ON HYPERBOLICITY OF SU(2)-EQUIVARIANT, PUNCTURED DISC BUNDLES OVER THE COMPLEX AFFINE QUADRIC

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Abstract. Given a holomorphic line bundle over the complex affine quadric Q^2, we investigate its Stein, $SU(2)$-equivariant disc bundles. Up to equivariant biholomorphism, these are all contained in a maximal one, say Ω_{max}. By removing the zero section from Ω_{max} one obtains the unique Stein, $SU(2)$-equivariant, punctured disc bundle over Q^2 which contains entire curves. All other such punctured disc bundles are shown to be Kobayashi hyperbolic.

1. Introduction

Consider a Reinhardt domain D_ρ in $\mathbb{C} \times \mathbb{C}^*$ of the form

$$\{ (w,z) \in \mathbb{C} \times \mathbb{C}^* \ : \ |z|\rho(|w|) < 1 \},$$

where $\rho : \mathbb{R} \to \mathbb{R}^{>0}$ is an even, upper semicontinuous function. Let S^1 act on D_ρ by $e^{it} \cdot (w,z) := (e^{it}w, z)$. Then D_ρ can be regarded as an S^1-invariant punctured disc bundle over $\mathbb{C}$, with S^1-equivariant projection $D_\rho \to \mathbb{C}$ given by $(w,z) \to w$. By rescaling the fiber coordinate one can normalize every D_ρ so that $\rho(0) = 1$.

Note that D_ρ is Stein if and only if ρ is logarithmically convex, i.e. if $\log \rho$ is convex. Under this assumption one has the extremal case $\rho \equiv 1$, corresponding to the trivial punctured disc bundle $D_{max} = \mathbb{C} \times \Delta^*$. Here Δ^* denotes the punctured unit disc in $\mathbb{C}$. All other Stein, normalized, punctured disc bundles are contained in D_{max}. These correspond to non constant, logarithmically convex ρ with $\rho(0) = 1$. In particular $\lim \rho(h) = \infty$ as $h \to \infty$ which, by a simple argument, implies that every non-maximal, Stein, punctured disc bundle D_ρ is Kobayashi hyperbolic. Then, by a result of Swonek ([**?**]), one also knows that D_ρ is biholomorphic to a bounded Reinhardt domain.

Let $U^{\mathbb{C}} = SL(2,\mathbb{C})$ and $K^{\mathbb{C}}$ be the universal complexifications of

$$U := SU(2) \quad \text{and} \quad K := \left\{ \begin{pmatrix} e^{iy} & 0 \\ 0 & e^{-iy} \end{pmatrix} \ : \ y \in \mathbb{R} \right\},$$

respectively. Here we are interested in U-equivariant disc bundles over the complex affine quadric $Q^2 \cong U^{\mathbb{C}}/K^{\mathbb{C}}$. In the sequel $K^{\mathbb{C}}$ is identified with $\mathbb{C}^*$ via

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the Lie group isomorphism given by

$$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \to \zeta\,.$$

One checks that every holomorphic line bundle over Q^2 is isomorphic to a homogeneous line bundle of the form (cf. Sect. 2)

$$L^m := U^{\mathbb{C}} \times_{\chi^m} \mathbb{C},$$

where $m \in \mathbb{Z}$ and the character $\chi^m : K^{\mathbb{C}} \to \mathbb{C}^*$ is defined by $\chi^m(\zeta) = \zeta^m$. Consider the symmetric decomposition $\mathfrak{k}\oplus\mathfrak{p}$ of the Lie algebra $\mathfrak{u}$ of U associated to the compact symmetric space $S^2 \cong U/K$, and let $\mathfrak{a}$ be the maximal abelian subalgebra in $\mathfrak{p}$ generated by a chosen element H in $\mathfrak{p}$. By a result of Mostow ([**?**]) one has the decomposition $U^{\mathbb{C}} = U\exp(i\mathfrak{a})K^{\mathbb{C}}$. Then a U-equivariant, punctured disc bundle in L^m is uniquely defined by (cf. Sect. 3)

$$\Omega_\rho := \{\, [g,z] \in L^m \ : \ |z||\zeta|^m\rho(h) < 1\,\}\,,$$

where $u\exp(ihH)\zeta^{-1}$ is a Mostow decomposition of g and $\rho : \mathbb{R} \to \mathbb{R}^{>0}$ is an even, upper semicontinuous function. Moreover one shows that Ω_ρ is Stein if and only if the function $U^{\mathbb{C}} \to \mathbb{R}$, given by $g \to |\zeta|^m\rho(h)$, is logarithmically plurisubharmonic (Prop. **??**). Warning: the function $g \to \log|\zeta|$ is not plurisubharmonic.

By acting fiberwise with a suitable element of $\exp(i\mathfrak{k})$ one can normalize Ω_ρ so that $\rho(0) = 1$. Then, for all $m \in \mathbb{Z}$ one finds a maximal Stein, U-equivariant disc bundle Ω_{max} defined by $\rho_{max}(h) := (\cosh(2h))^{|m|/2}$. It turns out that the associated punctured disc bundle Ω^*_{max}, which is obtained by removing the zero section, is not Kobayashi hyperbolic. Indeed its universal covering admits a proper $\mathbb{C}$-action. Moreover one shows (Thm. **??**)

All other normalized, Stein, U-equivariant, punctured disc bundles are contained in Ω^*_{max} and are Kobayashi hyperbolic.

As an application we give a new proof of a known characterization of the 3-dimensional, bounded symmetric domain of type IV (Thm. **??**).

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2. Line bundles over Q^2.

Here all holomorphic line bundles over the affine complex quadric Q^2 are shown to be isomorphic to homogeneous line bundles of the form $U^{\mathbb{C}} \times_{\chi^m} \mathbb{C}$, with χ^m a character of $K^{\mathbb{C}}$.

Recall that the homogeneous bundle $U \times_K \mathfrak{p} := (U \times \mathfrak{p})/K$, where K acts on $U \times \mathfrak{p}$ by $k\cdot(u,X) := (uk^{-1}, Ad_k X)$, can be identified with the tangent bundle of the compact symmetric space $S^2 \cong U/K$ via the U-equivariant diffeomorphism $U \times_K \mathfrak{p} \to TS^2$, defined by $[u, X] \to u_*(X)$. Here $\mathfrak{p}$ is identified with the tangent space of S^2 at the base point via the differential of the canonical projection $U \to S^2$.

As a consequence of Mostow's decomposition ([?], Lemma 4.1, cf. [?], Thm. D and [?], Sect. 9) one also has a U-equivariant identification of $U \times_K \mathfrak{p} \to U^{\mathbb{C}}/K^{\mathbb{C}} \cong Q^2$ given by $[u, X] \to u \exp(iX) K^{\mathbb{C}}$. Hence one obtains an identification of $U^{\mathbb{C}}/K^{\mathbb{C}}$ with the tangent bundle of U/K.

Realize the sphere $S^2 \cong U/K$ as the zero section of its tangent bundle via the immersion $\iota : U/K \to U^{\mathbb{C}}/K^{\mathbb{C}}$ defined by $uK \to uK^{\mathbb{C}}$. Let

$$B = \left\{ \begin{pmatrix} \zeta & 0 \\ \beta & \zeta^{-1} \end{pmatrix} \ : \ \zeta \in \mathbb{C}^*, \ \beta \in \mathbb{C} \right\}$$

be the isotropy at $[0:1]$ with respect to the standard linear $U^{\mathbb{C}}$-action on $\mathbb{P}^1$. Consider the projection $\pi : U^{\mathbb{C}}/K^{\mathbb{C}} \to U^{\mathbb{C}}/B$ given by $uK^{\mathbb{C}} \to uB$. One has the natural identifications $U^{\mathbb{C}}/B \cong \mathbb{P}^1 \cong S^2$ and $\pi \circ \iota = Id_{S_2}$. On the other hand the composition $\iota \circ \pi$ is the fiberwise projection onto the zero section in the tangent bundle $U^{\mathbb{C}}/K^{\mathbb{C}}$, therefore it is homotopic to $Id_{U^{\mathbb{C}}/K^{\mathbb{C}}}$. It follows that ι is a homotopic equivalence and consequently

$$\pi^* : H^2(\mathbb{P}^1, \mathbb{Z}) \to H^2(Q^2, \mathbb{Z})$$

is an isomorphism. Since $H^1(\mathbb{P}^1, \mathcal{O}^*) = H^2(\mathbb{P}^1, \mathbb{Z})$ and $H^1(Q^2, \mathcal{O}^*) = H^2(Q^2, \mathbb{Z})$, this gives an isomorphism among the groups of holomorphic line bundles

$$\pi^* : Pic(\mathbb{P}^1) \to Pic(Q^2)\,.$$

Now recall that

$$Pic(\mathbb{P}^1) = \{\hat{L}^m := U^{\mathbb{C}} \times_{\hat{\chi}^m} \mathbb{C} \ : \ m \in \mathbb{Z}\}\,,$$

where $\hat{\chi}^m$ is the character of B defined by $\begin{pmatrix} \zeta & 0 \\ \beta & \zeta^{-1} \end{pmatrix} \to \zeta^m$ and $U^{\mathbb{C}} \times_{\hat{\chi}^m} \mathbb{C}$ is the quotient of $U^{\mathbb{C}} \times \mathbb{C}$ with respect to the B-action defined by $b \cdot (g, z) = (gb^{-1}, \hat{\chi}^m(b) z)$. Indeed, since a homogeneous bundle is uniquely defined by the isotropy representation on the fiber at the base point, one has

$$\hat{L}^{m+n} = \hat{L}^m \otimes \hat{L}^n\,.$$

Then it is enough to note that the generator $\hat{L}^{-1}$ of the group $\{\hat{L}^m \ : m \in \mathbb{Z}\}$ is biholomorphic to the tautological line bundle $T \subset \mathbb{P}^1 \times \mathbb{C}^2$ over $\mathbb{P}^1$ (cf. [?]) via the the map

$$[g, z] \to \left(g \begin{bmatrix} 0 \\ 1 \end{bmatrix}, zg \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right),$$

where the action of $U^{\mathbb{C}}$ on $\mathbb{C}^2$ is the standard linear one.

Finally consider the homogeneous bundles over Q^2 of the form $L^m := U^{\mathbb{C}} \times_{\chi^m} \mathbb{C}$, where χ^m is the character on $K^{\mathbb{C}}$ defined by $\zeta \to \zeta^m$. One has the canonical projection $U^{\mathbb{C}}\times_{\chi^m}\mathbb{C}\to U^{\mathbb{C}}/K^{\mathbb{C}}$ given by $[g,z]\to gK^{\mathbb{C}}$ and the bundle projection $U^{\mathbb{C}}\times_{\chi^m}\mathbb{C}\to U^{\mathbb{C}}\times_{\hat{\chi}^m}\mathbb{C}$ defined by $[g,z]\to[g,z]$. Moreover the diagram

$$\begin{array}{ccc} U^{\mathbb{C}}\times_{\chi^m}\mathbb{C} & \to & U^{\mathbb{C}}\times_{\hat{\chi}^m}\mathbb{C} \\ \downarrow & & \downarrow \\ U^{\mathbb{C}}/K^{\mathbb{C}} & \xrightarrow{\pi} & U^{\mathbb{C}}/B\,, \end{array}$$

whose vertical maps are the canonical $U^{\mathbb{C}}$-equivariant projections, is commutative. It follows that $\pi^*(U^{\mathbb{C}}\times_{\hat{\chi}^m}\mathbb{C})=U^{\mathbb{C}}\times_{\chi^m}\mathbb{C}$ which, by the above remarks implies the following

Proposition 2.1. *Every holomorphic line bundle over the affine complex quadric Q^2 is isomorphic to a homogeneous line bundle $L^m:=U^{\mathbb{C}}\times_{\chi^m}\mathbb{C}$, for some $m\in\mathbb{Z}$.*

3. Stein, U-equivariant disc bundles over Q^2

Define a disc bundle in L^m as a subdomain whose intersection with every fiber of the canonical projection onto $Q^2\cong U^{\mathbb{C}}/K^{\mathbb{C}}$ consists of a disc of finite radius. As a consequence of Mostow's decomposition, one has

$$U^{\mathbb{C}}=U\exp(i\mathfrak{a})K^{\mathbb{C}}\,,$$

with $\mathfrak{a}$ a maximal abelian subalgebra of $\mathfrak{p}$ (cf. Sect. 1). Moreover every U-orbit in $U^{\mathbb{C}}/K^{\mathbb{C}}$ meets the "slice" $\exp(i\mathfrak{a})K^{\mathbb{C}}$ in an orbit of the Weyl group $W\cong\mathbb{Z}_2$. Here the non trivial element of the W-action is given by reflection in $\mathfrak{a}$.

In particular $U\backslash U^{\mathbb{C}}/K^{\mathbb{C}}$ is homeomorphic to $\mathfrak{a}/W$ and for every fixed $m\in\mathbb{Z}$ there is a one-to-one correspondence among U-equivariant disc bundles in L^m and even, upper semicontinuous, positive functions on $\mathfrak{a}$. Namely, let $\mathfrak{a}$ be generated by $H:=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. Then an even, upper semicontinuous, positive function $\rho:\mathbb{R}\to\mathbb{R}^{>0}$ defines a unique U-equivariant disc bundle in L^m by

$$\Omega_\rho:=\{\,[g,z]\in U^{\mathbb{C}}\times_{\chi^m}\mathbb{C}\;:\;|z||\zeta|^m\rho(h)<1\,\}\,,$$

where $u\exp(ihH)\zeta^{-1}$ is a Mostow decomposition of g. Let $U\times K$ act on $U^{\mathbb{C}}$ by $(u,k)\cdot g:=ugk^{-1}$.

It is easy to check that the $U\times K$-invariant function $U^{\mathbb{C}}\to\mathbb{R}^{>0}$, defined by $g\to|\zeta|^m\rho(h)$, does not depend on the chosen decomposition of g and consequently Ω_ρ is well defined. Also note that such a function defines a U-invariant hermitian norm on L^m.

Proposition 3.1. (i) *The U-equivariant disc bundle Ω_ρ is Stein if and only if the $U\times K$-invariant function $|\zeta|^m\rho(h)$ defined on $U^{\mathbb{C}}$ is logarithmically plurisubharmonic.*
(ii) *If Ω_ρ is Stein then ρ is logarithmically convex. In particular ρ is continuous and realizes a minimum at zero.*

Proof. (i) Let $\Pi : U^{\mathbb{C}} \times \mathbb{C} \to U^{\mathbb{C}} \times_{\chi^m} \mathbb{C}$ be the natural projection and $O_\rho := \Pi^{-1}(\Omega_\rho)$. Then O_ρ is a principal $\mathbb{C}^*$-bundle over Ω_ρ and, by a classical result of Serre (cf. [**?**], Thm. 4 and 6), if Ω_ρ is Stein so is O_ρ. On the other hand Ω_ρ is the quotient of O_ρ with respect to the twisted $K^{\mathbb{C}}$-action. Thus if O_ρ is Stein, so is Ω_ρ by Theorem 5 in [**?**].

Finally note that the generalized Reinhardt domain $O_\rho = \{\, (g,z) \in U^{\mathbb{C}} \times \mathbb{C} \ : \ |z| < |\zeta|^{-m}\rho(h)^{-1}\}$ is Stein if and only if the function $U^{\mathbb{C}} \to \mathbb{R}$, given by $g \to -\log(|\zeta|^{-m}\rho(h)^{-1})$, is plurisubharmonic (cf. [**?**], Sect. 19.4).

(ii) Let $f : \mathbb{C} \to U^{\mathbb{C}}$ be the holomorphic map defined by $x+iy \to \exp(x+iy)H$. By composing with the plurisubharmonic function $\log(|\zeta|^m\rho(h))$ one obtains the $\mathbb{R}$-invariant function $\mathbb{C} \to \mathbb{R}$, given by $x+iy \to \log\rho(y)$, whose subharmonicity is equivalent to convexity of $\log\rho$. The last part of the statement follows from elementary properties of convex, even functions on $\mathbb{R}$. □ □

Remark 3.2. By [**?**], Thm. 1, p. 367, the function $\rho : \mathbb{R} \to \mathbb{R}^{>0}$ is logarithmically convex if and only if the $U \times K^{\mathbb{C}}$-invariant function on $U^{\mathbb{C}}$, defined by $g \to \rho(h)$, is logarithmically plurisubharmonic.

Remark 3.3. In the definition of a disc bundle one could allow the fibers to have infinite radius, i.e. the function ρ to take values in $\mathbb{R}^{>0} \cup \{\infty\}$. Then, for a Stein, U-equivariant disc bundle Ω_ρ, the convexity of ρ would imply that either $\Omega_\rho = L^m$ or ρ is real valued as in the above setting. That is, no matter which definition one chooses, the above proposition describes all proper, Stein, U-equivariant disc bundles over Q^2.

4. Some coordinates

For later use we introduce some coordinates on the double quotient $U\backslash U^{\mathbb{C}}/K$. First consider the map

$$\Pi_1 : U^{\mathbb{C}} \to U^{\mathbb{C}}\,, \quad g \to \sigma_U(g)^{-1}g\,,$$

where $\sigma_U : U^{\mathbb{C}} \to U^{\mathbb{C}}$, given by $g \to^t \overline{g}^{-1}$, is the antiholomorphic involutive automorphism of $U^{\mathbb{C}}$ whose fixed point set is U. Let U act on $U^{\mathbb{C}}$ by left multiplication and note that every fiber of Π_1 consists of a single U-orbit. Thus $\Pi_1(U^{\mathbb{C}})$ is set theoretically equivalent to $U\backslash U^{\mathbb{C}}$ and

$$\Pi_1 : U^{\mathbb{C}} \to \Pi_1(U^{\mathbb{C}})$$

is a realization of the quotient map. Moreover, one checks that $\Pi_1(U^{\mathbb{C}})$ consist of the connected component of $\{\, g \in U^{\mathbb{C}} \ : \ \sigma_U(g) = g^{-1}\,\}$, explicitly given by

$$\mathcal{Q} := \left\{ \begin{pmatrix} s & b \\ \bar{b} & t \end{pmatrix} \ : \ s, t \in \mathbb{R}^{>0},\ b \in \mathbb{C} \ \text{and} \ st - |b|^2 = 1 \right\}.$$

Let us describe how the right K-action on $U^{\mathbb{C}}$ is transformed after applying Π_1. An element of K is given by $k = \exp(yC)$ for some real y and $C := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Then one has

$$\Pi_1(gk^{-1}) = \sigma_U(g\exp(-yC))^{-1} g\exp(-yC) =$$
$$\sigma_U(\exp(-yC))^{-1}\sigma_U(g)^{-1} g\exp(-yC) = k\Pi_1(g)k^{-1}$$

Therefore $\Pi_1 : U^{\mathbb{C}} \to \mathcal{Q}$ is K-equivariant, if one lets K act on $U^{\mathbb{C}}$ by right multiplication and on $\mathcal{Q}$ by conjugation, i.e.

$$\exp(yC) \cdot \begin{pmatrix} s & b \\ \bar{b} & t \end{pmatrix} := \begin{pmatrix} s & e^{2iy}b \\ e^{-2iy}\bar{b} & t \end{pmatrix},$$

for every $y \in \mathbb{R}$. In particular, after applying Π_1, the K-action reads as rotations on b. Let

$$\mathcal{P} := \{\, (s,t) \in \mathbb{R}^2 \ : \ st \geq 1\}$$

and define $\Pi_2 : \mathcal{Q} \to \mathcal{P}$ by

$$\begin{pmatrix} s & b \\ \bar{b} & t \end{pmatrix} \to (s,t)\,.$$

For every $(s,t) \in \mathcal{P}$ the inverse image $\Pi_2^{-1}(s,\, t)$ consists of a single K-orbit given by

$$\left\{ \begin{pmatrix} s & b \\ \bar{b} & t \end{pmatrix} \in \mathcal{Q} \ : \ |b|^2 = st - 1 \right\}.$$

Hence $\mathcal{P}$ is a realization of the quotient $\mathcal{Q}/K \cong U\backslash U^{\mathbb{C}}/K$ and (s,t) can be regarded as coordinates for $U\backslash U^{\mathbb{C}}/K$. Moreover the composition map $\Pi_2 \circ \Pi_1$ is a realization of the quotient map.

Now let $u\exp(ihH)\zeta^{-1}$ be a Mostow decomposition of an element g of $U^{\mathbb{C}}$, with $\zeta = e^{x+iy}$. One has

$$\Pi_2 \circ \Pi_1(g) = \Pi_1 \circ \Pi_2 \left(\exp(ihH) \begin{pmatrix} e^{-x} & 0 \\ 0 & e^{x} \end{pmatrix} \right) =$$

$$\Pi_2\left(\begin{pmatrix} e^{-x} & 0\\ 0 & e^{x}\end{pmatrix}\begin{pmatrix} 0 & -2ih\\ 2ih & 0\end{pmatrix}\begin{pmatrix} e^{-x} & 0\\ 0 & e^{x}\end{pmatrix}\right) = (e^{-2x}\cosh 2h, e^{2x}\cosh 2h)\,.$$

Then one can define the $U\times K$-invariant functions $|\zeta| = e^x$ and h in terms of the coordinates (s,t) on the quotient $\mathcal{P} \cong U\backslash U^{\mathbb{C}}/K$. For this it is convenient to choose h to be positive.

Lemma 4.1. *Let $u\exp(ihH)\zeta^{-1}$ be a Mostow decomposition of an element g in $U^{\mathbb{C}}$, with $h\geq 0$.*

(i) *The $U\times K$-invariant function $g\to|\zeta|$ on $U^{\mathbb{C}}$ pushes down on $\mathcal{P}$ to*

$$|\zeta| = \sqrt[4]{\frac{t}{s}}\,.$$

(i) *The $U\times K$-invariant function $g\to h$ on $U^{\mathbb{C}}$ pushes down on $\mathcal{P}$ to*

$$h = \frac{1}{2}\operatorname{arccosh}\sqrt{st}\,.$$

Remark 4.2. Note that if $g = \begin{pmatrix} z_1 & z_3\\ z_2 & z_4\end{pmatrix}$, then $(s,t) = (|z_1|^2+|z_2|^2, |z_3|^2+|z_4|^2)$. It follows that $\log t$ and $\log s$ are plurisubharmonic functions on $U^{\mathbb{C}}$.

5. Hyperbolicity

Given a U-equivariant disc bundle Ω_ρ as in section 4, the associated punctured disc bundle $\Omega^*_\rho := \{\,[g,z]\in U^{\mathbb{C}}\times_{\chi^m}\mathbb{C}^* \;:\; |z||\zeta|^m\rho(h) < 1\,\}$ is obtained by removing the zero section and can be regarded as a particular annular bundle (cf. [?]). Here we first show that, up to U-equivariant biholomorphism, every Stein, U-equivariant disc bundle Ω_ρ over Q^2 is contained in a maximal one, say Ω_{max}. Then we note that the universal covering of the associated punctured disc bundle Ω^*_{max} admits a proper $\mathbb{C}$-action. In fact Ω^*_{max} turns out to be the unique Stein, U-equivariant punctured disc bundle over Q^2 which is not Kobayashi hyperbolic. We need the following lemma. Let $\mathbb{C}^*$ act on L^m by fiberwise multiplication.

Lemma 5.1. *There exists a $\mathbb{C}^*$-equivariant biholomorphism $\varphi : L^m\to L^{-m}$ which maps U-equivariant, punctured disc bundles in L^m onto U-equivariant, punctured disc bundles in L^{-m}.*

Proof. Consider the basis of $\mathfrak{u}$ given by

$$C := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad H := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad W := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

and let $\hat{\varphi} : U^{\mathbb{C}} \to U^{\mathbb{C}}$ be the Lie group isomorphism associated to the Lie algebra isomorphism mapping $\{C, H, W\}$ into $\{-C, -H, W\}$. Extend $\hat{\varphi}$ to the isomorphism of $U^{\mathbb{C}} \times \mathbb{C}$ defined by $(g, z) \to (\hat{\varphi}(g), z)$. Since $\hat{\varphi}(k) = k^{-1}$ for all $k \in K^{\mathbb{C}}$, one has

$$\hat{\varphi}(gk^{-1}, \chi^m(k)z) = (\hat{\varphi}(g)k, \chi^m(k)z) = (\hat{\varphi}(g)(k^{-1})^{-1}, \chi^{-m}(k^{-1})z)\,.$$

This implies that $\hat{\varphi}$ pushes down to a biholomorphism $\varphi : L^m \to L^{-m}$. Moreover by construction $\hat{\varphi}(U) = U$, therefore every U-invariant domain of L^m is mapped onto a U-invariant domain of L^{-m}.

In order to avoid ambiguity, here we let $\Omega_{m,\rho}$ denote the U-equivariant disc bundle Ω_ρ contained in L^m. If $[g, z] \in \Omega_{m,\rho}$, with $g = u\exp(ihH)\zeta^{-1}$, one has

$$\varphi([g, z]) = [\varphi(u)\exp(-ihH)(\zeta^{-1})^{-1}, z]\,,$$

with $|z||\zeta^{-1}|^{-m}\rho(-h) = |z||\zeta|^m\rho(h) < 1$. Thus $\varphi(\Omega_{m,\rho}) = \Omega_{-m,\rho}$, implying the statement. □ □

Remark 5.2. Since $\hat{\varphi}(K^{\mathbb{C}}) = K^{\mathbb{C}}$, one can consider the induced biholomorphism $\hat{\varphi} : Q^2 \to Q^2$ and it is easy to check that $L^m = \hat{\varphi}^*(L^{-m})$ for all $m > 0$. However recall that L^m and L^{-m} are not isomorphic as line bundles over Q^2.

If $m \neq 0$, then $U^{\mathbb{C}}$ acts transitively on $U^{\mathbb{C}} \times_{\chi^m} \mathbb{C}^*$ by $g \cdot [g', z] := [gg', z]$ and the isotropy at $[e, 1]$ is the cyclic group $\Gamma_m = \{\, \zeta \in K^{\mathbb{C}} \;:\; \zeta^m = 1\}$. Therefore one has a commutative diagram

$$\begin{array}{ccc} U^{\mathbb{C}} & & \\ \downarrow & \searrow \pi & \\ U^{\mathbb{C}}/\Gamma_m & \cong & U^{\mathbb{C}} \times_{\chi^m} \mathbb{C}^*\,, \end{array}$$

where π is the orbit map given by $\pi(g) = [g, 1]$. It follows that $\widetilde{\Omega}^*_\rho := \pi^{-1}(\Omega^*_\rho) = \{g \in U^{\mathbb{C}} \;:\; |\zeta|^m\rho(h) < 1\}$ is a covering of Ω^*_ρ with m-sheets. In fact it is the universal covering of Ω^*_ρ, since it is homeomorphic to $U^{\mathbb{C}}$, which is simply connected. Indeed $\widetilde{\Omega}^*_\rho$ itself can be regarded as a disc bundle over Q^2 and one can apply a suitable fiberwise radial dilatation deforming $\widetilde{\Omega}^*_\rho$ onto $U^{\mathbb{C}}$.

For every $m \in \mathbb{Z}$ let Ω^*_{max} be the U-equivariant, punctured disc bundle in L^m associated to $\rho_{max} : \mathbb{R} \to \mathbb{R}^{>0}$ defined by $\rho_{max}(h) := (\cosh 2h)^{|m|/2}$.

Proposition 5.3. *The U-equivariant, punctured disc bundle Ω^*_{max} is Stein and its universal covering admits a proper $\mathbb{C}$-action. In particular Ω^*_{max} is not Kobayashi hyperbolic.*

Proof. For $m = 0$ one has $\Omega^*_{max} = Q^2 \times \Delta^*$ and the statement follows by considering the action on Q^2 of any one parameter subgroup in $U^{\mathbb{C}}$. Next, by Lemma **??** one has $|\zeta| = \sqrt[4]{\frac{t}{s}}$ and

$$h = \frac{1}{2}\operatorname{arccosh}\sqrt{st} = \frac{1}{2}\operatorname{arccosh} e^{\frac{1}{2}(\log s + \log t)}\,.$$

Define $\theta : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ by $\theta(\tau) := \log\rho(\frac{1}{2}\operatorname{arccosh} e^{\frac{\tau}{2}})$. Then

$$\log(|\zeta|^m\rho(h)) = m\log|\zeta| + \log\rho(h) = \frac{m}{4}(\log t - \log s) + \theta(\log t + \log s) =$$

$$\theta(\log t + \log s) - \frac{m}{4}(\log t + \log s) + \frac{m}{2}\log t.$$

Assume that $m > 0$ and fix $\rho_{max}(h) = (\cosh 2h)^{m/2}$, which corresponds to $\theta_{max}(\tau) = \frac{m}{4}\tau$. Then the above equation implies that $\log|\zeta|^m\rho_{max}(h) = \frac{m}{2}\log t$, which is plurisubharmonic by Remark **??**. Therefore Ω_{max} is Stein by Prop. **??** and so is the associated punctured disc bundle Ω^*_{max}.

Finally note that the function $U^{\mathbb{C}} \to \mathbb{R}^{>0}$ given by $g \to t$ is invariant with respect to the proper $\mathbb{C}$-action on $U^{\mathbb{C}}$ defined by (cf. Rem. **??**)

$$w \cdot g := g\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}.$$

Thus the universal covering $\widetilde{\Omega}^*_{max} = \{\, g \in U^{\mathbb{C}} \ : \ \frac{m}{2}\log t < 0 \,\}$ is a $\mathbb{C}$-invariant subdomain of $U^{\mathbb{C}}$, proving the statement for $m > 0$. A similar argument (or use Lemma **??**) applies to the case when $m < 0$. □ □

Note that the fiberwise multiplication by $\rho(0)$ on $U^{\mathbb{C}} \times_{\chi^m} \mathbb{C}$, given by $[g, z] \to [g, \rho(0)z]$, maps Ω_ρ biholomorphically and U-equivariantly onto $\Omega_{\rho/\rho(0)}$. It follows that one can always normalize Ω_ρ so that $\rho(0) = 1$.

Theorem 5.4.
(i) *Every Stein, normalized, U-equivariant, disc bundle Ω_ρ over Q^2 is contained in Ω_{max}, and*
(ii) *if Ω_ρ does not coincide with Ω_{max}, then the associated Stein, punctured disc bundle Ω^*_ρ is Kobayashi Hyperbolic.*

Proof. First assume $m=0$ and let $\Omega_\rho = \{\, (gK^{\mathbb{C}}, z) \in Q^2 \times \mathbb{C} \ : \ |z|\rho(h) < 1 \,\}$ be Stein. Since $\rho(0)=1$ and ρ is logarithmically convex by Prop. **??**, it follows that $\Omega_\rho \subset Q^2 \times \Delta = \Omega_{max}$, proving (i). For (ii) consider the associated (Stein) punctured disc bundle Ω^*_ρ and the projection $\pi : \Omega^*_\rho \to \Delta^*$ onto the second factor. Note that if ρ is non constant then $\rho(h)^{-1} \to 0$ as $h \to \infty$. As a consequence for every relatively compact domain A of Δ^* the preimage $\pi^{-1}(A)$ is contained in the product $U \exp(IiH)K^{\mathbb{C}} \times A$, for some relatively compact interval I in $\mathbb{R}$. In particular $\pi^{-1}(A)$, being relatively compact in a Stein manifold, is Kobayashi hyperbolic and so is Ω^*_ρ by Thm. 3.2.15 in [**?**].

If $m \neq 0$ we prove the inclusion in (i) for the universal coverings $\widetilde{\Omega}_\rho$ and $\widetilde{\Omega}_{max}$. As a consequence of Lemma **??** it is enough to consider the case $m>0$. Recall that

$$\widetilde{\Omega}_{max} = \{\, g \in U^{\mathbb{C}} \ : \ \frac{m}{2} \log t < 0 \,\}.$$

Note that

$$\widetilde{\Omega}_\rho = \{\, g \in U^{\mathbb{C}} \ : \ \delta(\log s + \log t) + \frac{m}{2} \log t < 0 \,\},$$

where $\delta : [0,\infty) \to \mathbb{R}$ is defined by

$$\delta(\tau) := \theta(\tau) - \frac{m}{4}\tau = \log \rho(\frac{1}{2} \operatorname{arccosh}(e^{\tau/2})) - \frac{m}{4}\tau .$$

Since $\delta(0)=0$, in order to prove (i) it is enough to show that δ is increasing. Indeed one has

Claim. The function δ is increasing. Moreover, if $\delta \not\equiv 0$ then $\delta(\tau) \to \infty$ as $\tau \to \infty$.

Proof. Since Ω_ρ is Stein, the $U \times K$-invariant function $U^{\mathbb{C}} \to \mathbb{R}^{>0}$, given by $|\zeta|^m \rho(h) = \delta(\log s + \log t) + \frac{m}{2} \log t$, is plurisubharmonic (cf. Proposition **??**). Then, by composing with the holomorphic map $\mathbb{C} \to U^{\mathbb{C}}$, defined by

$$x + iy \to \begin{pmatrix} 1 & 0 \\ e^{x+iy} & 1 \end{pmatrix},$$

one obtains an subharmonic, $i\mathbb{R}$-invariant function, namely $x+iy \to \delta(\log(1+e^{2x}))$. It follows that the function $x \to \delta(\log(1+e^{2x}))$ is convex. Then it is necessarily increasing, since it converges to 0 as $x \to -\infty$. Furthermore $x \to \log(1+e^{2x})$ is strictly increasing, therefore δ is also increasing, as claimed. Finally note that if $\delta \not\equiv 0$, then $x \to \delta(\log(1+e^{2x}))$ is non constant, convex and increasing. Then necessarily $\delta(\tau) \to \infty$ as $\tau \to \infty$, concluding the proof of the claim.

For (ii) note that by Theorem 3.2.8 in [**?**] the Stein, punctured disc bundle Ω^*_ρ is Kobayashi hyperbolic if and only if its covering $\widetilde{\Omega}^*_\rho \subset U^{\mathbb{C}}$ is hyperbolic.

Assume as above that $m > 0$ and consider the projection

$$P : \widetilde{\Omega}_\rho^* \to \mathbb{C}^2 \setminus \{(0,0)\}\,, \qquad \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix} \to (z_3, z_4).$$

Since $\delta \geq 0$ and $\delta(\log s + \log t) + \frac{m}{2} \log t < 0$ on $\widetilde{\Omega}_\rho^*$ it follows that $t = |z_3|^2 + |z_4|^2 < 1$ and consequently $P(\widetilde{\Omega}_\rho^*)$, being contained in the punctured unit ball $\mathbb{B}_1^*(0,0)$ of $\mathbb{C}^2$, is Kobayashi hyperbolic. Then, by Thm. 3.2.15 in [**?**], in order to show that $\widetilde{\Omega}_\rho^*$ is Kobayashi hyperbolic it is sufficient to show that for every fixed (z_3, z_4) in $P(\widetilde{\Omega}_\rho^*)$ there exists ε small enough such that $P^{-1}(\mathbb{B}_\varepsilon(z_3, z_4))$ is Kobayashi hyperbolic. Here $\mathbb{B}_\varepsilon(z_3, z_4)$ denotes the ball centered in (z_3, z_4) of radius ε. Choose ε such that $\mathbb{B}_\varepsilon(z_3, z_4)$ is relatively compact in $\mathbb{B}_1^*(0,0)$. Then there exists a real, positive constant C such that $-C < \log t$ and consequently $\delta(\log s + \log t) < \frac{m}{2} C$ on $P^{-1}(\mathbb{B}_\varepsilon(z_3, z_4))$. Since by assumption $\rho \not\equiv \rho_{max}$, i.e. $\delta \not\equiv 0$, the above claim implies that $\delta(\tau) \to \infty$ as $\tau \to \infty$. It follows that $\log s + \log t < D$ for some real constant D. Hence $\log s < D + C$ and consequently $s = |z_1|^2 + |z_2|^2$ is bounded. This implies that $P^{-1}(\mathbb{B}_\varepsilon(z_3, z_4))$ is contained in the product of two balls in $\mathbb{C}^4$, therefore it is Kobayashi hyperbolic. □ □

Remark 5.5. Note that the Stein, U-equivariant, punctured disc bundles Ω_ρ^* are not hyperconvex, in the sense of [**?**]. Assume by contradiction that there exists a bounded plurisubharmonic exhaustion φ defined on Ω_ρ^*. Since every fiber F is closed in Ω_ρ^*, the restriction $\varphi|_F$ of φ to F is a subharmonic exhaustion. In particular $\varphi|_F$ is not constant. However F is biholomorphic to a punctured disc and $\varphi|_F$ is bounded, therefore $\varphi|_F$ extends to a bounded, subharmonic function on the whole disc with a maximum at the origin. Hence $\varphi|_F$ is constant, giving a contradiction.

For later use we note the following fact.

Lemma 5.6. *Let $\Omega_\rho \subset L^m$ be a Stein, U-equivariant disc bundle over Q^2. If $m \neq 0$ then every automorphism of Ω_ρ leaves the zero section invariant.*

Proof. Note that if p belongs to the zero section $Z \cong U^{\mathbb{C}}/K^{\mathbb{C}}$, then for every X in the 2-dimensional tangent space $T_pZ \cong \mathfrak{p}^{\mathbb{C}}$ there exists an entire curve through p and tangent to X. Namely, $\exp(\mathbb{C}X) \cdot p$. Then it is enough to show that for $p \in \Omega_\rho^*$ the subspace of the elements of $T_p\Omega_\rho$ with this property is lower dimensional.

For this consider the free action of the cyclic group $\Gamma_m \subset K^{\mathbb{C}} \cong \mathbb{C}^*$ on the punctured unit ball $\mathbb{B}_1^*(0,0)$ in $\mathbb{C}^2$ given by $\gamma \cdot (z, w) := (\gamma z, \gamma w)$. Let

$P : \Omega_\rho^* \to \mathbb{B}_1^*(0,0)/\Gamma_m$ be the projection defined by (cf. the proof of Thm. **??**)

$$\left[\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}, 1\right] \to [z_3, z_4]$$

and let $\iota : \mathbb{B}_1^*(0,0)/\Gamma_m \to \Delta^3$ be the injective holomorphic map defined by $[z,w] \to (z^m, z^{m-1}w, w^m)$.

For an entire curve $f : \mathbb{C} \to \Omega_\rho$ through $p \in \Omega_\rho^*$ the inverse image $f^{-1}(Z)$ is a discrete set. Moreover the composition $\iota \circ P \circ f|_{\mathbb{C}\setminus f^{-1}(Z)} : \mathbb{C} \setminus f^{-1}(Z) \to \Delta^3$ defines a bounded holomorphic map. Thus it extends to a bounded holomorphic function on $\mathbb{C}$ which, by Liouville's theorem is constant. It follows that $f(\mathbb{C})$ is contained in the one dimensional fiber $P^{-1}(P(p))$ of P, which proves the statement. □ □

6. A characterization

A recent classification of holomorphic actions of classical simple, real Lie groups by Huckleberry and Isaev applies to show that the bounded symmetric domain $SO(3,2)/(SO(3) \times SO(2))$ is characterized among Stein manifolds by its complex dimension and by its automorphism group (see Thm. 8.1 in [**?**]). As an application of Theorem **??** we present a different proof of this fact. Here we follow the strategy pointed out in [**?**], where higher dimensional bounded symmetric domains of type IV were considered. We need a preparatory lemma. For notations and definitions we refer to [**?**].

Lemma 6.1. (cf. Prop. 4.7 in [**?**]) *Let X be a 3-dimensional Stein manifold such that $Aut(X)$ is isomorphic to $SO(3,2)$. Assume that X contains a minimal $SO(3) \times SO(2)$-orbit of dimension 3 which is $SO(3)$-homogeneous. Then X is biholomorphic to a $U \times K$-invariant domain in $U^{\mathbb{C}}/\Gamma_m$.*

Proof. Let $M = (SO(3,\mathbb{R}) \times SO(2,\mathbb{R}))/H$ be the minimal 3-dimensional orbit. Then the connected component H^e of the isotropy subgroup H at e is 1-dimensional and there exists an isomorphism $SO(2,\mathbb{R}) \to H^e$, say $t \to (\varphi(t), \psi(t))$. By (ii) of Lemma 4.6 in [**?**] the $SO(2,\mathbb{R})$-action on M is free, therefore the homomorphism φ is injective. Up to Lie group isomorphism we may assume that $\varphi(SO(2,\mathbb{R}))$ is the one parameter subgroup of $SO(3,\mathbb{R})$ generated by the element

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

belonging to the Lie algebra of $SO(3,\mathbb{R})$. By assumption $M = SO(3,\mathbb{R})/F$, where $F := H \cap SO(3)$ is finite. Since F is a subgroup of H, it normalizes H^e. As a consequence F is contained in the normalizer of $\varphi(SO(2,\mathbb{R}))$, which is given by $\varphi(SO(2,\mathbb{R})) \cup \gamma\varphi(SO(2,\mathbb{R})) \cong O(2,\mathbb{R})$, where $\gamma := diag(-1,-1,1)$. However, if F contains an element of the form $\gamma\varphi(t')$, then for all $t \in SO(2,\mathbb{R})$ one has $(\gamma\varphi(t')\varphi(t)\varphi(t')^{-1}\gamma^{-1},\, \psi(t)) = (\varphi(t)^{-1}, \psi(t)) \in H$ and consequently $(\varphi(t)^{-1}, \psi(t))(\varphi(t), \psi(t)) = (e, \psi(t)^2) \in H$. Since the $SO(2,\mathbb{R})$-action on M is free ((ii) of Lemma 4.6 in [?]), this implies that the homomorphism ψ is trivial and $M = SO(3,\mathbb{R})/O(2,\mathbb{R}) \times SO(2,\mathbb{R})$, contradicting the transitivity of the $SO(3,\mathbb{R})$-action. Hence F is a cyclic subgroup of $\varphi(SO(2,\mathbb{R}))$. Consider the commutative diagram

$$\begin{array}{ccc} SO(3,\mathbb{R}) & & \\ \downarrow & \searrow \Psi & \\ SO(3,\mathbb{R})/F & \cong & (SO(3,\mathbb{R}) \times SO(2,\mathbb{R}))/H = M\,, \end{array}$$

where the surjective orbit map Ψ is defined by $\Psi(g) = [g,e]$. Let $SO(3,\mathbb{R}) \times SO(2,\mathbb{R})$ act on $SO(3,\mathbb{R})$ by $(g',t)\cdot g := g'g\varphi(t)^{-1}$ and naturally on M (i.e., by left $SO(3,\mathbb{R}) \times SO(2,\mathbb{R})$-action). One has

$$\Psi(g'g\varphi(t)^{-1}) = [g'g\varphi(t)^{-1}, \psi(t)\psi(t)^{-1}] = [g'g,\ \psi(t)] = (g', \psi(t)) \cdot \Psi(g)\,.$$

Now recall that X is biholomorphic to an $SO(3,\mathbb{R}) \times SO(2,\mathbb{R})$-invariant domain in the complexified orbit $(SO(3,\mathbb{C}) \times SO(2,\mathbb{C}))/H^{\mathbb{C}}$ (cf. the beginning of Sect. 4.1 in [?]) and extend the isomorphism in the above diagram to $SO(3,\mathbb{C})/F \to (SO(3,\mathbb{C}) \times SO(2,\mathbb{C}))/H^{\mathbb{C}}$. Then, the analytic continuation principle and the above equivariance relation imply that the manifold X is biholomorphic to an $SO(3,\mathbb{R}) \times SO(2,\mathbb{R})$-invariant domain in $SO(3,\mathbb{C})/F$.

Finally let $\Pi : U^{\mathbb{C}} \to SO(3,\mathbb{C})$ be a universal covering of $SO(3,\mathbb{C})$ which maps U onto $SO(3,\mathbb{R})$ and K onto H^e. Then the finite subgroup $\Pi^{-1}(F)$ of K is cyclic and $SO(3,\mathbb{C})/F$ is equivariantly biholomorphic to $U^{\mathbb{C}}/\Pi^{-1}(F)$, implying the statement. □ □

Theorem 6.2. *Let X be a 3-dimensional Stein manifold such that $Aut(X)$ is isomorphic to $SO(3,2)$. Then X is biholomorphic to the bounded symmetric domain $SO(3,2)/(SO(3) \times SO(2))$.*

Proof. If the maximal compact subgroup $SO(3) \times SO(2)$ has a fixed point in X, then the statement follows from Prop. 3.1 in [?].

So let us assume by contradiction that $SO(3) \times SO(2)$ has no fixed points in X. Then, as a consequence of Lemma **??** above, Prop. 4.8 and 4.10 in [?], the manifold X is biholomorphic to a U-invariant domain in a line bundle either over the complex affine quadric Q^2 or over $Q^2/\mathbb{Z}_2$. Here we allow finite ineffectivity in order to replace the action of $SO(3,\mathbb{R})$ with the action of its universal covering $U = SU(2)$. If the base is Q^2, the line bundle is given by $L^m := U^{\mathbb{C}} \times_{\chi^m} \mathbb{C}^*$, for some $m \in \mathbb{Z}$, with projection $p : L^m \to Q^2$ given by $[g, z] \to gK^{\mathbb{C}}$ (cf. Sect.2). We distinguish several cases.

If $p(X)$ does not coincide with Q^2, then $p(X)$ is Kobayashi hyperbolic and so is X by Thm. 3.2.15 in [**?**] (cf. the proof of Thm. 5.5 in [**?**]). Then, as a consequence of Prop. 3.2 in [**?**] the group $SO(3) \times SO(2)$ has a fixed point, giving a contradiction.

If $p(X) = Q^2$ and $m = 0$, analogous arguments as in the proof of Thm. 5.5 in [**?**] imply that either X is Kobayashi hyperbolic or $Aut(X)$ is infinite dimensional, giving again a contradiction.

If $p(X) = Q^2$ and $m \neq 0$, then one checks that X is biholomorphic either to a disc bundle Ω_ρ or to a punctured disc bundle Ω^*_ρ. As a consequence of Lemma **??**, in both cases $SO(3, 2)$ acts on Ω^*_ρ. If $\Omega^*_\rho \neq \Omega^*_{max}$, then Ω^*_ρ is Kobayashi hyperbolic by Theorem **??** and one obtains a contradiction as above.

In the case when $\Omega^*_\rho = \Omega^*_{max}$ consider the projection $P : \Omega^*_{max} \to \mathbb{B}^*_1(0, 0)/\Gamma_m$ introduced in the proof of Lemma **??** and note that every fiber F of P is biholomorphic to $\mathbb{C}$. Then hyperbolicity of $\mathbb{B}^*_1(0, 0)/\Gamma_m$ implies that for every $g \in SO(3, 2)$ the composition $P \circ g|_F$ is constant. That is, g maps fibers to fibers and consequently the $SO(3, 2)$-action on Ω^*_{max} pushes down to an action on $\mathbb{B}^*_1(0, 0)/\Gamma_m$. By hyperbolicity of $\mathbb{B}^*_1(0, 0)/\Gamma_m$ such an action is necessarily proper and consequently every isotropy subgroup is contained in a copy of the maximal one. It follows that the minimal real dimension of every $SO(3, 2)$-orbit in $\mathbb{B}^*_1(0, 0)/\Gamma_m$ is six. Since $\mathbb{B}^*_1(0, 0)/\Gamma_m$ is a complex 2-dimensional manifold, this gives a contradiction.

Similar arguments apply to the case when X is biholomorphic to a U-invariant domain in a line bundle over $Q^2/\mathbb{Z}_2$ and we omit the details. □ □

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