A_r}{2}\Big) =$$
$$= \exp\Big(i\frac{\pi}{2}\big(\frac{1}{2}(A_1 + A_2 + \ldots + A_r)\big)\Big) = g_1$$

holds true, as claimed. □

From now on, we assume the space G/K to be of tube type. We refer to Remark 7.7 for some details about the non-tube case.

Lemma 7.2. *Let G/K be an irreducible symmetric space of tube type. Then the G-orbit of the point $x_1 = g_1 \cdot x_0$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$ is a semisimple symmetric space G/H of Cayley type, with involution $\tau = \mathrm{Ad}_{g_1^2}\theta$ and $H = G^\tau$. The space G/H has the same rank, real rank and dimension as G/K.*

Proof. In the tube case $\omega_r = \frac{1}{2}(A_1 + A_2 + \ldots + A_r)$. One easily verifies that $\alpha(\frac{\pi}{2}\omega_r) \in \mathbb{Z}\frac{\pi}{2}$, for every $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$, i.e. g_1 satisfies conditions (5) in [Gea12]. Then the orbit $G \cdot x_1$, with the involution $\tau = \mathrm{Ad}_{g_1}\theta\mathrm{Ad}_{g_1^{-1}} = \mathrm{Ad}_{g_1^2}\theta$, is a pseudo-Riemannian symmetric space, say G/H, of the same rank, real rank and dimension as G/K. In addition, G/H is a totally real submanifold of $G^{\mathbb{C}}/K^{\mathbb{C}}$ of maximal dimension (see [Gea12], Lemma 2.2). Since x_1 lies on the semisimple boundary of Ξ, by [GiKr02], Thm. B, the space G/H is a non-compactly causal symmetric space.

To prove that G/H is also compactly causal, we use the characterisation of Theorem 4.1 in [FaOl95], stating that an irreducible symmetric space $(G/H, \tau)$ is compactly causal if and only if G/K is a non-compact Hermitian symmetric space and the involution $\tau\colon G/K \to G/K$ is antiholomorphic. Since τ defines an involution of $\mathfrak{g}$ commuting with θ, it also determines an involution of G/K. It remains to show that, the action of τ on $\mathfrak{p}$ anticommutes with the complex structure $J_0 = ad_{Z_0}$, where $Z_0 = \frac{1}{2}\sum_j T_j$ (see Rem. 2.2). From the definition of τ and Lemma 7.1, one can see that the further conditions $\theta E_j = -\tau E_j$, for $j = 1, \ldots, r$, are satisfied. Consequently, all the vectors $T_j := E_j + \theta E_j$, and in particular $Z_0 = \frac{1}{2}\sum_j T_j$, are contained in $\mathfrak{q} \cap \mathfrak{k}$. Then, for all $X \in \mathfrak{p}$, one has

$$ad_{Z_0}\tau(X) = [Z_0, \tau(X)] = \tau[\tau(Z_0), X] = -\tau[Z_0, X] = -\tau\big(ad_{Z_0}(X)\big),$$

as wished. This concludes the proof of the lemma. □

Let $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \tau)$ be the symmetric algebra associated to the symmetric space G/H and let $W^\pm$ denote the *maximal* proper, open, convex, Ad_H-invariant, elliptic cones in $\mathfrak{q}$.

It is important to observe that for the Cayley type symmetric space G/H, the *maximal* and the *minimal* proper, open, convex, Ad_H-invariant elliptic cones in $\mathfrak{q}$ coincide: under the Adjoint action of H, the space $\mathfrak{q}$ decomposes as the direct sum of irreducibles subspaces $\mathfrak{q}^+ \oplus \mathfrak{q}^-$, with the property that $\mathfrak{q}^- = -\theta\mathfrak{q}^+$. Each summand contains closed, convex, Ad_H-invariant cones $\pm C_+ \subset \mathfrak{q}^+$ and $\pm C_- \subset$

$\mathfrak{q}^-$, with the property that the minimal elliptic and hyperbolic closed cones in $\mathfrak{q}$ are given by $\pm(C_+ - C_-)$ and $\pm(C_+ + C_-)$, respectively (cf. [HiOl97], p.53). In particular, for the minimal closed, Ad_H-invariant elliptic cone $\overline{W^+_{min}}$, there is an isomorphism $\overline{W^+_{min}} \cong C_+ + C_+$.

Denote by C^0_+ the interior of C_+. Since the symmetric space G/K is biholomorphic to the tube domain $\mathfrak{q}^+ + iC^0_+$ (see [HiOl97], Rem.2.6.9, p.55), the cone C_+ is selfadjoint (i.e. it coincides with its dual cone). As a consequence, the minimal proper, closed, convex, Ad_H-invariant, elliptic cone in $\mathfrak{q}$ is selfadjoint and coincides with the maximal one, which by definition is its dual cone $\left(\overline{W^+_{min}}\right)^*$. The same is true for the respective interior parts.

The domains $G \exp iW^\pm \cdot x_1$ are G-invariant Stein domains in $G^{\mathbb{C}}/H^{\mathbb{C}}$, where $H^{\mathbb{C}} = g_1 K^{\mathbb{C}} g_1^{-1}$ is the isotropy subgroup of x_1 in $G^{\mathbb{C}}$ (cf. [Nee99], Thm. 3.5, p. 205). Under the G-equivariant biholomorphism

$$G^{\mathbb{C}}/H^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}, \quad gH^{\mathbb{C}} \to gg_1K^{\mathbb{C}},$$

they can be identified with the G-invariant Stein domains $S^\pm := G \exp iW^\pm g_1 \cdot x_0$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$.

Since the involutions θ and τ commute, $\mathfrak{g}$ has a joint eigenspace decomposition $\mathfrak{g} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}) \oplus (\mathfrak{q} \cap \mathfrak{k}) \oplus (\mathfrak{q} \cap \mathfrak{p})$. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{q} \cap \mathfrak{p}$. Then $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ and in $\mathfrak{q}$ (see [HiOl97], Prop. 3.1.11, p.77).

Fix a set of commuting $\mathfrak{sl}(2,\mathbb{R})$-triples $\{E_j, \theta E_j, A_j\}$ as in (5). As we remarked in the proof of Lemma 7.2, each $T_j := E_j + \theta E_j$ is contained in $\mathfrak{q} \cap \mathfrak{k}$ and $\mathfrak{c} := \mathrm{span}_{\mathbb{R}}\{T_1, \ldots, T_r\}$ is a compact Cartan subspace in $\mathfrak{q}$. In particular, $\mathfrak{c}$ contains the element $Z_0 = \frac{1}{2}(T_1 + \ldots + T_r) \in Z(\mathfrak{k})$ (see Rem. 2.2).

Lemma 7.3. *One has*

$$S^+ = G\left(\exp i \bigoplus_{j=1}^{r} (0,\infty) T_j\right) g_1 \cdot x_0.$$

Proof. A proper, closed, convex, Ad_H-invariant, elliptic cone in $\mathfrak{q}$ intersects the compact Cartan subspace $\mathfrak{c}$ in a proper, closed, convex, $W_H(\mathfrak{c})$-invariant, elliptic cone. Here $W_H(\mathfrak{c}) := N_H(\mathfrak{c})/Z_H(\mathfrak{c})$. Since the cone $\overline{W^+}$ is selfadjoint (i.e. both maximal and minimal), we can identify the intersection $\overline{W^+_{\mathfrak{c}}} := \overline{W^+} \cap \mathfrak{c}$ with a minimal proper, closed, convex, $W_H(\mathfrak{c})$-invariant, elliptic cone in $\mathfrak{c}$. We prove the lemma by showing that

$$\overline{W^+_{\mathfrak{c}}} = \bigoplus_{j=1}^{r} [0,\infty) T_j.$$

In order to do this we first observe that

$$W_H(\mathfrak{c}) \cong W_{H\cap K}(\mathfrak{c}) \cong W_{H^0 \cap K}(\mathfrak{c}),$$

where the second isomorphism follows from the fact that the c-dual symmetric space G^c/H is non-compactly causal. In addition, $i\mathfrak{c}$ is a hyperbolic maximal abelian subspace in $i\mathfrak{q}$. Then, by [HiOl97], Thm. 3.1.18 and Thm. 3.1.20, the group H is essentially connected, i.e. $H = H^0 Z_{H\cap K}(i\mathfrak{c})$ (see [HiOl97], Def. 3.1.16).

Next we recall the characterization of the minimal proper, closed, convex, $W_{H^0}(\mathfrak{c})$-invariant, elliptic cones in $\mathfrak{c}$ (see [KrNe96]). Consider the restricted root system $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$ of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{c}^{\mathbb{C}}$. Define the Lie subalgebra $\mathfrak{r} = \mathfrak{q} \cap \mathfrak{k} \oplus [\mathfrak{q} \cap \mathfrak{k}, \mathfrak{q} \cap \mathfrak{k}] \subset \mathfrak{k}$. A root $\alpha \in \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$ is called compact if $\mathfrak{g}^\alpha \cap \mathfrak{r}^{\mathbb{C}} \neq \{0\}$, and non-compact otherwise. Denote by $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})_c$ and $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})_n$ the compact and non-compact roots in $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$, respectively. The root system $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$ is called split if $\mathfrak{g}^\alpha \subset \mathfrak{k}^\mathbb{C}$, for all compact roots α. The Weyl group $W_{H^0\cap K}(\mathfrak{c})$ is isomorphic to the group W_c generated by the reflections in the compact roots ([KrNe96], Def.III.9 and Prop. V.2.i). If the positive non-compact roots $\Delta(\mathfrak{g}^\mathbb{C}, \mathfrak{c}^\mathbb{C})_n$ are stable under the group W_c, the system $\Delta(\mathfrak{g}^\mathbb{C}, \mathfrak{c}^\mathbb{C})^+$ is called $\mathfrak{r}$-adapted.

If the symmetric algebra $(\mathfrak{g}, \tau)$ is compactly causal then the restricted root system $\Delta(\mathfrak{g}^\mathbb{C}, \mathfrak{c}^\mathbb{C})$ is split and admits an $\mathfrak{r}$-adapted positive system. Moreover the minimal proper, closed, convex, $W_{H^0\cap K}(\mathfrak{c})$-invariant, elliptic cones in $\mathfrak{c}$ have the following characterization

$$\overline{iW_\mathfrak{c}^\pm} := \pm\text{cone}(\{h_\alpha\}_{\alpha\in\Delta(\mathfrak{g}^\mathbb{C},\mathfrak{c}^\mathbb{C})_n}),$$

where $h_\alpha \in i\mathfrak{c}$ is defined by $\alpha(H) = B(H, h_\alpha)$.

Now we come to our situation: since $\mathfrak{c}$ is the image of $\mathfrak{a}$ under a Cayley transform, the root system $\Delta(\mathfrak{g}^\mathbb{C}, \mathfrak{c}^\mathbb{C})$ is isomorphic to the ordinary restricted root system $\Delta(\mathfrak{g}, \mathfrak{a})$, and is of type C_r. For simplicity, identify $\mathfrak{c}_\mathbb{R} = i\mathfrak{c}$ with $\mathfrak{c}_\mathbb{R}^*$ using the Killing form. Since the restrictions to $\mathfrak{c}^\mathbb{C}$ of the roots $\tilde\lambda_1, \ldots, \tilde\lambda_r$ defined in Remark 2.2 are non-compact in $\Delta(\mathfrak{g}^\mathbb{C}, \mathfrak{c}^\mathbb{C})$, one has the inclusion

$$\text{cone}(\{2e_j\}_j) \subset \overline{iW_\mathfrak{c}^+}.$$

The fact that the image of $\text{cone}(\{2e_j\}_{j=1,\ldots,r})$ under the reflections with respect to roots of the form $\pm(e_i + e_j)$, for $1 \leq i < j \leq r$, is not contained in any regular cone in $i\mathfrak{c}$, implies that such roots are necessarily non-compact. It follows that

$$\text{cone}(\{2e_j\}_j) = \text{cone}(\{2e_j, (e_i + e_k)\}_{j,\, i\neq k}).$$

We claim that all roots of the form $\pm(e_i - e_j)$, for $1 \leq i < j \leq r$, are necessarily compact. In order to see this, first observe that the compact roots are a non-empty proper subset of $\Delta(\mathfrak{g}^\mathbb{C}, \mathfrak{c}^\mathbb{C})$. Then assume by contradiction that there is a non-compact root of the form $e_i - e_k$, for some $i < k$. Without loss of generality, we may also assume that either $e_i - e_j$, for some $i < j$, or $e_j - e_k$, for some $j < k$, is compact. From the W_c-invariance of the cone $\overline{iW_\mathfrak{c}^+}$ and the relations

$$r_{e_i - e_j}(e_i - e_k) = e_j - e_k \quad \text{and} \quad r_{e_j - e_k}(e_i - e_k) = e_i - e_j\,,$$

we deduce that either $e_j - e_k$ or $e_i - e_j$ is a non-compact root and lies in $\overline{iW_\mathfrak{c}^+}$ as well. From $(e_i - e_j) + (e_j - e_k) = (e_i + e_j) - 2e_k$, we obtain that $\mathbb{R}2e_k \subset \overline{iW_\mathfrak{c}^+}$; similarly, from $(e_i - e_k) + (e_i - e_j) = 2e_i - (e_k + e_j)$, we obtain that $\mathbb{R}(e_k + e_j) \subset \overline{iW_\mathfrak{c}^+}$. In both cases the assumption that $\overline{iW_\mathfrak{c}^+}$ is a proper cone is violated. Hence

$$\text{cone}(\{2e_j\}_j) = \overline{iW_\mathfrak{c}^+},$$

as desired. □

The next lemma proves that S^+ is contained Ξ^+ in the rank-one case. It also provides the main tool for the proof of the same inclusion in the higher rank case, which is based on the rank-one reduction. Fix the basis of $\mathfrak{sl}(2)$ given in (8), normalized as in (6), and set $T := E + \theta E$.

Lemma 7.4. *Set* $k_0 = \exp\frac{\pi}{4}T$.

(i) *For* $t \in (-\pi/4, \pi/4)$ *define* $a_1(t) = \exp\left(\ln(\frac{1}{\sqrt{\cos 2t}})A\right)$. *One has*

$$\exp itA \cdot x_0 = k_0 a_1(t) \exp i \sin 2tE \cdot x_0\,. \tag{18}$$

In particular $\exp itA \cdot x_0 \in G \exp i \sin 2tE \cdot x_0$ *and*

$$\Xi = G \exp i[0, 1)E \cdot x_0.$$

(ii) *For* $t \in (0,\infty)$ *define* $a_2(t) = \exp\big(\ln(\frac{1}{\sqrt{\sinh 2t}})A\big)$. *One has*

$$\exp itT\, g_1 \cdot x_0 = k_0 a_2(t) \exp i\cosh 2tE \cdot x_0\,. \tag{19}$$

In particular $\exp itT\, g_1 \cdot x_0 \in G\exp i\cosh 2tE\cdot x_0$ *and*

$$S^+ = G\exp i(1,\infty)E\cdot x_0.$$

Proof. Formula (18) is proved by showing that

$$\exp itA = k_0 a_1(t)\exp(i\sin 2tE)\, k,$$

for some $k \in SO(2,\mathbb{C})$. The above identity follows from a simple matrix computation with

$$\exp itA = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it}\end{pmatrix}, \quad k_0 = \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{pmatrix}, \quad a_1(t) = \begin{pmatrix} \frac{1}{\sqrt{\cos 2t}} & 0 \\ 0 & \sqrt{\cos 2t}\end{pmatrix}$$

$$\exp i\sin 2tE = \begin{pmatrix} 1 & i\sin 2t \\ 0 & 1\end{pmatrix}, \quad k = \frac{1}{\sqrt{2\cos 2t}}\begin{pmatrix} e^{-it} & -e^{it} \\ e^{it} & e^{-it}\end{pmatrix}.$$

The second statement in (i) follows directly from equation (18) and the definition of Ξ. An analogous computation was carried out in [KrOp08], Sect. 3.2, for the crown domain using the hyperbolic model $SO_0(1,2,\mathbb{C})/SO(2,\mathbb{C})$.

Formula (19) is proved by showing that

$$k = g_1^{-1}\,(\exp itT)^{-1}\, k_0\, a_2(t)\, \exp(i\cosh 2tE)$$

is an element of $SO(2,\mathbb{C})$. The above identity follows from a simple matrix computation with

$$g_1^{-1} = \begin{pmatrix} \frac{1-i}{\sqrt2} & 0 \\ 0 & \frac{1+i}{\sqrt2}\end{pmatrix}, \quad (\exp itT)^{-1} = \begin{pmatrix} \cosh t & -i\sinh t \\ i\sinh t & \cosh t\end{pmatrix}, \quad k_0 = \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{pmatrix}$$

$$a_2(t) = \begin{pmatrix} \frac{1}{\sqrt{\sinh 2t}} & 0 \\ 0 & \sqrt{\sinh 2t}\end{pmatrix}, \quad \exp i\cosh 2tE = \begin{pmatrix} 1 & i\cosh 2t \\ 0 & 1\end{pmatrix}.$$

The second statement in (ii) follows directly from equation (19) and Lemma 7.3. □

Proposition 7.5. *Let* G/K *be an irreducible Hermitian symmetric space of tube type. Then the domain* Ξ^+ *contains the crown*

$$\Xi = G\exp i\bigoplus_{j=1}^{r}[0,1)E_j\cdot x_0\,,$$

and the domain

$$S^+ = G\exp i\bigoplus_{j=1}^{r}(1,\infty)E_j\cdot x_0.$$

Proof. The first equality was proved in [KrOp08]. The second one follows from G-invariance, and rank-1 reduction. Indeed by Lemma 7.3 and Lemma 7.4, we have

$$S^+ = G\left(\prod_{j=1}^{r}\exp i(0,\infty)T_j\right) g_1\cdot x_0 = G\left(\prod_{j=1}^{r}\exp i(0,\infty)T_j\right)\prod_{j=1}^{r} g_{1,j}\cdot x_0 =$$

$$= G\left(\prod_{j=1}^{r}\exp i(0,\infty)T_j g_{1,j}\right)\cdot x_0 = G\prod_{j=1}^{r}\exp i(1,\infty)E_j\cdot x_0,$$

as claimed. □

Recall that the domain Ξ^+ is G-equivariantly diffeomorphic to the anti-holomorphic tangent bundle $G \times_K \mathfrak{p}^{0,1}$. From Lemma 4.5, we obtain another natural description of the crown Ξ and of the domain S^+ inside Ξ^+, by means of their intersections with the slice defined by $\tau(\mathfrak{a})$ in $\mathfrak{p}^{0,1}$ (see (12)).

Corollary 7.6. *One has*

$$\Xi = G\exp i\Big(\bigoplus_{j=1}^{r}[0,1)\frac{1}{2}(A_j + iJ_0A_j)\Big)\cdot x_0 = G\exp i\Big(\bigoplus_{j=1}^{r}(-1,1)\frac{1}{2}(A_j + iJ_0A_j)\Big)\cdot x_0$$

and

$$S^+ = G\exp i\Big(\bigoplus_{j=1}^{r}(1,\infty)\frac{1}{2}(A_j + iJ_0A_j)\Big)\cdot x_0 =$$

$$G\exp i\bigoplus_{j=1}^{r}\big((-\infty,-1)\cup(1,\infty)\big)\frac{1}{2}(A_j + iJ_0A_j)\cdot x_0\,.$$

Remark 7.7. If G/K is an irreducible Hermitian symmetric space, which is not of tube type, then the element g_1 in (17) satisfies conditions (3) in [Gea12] (while it does not satisfy conditions (5) therein). Then, by Lemma 2.1 in [Gea12], the orbit $G\cdot x_1$ of the point $x_1 = g_1\cdot x_0$ is not a symmetric space. However, the orbit $\widehat{G}\cdot x_1$, under the action of the proper reductive subgroup $\widehat{G} := Z_G(g_1^4)$ of G, is a reductive symmetric space with involution $\tau = \mathrm{Ad}_{g_1^2}\theta$. The space $\widehat{G}\cdot x_1$ has the same rank and real rank as G/K, but strictly smaller dimension. The isotropy subgroups of x_1 in G and in $\widehat{G}$ coincide and the slice representation at x_1 with respect to the G-action is equivalent to the isotropy representation of $\widehat{G}\cdot x_1$.

The orbit $\widehat{G}\cdot x_1$ is diffeomorphic to the Cayley symmetric space associated to the Hermitian symmetric space of tube type contained in G/K. In order to see this, observe that $\mathrm{Ad}_{g_1^4}$ is an involution of $G^{\mathbb{C}}$ which commutes both with the Cartan involution of $G^{\mathbb{C}}$ and the conjugation defining G. Since $G^{\mathbb{C}}$ is simply connected, $\widehat{G}^{\mathbb{C}} = Z_{G^{\mathbb{C}}}(g_1^4)$ is connected. Moreover it is reductive, being the complexification of $\widehat{U} = Z_U(g_1^4)$, the fixed point subgroup of the restriction of $\mathrm{Ad}_{g_1^4}$ to the simply connected compact real form U of $G^{\mathbb{C}}$. From the classification of simply connected, compact symmetric spaces one sees that the following three cases occur:

$G = SU(r,s),\ (r<s)$	$G^{\mathbb{C}} = SL(r+s,\mathbb{C})$	$\widehat{G}^{\mathbb{C}} = S(GL(s-r,\mathbb{C})\times GL(2r,\mathbb{C}))$
$G = Spin^*(2r)$	$G^{\mathbb{C}} = Spin^*(2r,\mathbb{C})$	$\widehat{G}^{\mathbb{C}} = \mathbb{C}^*Spin^*(2(r-1),\mathbb{C})$
$G = E_{6(-14)},\ (r=2)$	$G^{\mathbb{C}} = E_6$	$\widehat{G}^{\mathbb{C}} = \mathbb{C}^*Spin(10,\mathbb{C})$.

One can show that $\widehat{G}^{\mathbb{C}}$ can be written as the commuting product $\widehat{G}^{\mathbb{C}} = M^{\mathbb{C}}G^{\mathbb{C}}_{tube}$, where $M^{\mathbb{C}}$ is a subgroup of $Z_{K^{\mathbb{C}}}(\mathfrak{a}^{\mathbb{C}})$ and $G^{\mathbb{C}}_{tube}$ denotes the simply connected complexification of the connected, Hermitian simple group acting on the tube-type symmetric space contained in G/K. Moreover there are isomorphisms of coset spaces $\widehat{G}^{\mathbb{C}}/(\widehat{G}^{\mathbb{C}})^\tau \cong G^{\mathbb{C}}_{tube}/(G^{\mathbb{C}}_{tube})^\tau$ and $\widehat{G}/\widehat{G}^\tau \cong G_{tube}/(G_{tube})^\tau$.

Recall that in the non-tube case the element $Z_0 \in Z(\mathfrak{k})$ determining the complex structure of G/K can be written as $Z_0 = S + T_0$, where $S \in Z_K(\mathfrak{a})$ and $T_0 = \frac{1}{2}\sum T_j$, with $T_j = E_j + \theta E_j$. Hence Z_0 lies in $\widehat{\mathfrak{g}}$ and T_0 lies in $\widehat{\mathfrak{g}}_{tube}$. Denote by W^+ the maximal proper, open, convex, $\mathrm{Ad}_{(G_{tube})^\tau}$-invariant elliptic cone in $T_{x_1}(\widehat{G}_{tube}\cdot x_1)$, which satisfies $\overline{W^+} = \overline{conv\left(\mathrm{Ad}_{(G_{tube})^\tau}(\mathbb{R}^+T_0)\right)}$. Then

$$\Omega^+ = G\exp iW^+\cdot x_1 = G\exp iW^+g_1\cdot x_0$$

is an open G-invariant domain in $G^{\mathbb{C}}/K^{\mathbb{C}}$ and, by similar considerations as in the tube case, an analogue of Proposition 7.5 holds true. Namely

$$\Omega^+ = G \exp i \bigoplus_{j=1}^{r} (0,\infty) T_j \, g_1 \cdot x_0 .$$

It turns out that Ω^+ is not Stein and contains no proper G-invariant Stein subdomains (see [GeIa13], Thm. 5.1, Case(2)).

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