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Singular Traces and Compact Operators

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We give a necessary and sufficient condition on a positive compact operator T for the existence of a singular trace (i.e. a trace vanishing on the finite rank operators) which takes a finite non-zero value on T. This generalizes previous results by Dixmier and Varga. We also give an explicit description of these traces and associated ergodic states on $l^\infty(\mathbb{N})$ using tools of non standard analysis in an essential way. © 1996 Academic Press, Inc.

1. INTRODUCTION

In 1966, Dixmier proved that there exist traces on $B(H)$, the bounded linear operators on a separable complex Hilbert space, which are not normal [D2]. Dixmier traces have the further property to be "singular", i.e. they vanish on the finite rank operators.

The importance of this type of traces is well-known due to their applications in non commutative geometry and quantum field theory (see [C]).

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Since every trace can be decomposed in a sum of a normal and a singular part, and since the normal traces on $B(H)$ coincide with the usual trace up to a constant, it is natural to investigate the structure of the class of the singular traces. In this paper we study this problem at a local level, i.e. we fix an operator T in $B(H)$, and ask the following questions:

(i) Do there exist singular traces which are non-trivial "on T", i.e. on the two-sided ideal generated by T in $B(H)$?

(ii) Is it possible to give an explicit description of these traces, when they exist?

We shall prove that the answer to the first question is positive if and only if the operator T is generalized eccentric (see definition 2.7).

In relation to the second question, we describe explicitly some classes of singular traces, which may possibly be extended to larger ideals, and we give a detailed analysis of the structure of such classes.

As already mentioned, the first idea for constructing singular traces is due to Dixmier, who considered compact operators for which the sum of the first n singular values diverges at a given suitable rate. The singular trace is then obtained evaluating on these partial sums, appropriately rescaled, a state on $l^{\infty}(\mathbb{N})$ which is invariant under "2-dilations".

More recently, the general question which operator ideals in $B(H)$ support traces has been studied by Varga [V]. The procedure used by Varga in order to describe traces differs from that of Dixmier in the choice of the states on $l^{\infty}(\mathbb{N})$.

In two preceding papers [AGPS1], [AGPS2] we gave explicit formulas for Dixmier-type traces and introduced a new class of singular traces. The result concerning generalized eccentric operators has been announced in [AGPS3].

The present paper is organized as follows. In Section 2 we introduce the basic definition of generalized eccentric operators and prove the main theorem about singular traceability. We remark that while the question of mere traceability has a trivial answer when restricted to the trace class operators, this is not the case for singular traceability. Therefore our analysis turns out to be an extension of the theory of Varga. Moreover, we illustrate two different techniques to construct singular traces. These techniques are a generalization of those in [D2] and [V].

In Section 3 we describe ergodic states giving rise to both kinds of singular traces introduced in Section 2. The basic technique we use is related to non standard analysis (NSA). Section 3 also involves the representation of Banach–Mazur limits by NSA. Such representation has been discussed before – e.g. in [KM], [L].

In Section 4 we work out explicitly the computation of the Dixmier traces of an operator, again using the NSA framework in an essential way.

Let us finally mention that there are still several problems which deserve attention. For instance under which conditions a singular trace can be extended to a larger ideal, and possibly the existence of a maximal ideal in this context. Another interesting problem is to find out a general representation formula for all singular traces.

2. Singular Traces and Generalized Eccentric Operators

Let $\mathscr{R}$ a von Neumann algebra and $\mathscr{R}^+$ the cone of its positive elements. A *weight* on $\mathscr{R}$ is a linear map

$$\phi: \mathscr{R}^+ \to [0, +\infty]$$

Any weight can be extended by linearity to the natural domain given by the linear span of $\{T \in \mathscr{R}^+ \mid \phi(T) < +\infty\}$

A weight τ which has the property:

$$\tau(T^*T) = \tau(TT^*) \qquad \forall T \in \mathscr{R}$$

is called a *trace* on $\mathscr{R}$.

The natural domain of a trace τ is a two-sided ideal denoted by $\mathscr{I}_\tau$. For instance the natural domains of the trivial traces on $\mathscr{R}$ given by $\tau \equiv 0$ and $\tau \equiv +\infty$ are respectively the ideals $\mathscr{R}$ and $\{0\}$ while the usual trace on $B(H)$, the bounded linear operators on a complex, separable Hilbert space H, is associated with the ideal $L^1(H)$ of the trace class operators.

A weight ϕ on $\mathscr{R}$ is called *normal* if for every monotonically increasing generalized sequence $\{T_\alpha, \alpha \in I\}$ of elements of $\mathscr{R}^+$ such that $T = \sup_\alpha T_\alpha$ one has

$$\phi(T) = \lim_\alpha \phi(T_\alpha).$$

From now on the von Neumann algebra $\mathscr{R}$ will be fixed to be $B(H)$.

A classical result [D1] concerning normal traces on $B(H)$ is the following:

2.1. Theorem. *Every non trivial normal trace on $B(H)$ is proportional to the usual trace.*

By a theorem of Calkin (see [GK]), each proper two-sided ideal in $B(H)$ contains the finite rank operators and is contained in the ideal $K(H)$ of the compact linear operators on H. Therefore all traces on $B(H)$ live on the compact operators, and the following definition makes sense:

2.2. Definition. A trace τ on $B(H)$ will be call *singular* if it vanishes on the set $F(H)$ of finite rank operators.

2.3. Proposition. *Any trace τ on $K(H)$ can be uniquely decomposed as $\tau=\tau_1+\tau_2$, where τ_1 is a normal trace and τ_2 is a singular trace.*

Proof. If $\tau(F(H))\equiv 0$ then the result is obvious. Let us suppose there exists $A\in F(H)^+$ such that $\tau(A)=1$.

Since $A=\sum_1^N \lambda_i E_i$, where $\{E_i\}$ is a set of rank one projectors, $\tau(E_{i_0})=C>0$ for some i_0.

Since all rank one projectors are unitarily equivalent then $\tau(E)=C$ for each rank one projector E. As a consequence $\tau=C\,\mathrm{tr}$ ($\mathrm{tr}(\cdot)$ denoting the usual trace) on rank one projectors and therefore, by linearity, on all $F(H)$.

Let us set $\tau_2\equiv\tau-C\,\mathrm{tr}$ and $\tau_1\equiv C\,\mathrm{tr}$, then $\tau_2(F(H))\equiv 0$. It remains only to show the positivity of τ_2. For $A\in K(H)^+$ there exists a sequence $\{A_n\}$ of finite rank positive operators such that $A=l.u.b.\ A_n$. For this sequence we have $\tau(A_n)=\mathrm{tr}(A_n)$ hence $\tau(A)\geqslant l.u.b.\ \tau(A_n)=C\ (l.u.b.\,\mathrm{tr}(A_n))=C\,\mathrm{tr}\,(A)$, since $\mathrm{tr}(\cdot)$ is normal. From the previous inequality we get $\tau_2(A)\geqslant 0$. ∎

In view of this proposition, in the rest of the paper we shall restrict our attention to the singular traces.

For T a compact operator on H, $\{\mu_n(T)\}_{n=1}^{\infty}$ will denote the non increasing sequence of the eigenvalues of $|T|$ with multiplicity.

We shall also set $\sigma_n(T)\equiv\sum_{k=1}^{n}\mu_r(T)$.

2.4. Definition. Let T be a compact operator. We call *integral sequence* of T the sequence $\{S_n(T)\}_{n=0}^{\infty}$ which is an indefinite integral (w.r.t. the counting measure) of $\{\mu_n(T)\}_{n=1}^{\infty}$, i.e. $S_n(T)-S_{n-1}(T)=\mu_n(T)$, $n\geqslant 1$, and such that

$$S_0(T)\equiv\begin{cases}0 & T\notin L^1(H)\\ -\mathrm{tr}(T) & T\in L^1(H)\end{cases}$$

Notice that if $T\notin L^1(H)$, $S_n(T)=\sigma_n(T)$, $n\geqslant 1$, while if $T\in L^1(H)$, then $S_n(T)=\sigma_n(T)-\mathrm{tr}(T)\to 0$ as $n\to\infty$.

2.5. *Remark.* If T does not belong to $L^1(H)$ and τ is a trace which is finite and non-zero on $|T|$ then τ is necessarily singular, that is, the existence of traces which are non trivial on T is equivalent with the existence of non trivial singular traces on T. Since for $T\in L^1(H)$ the existence of a non trivial trace is obvious, it follows that the relevant question is not the mere “traceability” of a compact operator T, but the existence of a singular trace which is non trivial on $|T|$.

Let us also notice that a trace τ is finite on $|T|$ if and only if the principal ideal $\mathscr{I}(T)$, i.e. the (two-sided) ideal generated by T in $B(H)$, is contained in $\mathscr{I}_\tau$.

2.6. LEMMA. *Let T be a compact operator. The following are equivalent*:

(i) 1 *is a limit point of the sequence* $\{S_{2n}(T)/S_n(T)\}_{n=0}^{\infty}$

(ii) *There exists an increasing sequence of natural numbers* $\{p_k\}$ *such that* $\lim_{k\to\infty}(S_{kp_k}(T)/S_{p_k}(T))=1$.

Proof. (ii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (ii). First we exploit the concavity of the sequence $S_n := S_n(T)$:

$$\frac{k-2}{k-1}S_n+\frac{1}{k-1}S_{kn}\leqslant S_{2n} \qquad n\in\mathbb{N},\quad k\geqslant 2.$$

From that, with simple manipulations, we get

$$\left|1-\frac{S_{kn}}{S_n}\right|\leqslant (k-1)\left|1-\frac{S_{2n}}{S_n}\right| \qquad k, n\in\mathbb{N}.$$

By hypothesis (i), for each $k\in\mathbb{N}$ there exists $p_k\in\mathbb{N}$ such that

$$\left|1-\frac{S_{2p_k}}{S_{p_k}}\right|\leqslant\frac{1}{k^2}.$$

Therefore

$$\left|1-\frac{S_{kp_k}}{S_{p_k}}\right|\leqslant\frac{k-1}{k^2}$$

and the thesis follows. ∎

2.7. DEFINITION. A compact operator T which satisfies one of the equivalent properties of lemma 2.6 will be called *generalized eccentric*.

2.8. *Remark*. The class of generalized eccentric operators which are not in $L^1(H)$ coincides with the class of eccentric operators considered in [V].

Let T be a compact operator. Then it is clear that $\mathscr{I}(T)=\bigcup_{r=1}^{+\infty}\mathscr{I}_r(T)$ where $\mathscr{I}_r(T)$ is the set of all bounded operators of the form

$$A=\sum_{i=1}^{r}X_iTY_i, \qquad X_1, ..., X_r, Y_1, ..., Y_r\in B(H).$$

The estimates below will be crucial for the rest of the section.

2.9. PROPOSITION. (i) *An operator A belongs to $\mathcal{I}_r(T)$ if and only if*

$$\exists K \in \mathbb{R}: \mu_{r(n-1)+1}(A) \leqslant K\mu_n(T), \qquad n \in \mathbb{N}$$

(ii) *Given A, B positive compact operators, then*

$$\sigma_n(A+B) \leqslant \sigma_n(A) + \sigma_n(B) \leqslant \sigma_{2n}(A+B), \qquad n \in \mathbb{N}$$

Proof. See [V] for (i). See [GK] or [S] for (ii). ▮

We can now state and prove the main result of this section.

2.10. THEOREM. *Let T be a compact operator. Then the following are equivalent*:

(a) *There exists a singular trace τ such that $0 < \tau(|T|) < +\infty$.*

(b) *T is generalized eccentric.*

Let us remark that the condition (i) in Lemma 2.6 gives some information on the rate of convergence of the sequence $\{\mu_n(T)\}$. For instance, notice that if $\mu_n(T) \sim n^\alpha$ as $\to \infty$, condition (i) implies $\alpha = -1$. Other natural examples of sequences $\{\mu_n\}$ satisfying condition (i) are $\mu_n \equiv (\log n)^\alpha / n$. Hence, by Theorem 2.10, it follows that if $T \in K(H)$ and $\mu_n(T) = (\log n)^\alpha / n$ there exists a singular trace τ such that $\tau(T) = 1$. When $\alpha < -1$, the domain of the associated singular traces is contained in $L^1(H)$ (cf. Remark 2.14).

Proof. (a) $\Rightarrow$ (b). As we observed in Remark 2.5, when $T \notin L^1(H)$ the existence of a non trivial singular trace on $|T|$ is equivalent to the existence of a non trivial trace "tout court" on $|T|$. Therefore, in this case, the result is given by Theorem 1 in [V].

We are left with $T \in L^1(H)$. Let us notice that, in this case, we have only to show that

$$\sup_n \frac{S_{kn}(T)}{S_n(T)} > \frac{1}{3} \qquad \forall k \in \mathbb{N}.$$

Indeed, if 1 is not a limit point of $\{S_{2n}(T)/S_n(T)\}$, then $\sup_n(S_{2n}(T)/S_n(T)) = l < 1$. As a consequence, $S_{2^m n}(T)/S_n(T) \leqslant l^m < \frac{1}{3}$ when $n \in \mathbb{N}$ and $m > |\log 3 / \log l|$.

The proof will be given by contradiction, that is we assume $\sup_n(S_{kn}(T)/S_n(T)) \leqslant \frac{1}{3}$ for some fixed $k \in \mathbb{N}$ and then we prove that any singular trace is trivial on $\mathcal{I}(T)$.

Since a trace is trivial on a principal ideal $\mathscr{I}$ *iff* it is trivial on a positive operator S which generates $\mathscr{I}$, we may consider the following operator S, realized averaging over the eigenvalues of T,

$$\mu_n(S) \equiv \begin{cases} \dfrac{S_{k^{L(n)}}(T) - S_{k^{L(n)-1}}(T)}{k^{L(n)} - k^{L(n)-1}} & n > 1 \\ \mu_1(T) & n = 1 \end{cases}$$

where $L(n)$ is the integer defined by $k^{L(n)-1} < n \leqslant k^{L(n)}$.

From (i) of Proposition 2.9 it follows easily that $S \in \mathscr{I}_k(T)$ and vice versa $T \in \mathscr{I}_k(S)$, therefore $\mathscr{I}(S) = \mathscr{I}(T)$.

Next we notice that the following eigenvalue estimate holds:

$$\mu_n(S) \geqslant 2k\mu_{k(n-1)+1}(S). \tag{2.1}$$

Indeed, this follows from the definition of $\mu_n(S)$ and the inequality

$$S_{k^{L(n)}}(T) - S_{k^{L(n)-1}}(T) \geqslant 2(S_{k^{L(n)+1}}(T) - S_{k^{L(n)}}(T))$$

which is a consequence of the assumptions on T.

By means of a k-dilation procedure, we now construct another compact positive operator $\tilde{S}$ such that $\tau(S) = \tau(\tilde{S})$ for any τ.

We fix an orthonormal basis of H and describe the operators which are diagonal w.r.t. this basis by means of the corresponding eigenvalue sequences

$$S \equiv \mu_1, \mu_2 \cdots$$

$$S_1 \equiv \mu_1, \underbrace{0 \cdots 0}_{(k-1)\text{ times}}, \mu_2, \underbrace{0 \cdots 0}_{(k-1)\text{ times}}, \ldots$$

$$S_2 \equiv 0, \mu_1, \underbrace{0 \cdots 0}_{(k-2)\text{ times}}, 0, \mu_2, \underbrace{0 \cdots 0}_{(k-2)\text{ times}}, \ldots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$S_k \equiv \underbrace{0 \cdots 0}_{(k-1)\text{ times}}, \mu_1, \underbrace{0 \cdots 0}_{(k-1)\text{ times}}, \mu_2, \ldots .$$

Then we define

$$\tilde{S} = \frac{1}{k} \sum_{i=1}^{k} S_i.$$

By linearity and unitary invariance $\tau(\tilde{S}) = \tau(S)$ for each trace τ. Moreover, by construction.

$$\mu_{k(n-j)+j}(\tilde{S}) = \frac{1}{k} \mu_n(S) \qquad \forall n \in \mathbb{N}, \quad j = 1, ..., k.$$

Hence,

$$\mu_{k(n-1)+j}(\tilde{S}) \geqslant 2\mu_{k(n-1)+1}(S) \geqslant 2\mu_{k(n-1)+j}(S) \tag{2.2}$$

by (2.1).

It is evident that (2.2) implies

$$\tau(\tilde{S}) \geqslant 2\tau(S) = 2\tau(\tilde{S})$$

which is impossible if $\tau(S)$ is finite non zero.

The proof of (b) $\Rightarrow$ (a) follows immediately by Theorem 2.11 (see below). ∎

Let us now discuss possible procedures to construct singular traces on $K(H)$.

Our first step is a generalization of a method suggested in [V], in order to built up singular traces τ associated with generalized eccentric operators T. To this aim it is useful to introduce a triple $\Omega = (T, \varphi, \{n_k\})$, where T is generalized eccentric, φ is a state on $l^\infty(\mathbb{N})$ which vanishes on c_0, the space of infinitesimal sequences, and $n_k = kp_k$, $k \in \mathbb{N}$, where $\{p_k\}$ is the sequence of natural numbers given in Lemma 2.6.

With such a triple Ω we associate a functional τ_Ω on the positive part of the ideal $\mathscr{I}(T)$:

$$\tau_\Omega(A) \equiv \varphi\left(\left\{\frac{S_{n_k}(A)}{S_{n_k}(T)}\right\}\right), \qquad A \in \mathscr{I}(T)^+ \tag{2.3}$$

2.11. Theorem. *Let T be a generalized eccentric operator. The functional τ_Ω defined in* (2.3) *extends linearly to a singular trace on the ideal $\mathscr{I}(T)$*

Proof. The positivity, homogeneity and unitary invariance of τ_Ω are obvious. It suffices to check additivity on positive elements.

We take $C \in \mathscr{I}(T)^+$. Then $C \in \mathscr{I}_r(T)$ for some $r \geqslant 2$ and, by Proposition 2.9(i), if $2 \leqslant r \leqslant k$

$$\begin{aligned}
S_{2kp_k}(C) - S_{kp_k}(C) &= \sigma_{2kp_k}(C) - \sigma_{kp_k}(C) \\
&\leqslant \sigma_{rkp_k}(C) - \sigma_{rp_k}(C) \\
&= \sum_{j=rp_k+1}^{rkp_k} \mu_j(C) = \sum_{j=p_k+1}^{kp_k} \sum_{i=1}^{r} \mu_{r(j-1)+i}(C) \\
&\leqslant \sum_{j=p_k+1}^{kp_k} Kr\mu_j(T) = Kr(S_{kp_k}(T) - S_{p_k}(T)).
\end{aligned}$$

As a consequence,

$$\left|\frac{S_{2n_k}(C)-S_{n_k}(C)}{S_{n_k}(T)}\right| \leqslant Kr\left|1-\frac{S_{p_k}(T)}{S_{kp_k}(T)}\right| \xrightarrow[k\to\infty]{} 0.$$

Now suppose $T \in L^1(H)$ and $A, B \in \mathscr{I}(T)^+$. Then $A, B \in L^1(H)$, and Proposition 2.9(ii) implies

$$\frac{S_{n_k}(A+B)}{S_{n_k}(T)} \geqslant \frac{S_{n_k}(A)}{S_{n_k}(T)} + \frac{S_{n_k}(B)}{S_{n_k}(T)} \geqslant \frac{S_{2n_k}(A+B)}{S_{n_k}(T)}$$

Since φ is positive and vanishes on infinitesimal sequences,

$$\begin{aligned}\tau_\Omega(A+B) &\geqslant \tau_\Omega(A)+\tau_\Omega(B)\\ &\geqslant \tau_\Omega(A+B)+\varphi\left(\left\{\frac{S_{2n_k}(A+B)-S_{n_k}(A+B)}{S_{n_k}(T)}\right\}\right)\\ &= \tau_\Omega(A+B)\end{aligned}$$

i.e. τ_Ω is additive.

The proof for $T \notin L^1(H)$ is analogous. ∎

The singular traces τ_Ω given by (2.3) which are associated with generalized eccentric operators T give rise to a constructive proof of the implication (b) ⇒ (a) of Theorem 2.10.

We shall call such traces *generalized Varga traces*. Indeed, in the case $T \notin L^1(H)$, the traces τ_Ω correspond to traces constructed in [V].

Let us notice that the traces given by (2.3) can be writen also as

$$\tau_\Omega(A) = \varphi^{\{n_k\}}\left(\left\{\frac{S_n(A)}{S_n(T)}\right\}\right), \qquad A \in \mathscr{I}(T) \tag{2.4}$$

where $\varphi^{\{n_k\}}$ is the (non normal) state on $l^\infty(\mathbb{N})$ defined by

$$\varphi^{\{n_k\}}(\{a_n\}) \equiv \varphi(\{a_{n_k}\}).$$

We remark that if φ is an extremal state on $l^\infty(\mathbb{N})$, so is $\varphi^{\{n_k\}}$.

Let us now consider in general the functional

$$\tau_\psi(A) \equiv \psi\left(\left\{\frac{S_n(A)}{S_n(T)}\right\}\right), \qquad A \in \mathscr{I}(T) \tag{2.5}$$

where ψ is a generic state on $l^\infty(\mathbb{N})$.

The above remarks show that, if ψ is chosen as the state $\varphi^{\{n_k\}}$, then (2.5) gives rise to a singular trace.

Other singular traces can be obtained by choosing states ψ in (2.5) with suitable invariance properties. Generalizing an idea of Dixmier [D2], we shall prove the following theorem.

2.12. THEOREM. *If* ψ *is a two-dilation invariant state and* $\lim_{n\to+\infty} S_{2n}(T)/S_n(T)=1$, *then* τ_ψ *is a trace on* $\mathscr{I}(T)$. *Moreover, in this case formula* (2.5) *gives rise to a singular trace* (*which will be denoted by* τ_ψ *as well*) *even on the* (*larger*) *ideal*

$$\mathscr{I}_m(T)\equiv\left\{A\in K(H)\;\middle|\;\left\{\frac{S_n(A)}{S_n(T)}\right\}\in l^\infty\right\}.$$

We would like to point out that, when $T\notin L^1(H)$, the ideal $\mathscr{I}_m(T)$ is a maximal norm ideal in the sense of Schatten [S] (see also [GK]).

2.13. *Remark.* A 2-dilation invariant state is necessarily not normal, more precisely it is zero on the space c_0 of infinitesimal sequences. Indeed if $\{a_n\}$ has only a finite number of non zero elements then $\varphi(\{a_n\})=\varphi(\{a_{2^k n}\})=\varphi(\{0\})=0$ for a sufficiently large k.

By continuity this result extends to c_0.

Proof of Theorem 2.12. Unitary invariance is obvious. We prove positivity. To this aim let A be a positive operator. If $A, T\in L^1(H)$ or $A, T\notin L^1(H)$ then $S_n(A)/S_n(T)\geqslant 0$ and positivity follows.

For $A\in L^1(H)$, $T\notin L^1(H)$, we have $S_n(A)/S_n(T)\to 0$ as $n\to\infty$, so $\tau_\psi(\{S_n(A)/S_n(T)\})=0$ by Remark 2.13.

Finally if $A\notin L^1(H)$, $T\in L^1(H)$ then $A\notin\mathscr{I}(T)$ and therefore $\tau_\psi(A)=+\infty$. Now we prove linearity.

First observe that if A, B are such that $A+B\notin\mathscr{I}(T)$ then at least one of them, say A, does not belong to $\mathscr{I}(T)$. Therefore $\tau_\psi(A+B)=\tau_\psi(A)+\tau_\psi(B)(=+\infty)$.

From Proposition 2.9(ii) it follows

$$S_n(A+B)\leqslant S_n(A)+S_n(B)\leqslant S_{2n}(A+B)$$

when $A, B\in L^1(H)$ or $A, B\notin L^1(H)$. In both cases if $T\notin L^1(H)$ we get

$$\begin{aligned}\psi\left(\left\{\frac{S_n(A+B)}{S_n(T)}\right\}\right)&\leqslant\psi\left(\left\{\frac{S_n(A)}{S_n(T)}\right\}\right)+\psi\left(\left\{\frac{S_n(B)}{S_n(T)}\right\}\right)\\&\leqslant\psi\left(\left\{\frac{S_{2n}(A+B)}{S_n(T)}\right\}\right)\end{aligned}\tag{2.6}$$

while if $T\in L^1(H)$ we get the reversed inequalities.

Let us now remark the following property of the state ψ: if $\{b_n\} \in l^\infty(\mathbb{N})$ and $a_n/b_n \to 1$ as $n \to \infty$ then $\psi(\{a_n\}) = \psi(\{b_n\})$. This follows from the fact that $\{a_n - b_n\} \in c_0$. Applying these properties we get

$$\begin{aligned}\psi\left(\left\{\frac{S_{2n}(A+B)}{S_n(T)}\right\}\right) &= \psi\left(\left\{\frac{S_{2n}(A+B)}{S_{2n}(T)}\right\}\left\{\frac{S_{2n}(T)}{S_n(T)}\right\}\right)\\ &= \psi\left(\left\{\frac{S_{2n}(A+B)}{S_{2n}(T)}\right\}\right)\\ &= \psi\left(\left\{\frac{S_n(A+B)}{S_n(T)}\right\}\right) \end{aligned} \tag{2.7}$$

the last equality being a consequence of 2-dilation invariance.

Therefore by (2.6), (2.7) and the definition of τ_ψ we get

$$\tau_\psi(A+B) = \tau_\psi(A) + \tau_\psi(B)$$

when $A, B \in L^1(H)$ (or $A, B \notin L^1(H)$) and either $T \in L^1(H)$ or not.

The case $A \in L^1(H)$, $B \notin L^1(H)$ and $T \in L^1(H)$ implies $A + B \notin L^1(H)$ and therefore $A + B \notin \mathscr{I}(T)$, a situation already discussed. It remains the possibility $A \in L^1(H)$, $B \notin L^1(H)$ and $T \notin L^1(H)$. In such a case we have

$$\frac{S_n(A+B)}{S_n(T)} \leqslant \frac{S_n(A) + \operatorname{tr}(A) + S_n(B)}{S_n(T)} \leqslant \frac{S_{2n}(A+B)}{S_{2n}(T)}$$

and therefore, since $\operatorname{tr}(A)/S_n(T) \to 0$ as $n \to \infty$, we obtain that linearity holds once again. This ends the proof. ∎

2.14. *Remark.* We notice that if the operator $T \notin L^1(H)$, then the traces described in Theorem 2.12 are exactly the traces discussed by Dixmier in [D2]. Indeed the sequence $\{S_n(T)\}$ has all the properties of the sequence $\{\alpha_n\}$ listed in the paper of Dixmier. On the other hand given any sequence $\{\alpha_n\}$ with the properties required by Dixmier there exists a generalized eccentric operator T for which $S_n(T) = \alpha_n$ (see e.g. [AGPS1]).

In the case $T \in L^1(H)$ our Theorem 2.12 produces a new class of non normal traces, which is in a sense the inverse image inside $L^1(H)$ of the class of Dixmier traces. For such a reason we shall call *generalized Dixmier traces* the singular traces given by Theorem 2.12.

The existence of this new type of traces was announced in [AGPS2].

2.15. *Remark.* According to the decomposition in Proposition 2.3, a trace is non normal *iff* τ_2 is non-zero, and it is faithful *iff* τ_1 is non-zero. On the other hand if τ_2 vanishes on $L^1(H)$ it gives no contribution to the sum. Therefore the traces which come from generalized eccentric operators inside $L^1(H)$, summed with the usual trace, give the first example of non-normal, faithful traces.

3. Two-Dilation Invariant States and Ergodicity

The main problem we are going to discuss in this section concerns extremal (ergodic) states which give rise to singular traces. In our opinion it is non standard analysis (NSA) which supplies the most convenient tools for this purpose.

Recall that if $\{a_n\}_{n \in \mathbb{N}}$ a standard sequence of real numbers, $\{{}^*a_n\}_{n \in {}^*\mathbb{N}}$ will denote its non standard extension.

As always, if x is a finite element in ${}^*\mathbb{R}$ then ${}^\circ(x) \in \mathbb{R}$ will denote the standard part of x.

We first briefly discuss extremal states corresponding to the generalized Varga traces described in Theorem 2.11.

Let us denote by $\Delta_{\{n_k\}}$ the set of all extremal points in the set of non normal states of the form (2.4).

3.1. Proposition. *The set $\Delta_{\{n_k\}}$ consists of the states*

$$\psi(\{a_n\}) \equiv {}^\circ({}^*a_{n_m})$$

for some $m \in {}^*\mathbb{N}_\infty = {}^*\mathbb{N} - \mathbb{N}$.

Proof. It immediately follows from the fact that extremal states on l^∞ are all of the form

$$\varphi(\{a_n\}) = {}^\circ({}^*a_m), \qquad m \in {}^*\mathbb{N}. \tag{3.1}$$

Since additionally φ must vanish on the set c_0 then m becomes infinitely large. ▮

3.2. *Remark.* Of course, instead of infinitely large numbers one can equivalently use the Stone–Čech compactification $\bar{\mathbb{N}}$ of $\mathbb{N}$ and the isomorphism $l^\infty(\mathbb{N}) \simeq C(\bar{\mathbb{N}})$ given by the Gelfand transform in order to describe extremal states in Proposition 3.1. Namely, they will be given by Dirac measures supported by the set $\bar{\mathbb{N}} - \mathbb{N}$. On the contrary, the classification of ergodic 2-dilation invariant states *does* require NSA (see e.g. the Remark 3.6).

Now we come to the much more difficult problem of classification of two dilation invariant states. First, we remark that in order to prove the existence of such states, Dixmier invoked the amenability of the affine group. As promised, we shall adopt here an alternative point of view, which relies on the use of NSA and related methods (see e.g. [HL], [AFHKL]).

3.3. THEOREM. *The map $\omega \to \varphi_\omega$, $\omega \in {}^*\mathbb{N}_\infty$, defined by*

$$\varphi_\omega(a) \equiv {}^\circ\left(\frac{1}{\omega} \sum_{k=1}^{\omega} {}^*a_{2^k}\right) \tag{3.2}$$

takes values in the convex set of 2*-dilation invariant states over* l^∞.

Proof. Let $b_n \equiv 1/n \sum_{k=1}^{n} a_{2^k}$. Since $\{a_n\}$ is bounded $\{b_n\}$ is also bounded so that $\varphi_\omega(a) = {}^\circ({}^*b_\omega)$ is well defined for all ω. Obviously, φ_ω is a state. It is also 2-dilation invariant since:

$$\varphi_\omega(\{a_{2n}\}) - \varphi_\omega(\{a_n\}) = {}^\circ\left(\frac{1}{\omega}({}^*a_{2^{\omega+1}} - {}^*a_2)\right) = 0. \quad \blacksquare$$

A consequence of this theorem is that an explicit formula for the previously introduced traces can easily be given.

3.4. COROLLARY. *If T is an operator verifying* $\lim_n S_{2n}(T)/S_n(T) = 1$ *and ω is an infinite hypernatural number then*

$$\tau_\omega(A) \equiv {}^\circ\left(\frac{1}{\omega} \sum_{k=1}^{\omega} \frac{{}^*S_{2^k}(A)}{{}^*S_{2^k}(T)}\right) \qquad A \in \mathcal{I}_m(T) \tag{3.3}$$

is one of the singular traces described in Theorem 2.12.

The proof of this corollary follows immediately from Theorems 2.12 and 3.3.

There is a simple generalization of the formula (3.2) which describes 2-dilation invariant states. If $j \in {}^*\mathbb{N}$ and $n \in {}^*\mathbb{N}_\infty$ the map

$$\{a_k\} \to {}^\circ\left(\frac{1}{n} \sum_{i=1}^{n} {}^*a_{j2^i}\right) \tag{3.4}$$

is a 2-dilation invariant state over l^∞ and therefore gives rise to a singular trace.

Since any hypernatural j can be written in a unique way as a product of an odd number and a power of 2, $j = (2m-1)\,2^{k-1}$, we may rewrite the previous states as

$$\varphi_{k,m,n}(a) = {}^\circ\left(\frac{1}{n} \sum_{i=k+1}^{k+n} {}^*a_{(2m-1)\,2^{i-1}}\right) k, m \in {}^*\mathbb{N}, n \in {}^*\mathbb{N}_\infty. \tag{3.5}$$

In the rest of this section we shall study states of the form (3.5) in relation to the problem of ergodicity.

Let $\Delta\colon \mathbb{N} \to \mathbb{N}$ be the multiplication by 2, Δ_* the corresponding morphism on $l^\infty(\mathbb{N})$, $\Delta_*(\{a_n\}) = \{a_{2n}\}$, we shall say that the state φ is

Δ-invariant if $\varphi \circ \Delta_* = \varphi$. We shall give necessary conditions for extremality in the (convex compact) set of Δ-invariant states in terms of NSA.

It is known (see e.g. [E, p. 113]) that the states on $l^\infty(\mathbb{N})$ can be identified with the finitely additive probability measures on $\mathbb{N}$, therefore we shall denote any such a state by μ, and the notation $\mu(A)$ with $A \subset \mathbb{N}$ makes sense.

Moreover extremality of a Δ-invariant state μ can be expressed in terms of ergodicity of μ seen as a measure, i.e. μ is ergodic if, for each $A \subset \mathbb{N}$ such that $\Delta A = A \mu$-a.e., one has $\mu(A) = 0$ or 1.

3.5. *Remark.* Using the Stone–Čech compactification $\bar{\mathbb{N}}$ of $\mathbb{N}$ and the isomorphism $l^\infty(\mathbb{N} \simeq C(\bar{\mathbb{N}})$ given by the Gelfand transform once again we get an identification of the states on $l^\infty(\mathbb{N})$ with the σ-additive probability Radon measures on $\bar{\mathbb{N}}$. On the other hand a transformation on $\mathbb{N}$ extends to a continuous transformation on $\bar{\mathbb{N}}$. We shall denote with $\bar{\mu}, \bar{\Delta}$ the measure and the transformation on $\bar{\mathbb{N}}$ induced by μ and Δ respectively. It turns out that ergodicity of $\bar{\mu}$ is equivalent to ergodicity of the finitely additive measure μ. This equivalence can be shown using well known criteria for ergodicity (see e.g. [Ma]).

3.6. *Remark.* Let us consider the correspondence η: $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $(m, n) \to (2m-1)\, 2^{n-1}$, which is a bijection. It induces an isomorphism: $\eta_*: l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N} \times \mathbb{N})$ given by

$$(\eta_* a)_{m,n} \equiv a_{\eta(m,n)},$$

$\Delta_* \equiv \eta_*^{-1} \Delta \eta_*$ becoming the translation T in the second variable: $\Delta_*(m, n) = (m, n+1) \equiv (m, Tn)$. It might be thought that the isomorphism η_* gives rise to the splitting of the dynamical system $(\bar{\mathbb{N}}, \bar{\Delta})$ in a product of two dynamical systems $(\bar{\mathbb{N}}, id)$ and $(\bar{\mathbb{N}}, \bar{T})$, thus furnishing a standard approach to the considerations we shall make below. Unfortunately this is not true since the spaces $\bar{\mathbb{N}} \times \bar{\mathbb{N}}$ and $\overline{\mathbb{N} \times \mathbb{N}}$ are not homeomorphic (see e.g. [G]). Our idea is to exploit the advantages of NSA, in particular the nice functorial property ${}^*\mathbb{N} \times {}^*\mathbb{N} = {}^*(\mathbb{N} \times \mathbb{N})$.

Let M_Δ denote the set of extremal Δ-invariant (i.e. ergodic) states on $l^\infty(\mathbb{N})$.

3.7. PROPOSITION. *Any $\mu \in M_\Delta$ coincides with one of the states $\varphi_{k,m,n}$ for some $k, m \in {}^*\mathbb{N}$, $n \in {}^*\mathbb{N}_\infty$.*

Proof. Using the fact that μ is a finitely additive probability measure on $\mathbb{N}$ and due to the lifting results for measures (see e.g. [AFHKL, Ch. 3]), for an infinitely large natural number s, there exists a *-finitely additive hyperfinite probability measure $\bar{\mu}$ (a lifting of μ) such that $\bar{\mu}(a) \approx \mu(a)$ for each $a \in l^\infty(\mathbb{N}) \equiv l^\infty$. On the other hand, $\bar{\mu}$ as any *-finitely additive hyperfinite probability measure has the following representation:

$$\text{(i)} \quad \bar{\mu}(c) = \sum_{j=1}^{s} \lambda_j c_j \qquad \text{(ii)} \quad \lambda_j \geqslant 0, \qquad \text{(iii)} \quad \sum_{j=1}^{s} \lambda_j = 1,$$

where $c = (c_1, ..., c_s)$, so that

$$\mu(a) \approx \sum_{j=1}^{s} \lambda_j {}^*a_j \qquad (\forall a \in l^\infty).$$

Since μ is 2-dilation invariant, for all finite n

$$\mu(a) \approx \sum_{j=1}^{s} \lambda_j \frac{1}{n} \sum_{i=1}^{n} {}^*a_{j2^{i-1}} \qquad (\forall a \in l^\infty),$$

and therefore, for each finite dimensional space $F \subset l^\infty$, and each $n \in \mathbb{N}$ we have

$$\alpha_{n, {}^*F} \equiv \sup_{a \in {}^*F, \, \|a\| \leqslant 1} \; \sup_{1 \leqslant p \leqslant s} \left\{ \left| {}^*\mu(a) - \sum_{j=1}^{s} \lambda_j \frac{1}{n} \sum_{i=1}^{n} a_{j2^{i-1}} \right| + \left| {}^*\varphi_{k, m, n}(a) - \frac{1}{n} \sum_{i=1}^{n} a_{p2^{i-1}} \right| \right\} \approx 0 \tag{3.6}$$

with m and k given by $p = (2m-1)\, 2^{k-1}$. By saturation, there is a hyperfinite dimensional space F, $l^\infty \subset F \subset {}^*l^\infty$ and a number $n \in {}^*\mathbb{N}_\infty$ such that (3.6) remains valid. This means that

$${}^*\mu(a) \approx \sum_{j=1}^{s} \lambda_j {}^*\varphi_{m, k, n}(a) \qquad (j = (2m-1)\, 2^{k-1}) \tag{3.7}$$

for all $a \in l^\infty$ and immediately implies that the set of all T-invariant states coincides with the closed convex hull $\overline{\text{co}}\, M$ of the set $M \equiv \{\mu_{k, n} \mid k \in {}^*\mathbb{N}, n \in {}^*\mathbb{N}_\infty\}$.

Finally, we show that M is closed which, according to [E, p. 708], would imply the inclusion $M_\Delta \subset M$. Consider a directed set $\Gamma \subset {}^*\mathbb{N}^3$ and assume that $\varphi_{k, m, n} \xrightarrow{\Gamma} \mu$. Clearly, for every finite dimensional $E \subset l^\infty$ there exists a subsequence $\{(k_i, m_i, n_i)\} \in \Gamma$ such that

$${}^*\varphi_{k_i, m_i, n_i}(a) \approx {}^*\mu(a) \quad \text{for some} \quad i \in {}^*\mathbb{N}_\infty \quad \text{and for all} \quad a \in {}^*E, \|a\| \leqslant 1.$$

If μ is Δ invariant then by saturation the latter relation holds for a certain hyperfinite dimensional $E \supset l^\infty$ with $k \in {}^*\mathbb{N}$ and $n \in {}^*\mathbb{N}_\infty$. This concludes the proof. ▮

3.8. COROLLARY. *The problems of description of extremal 2-dilation invariant and extremal translation invariant states are equivalent.*

Proof. Recalling the map η_* given by $(\eta_* a)_{m,n} = a_{\eta(m,n)}$ where $\eta_*(m,n) = (2m-1)\,2^{n-1}$ and applying the proposition just proved we conclude that for each fixed m the states $\varphi_{k,m,n} \circ \eta_*$ coincide with the translation invariant states $\mu_{k,n}$ on l^∞ defined by

$$\mu_{k,n}(a) \equiv {}^\circ\!\left(\frac{1}{n} \sum_{i=k+1}^{k+n} {}^*a_i\right),$$

so that any $\mu \in \eta_*^{-1}(M_\Delta)$ is contained in one of the sets $\{\delta_m \otimes \nu \mid \nu \in M_T \cap M\}$ where δ_m is given by $\delta_m(a) \equiv {}^\circ({}^*a_m)$, $m \in {}^*\mathbb{N}$, M_T stands for the set of extremal translation invariant states on l^∞ and $M \equiv \{\mu_{k,n} \mid k \in {}^*\mathbb{N}, n \in {}^*\mathbb{N}_\infty\}$. On the other hand it is known (see [KM], [L], or also [AGPS1] where the result was proved independently) that $M_T \subset M$, which completes the proof. ▮

3.9. *Remark.* Of course, the representation given in Proposition 3.7 is not unique due to Corollary 3.8 and the following trivial

3.10. PROPOSITION. *If $(k-l)/n \approx 0$ and $p-n/n \approx 0$, then $\mu_{k,n} = \mu_{l,p}$.*

Now we formulate the main result in this section more precisely.

3.11. THEOREM. *If μ is an extremal 2-dilation invariant state on l^∞ then $\mu = \varphi_{k,m,n}$ (with $\varphi_{k,m,n}$ defined by (3.5)) for some $m \in {}^*\mathbb{N}$ and infinitely large hypernaturals k and n such that $n/k \approx 0$.*

Proof. By virtue of Corollary 3.8 it suffices to show that if $\nu \in M_T$ then $\nu = \mu_{k,n}$ for some $k, n \in {}^*\mathbb{N}_\infty$. We first prove that $k \in {}^*\mathbb{N}_\infty$. Suppose it is not the case and k is finite. Without loss of generality we can assume $k = 1$, and, due to Proposition 3.10, $n = 2m$. If we show that $\mu_{1,n} \neq \mu_{1,m}$ then the representation $\mu_{1,n} = \frac{1}{2}(\mu_{1,m} + \mu_{m,m})$ implies $\mu_{1,n}$ is not extremal.

For $b \equiv m2^{-2p}$, choose $p \in {}^*\mathbb{N}_\infty$ such that $0 < {}^\circ b < \infty$ and define a sequence $\{c_j\}$ of natural numbers by putting $c_j \equiv [2^j\,{}^\circ b]$ ($[\cdot]$ denotes the integer part). Since

$$c_j \leqslant \tfrac{1}{2}[2^{j+1}\,{}^\circ b] = \tfrac{1}{2} c_{j+1},$$

one gets $c_{j+1} \geqslant 2c_j$. At the same time,

$$^{\circ}\left(\frac{{}^*c_{2p}-m}{m}\right) = {}^{\circ}\left(\frac{{}^*c_{2p}-m}{2^{2p}}\right) {}^{\circ}\left(\frac{2^{2p}}{m}\right) = {}^{\circ}\left(\frac{[2^{2p\circ}b]}{2^{2p}} - b\right)\frac{1}{{}^{\circ}b}$$

$$\leqslant \frac{1}{{}^{\circ}b}\,{}^{\circ}({}^{\circ}b - b) = 0.$$

By the same reason, ${}^*c_{2p+1} - n/n \approx 0$. Applying Proposition 3.10 once again we obtain

$$\begin{aligned} \frac{1}{m}\sum_{i=1}^{m} {}^*a_i &\approx \frac{1}{{}^*c_{2p}}\sum_{i=1}^{{}^*c_{2p}} {}^*a_i; \\ \frac{1}{n}\sum_{i=1}^{n} {}^*a_i &\approx \frac{1}{{}^*c_{2p+1}}\sum_{i=1}^{{}^*c_{2p+1}} {}^*a_i \qquad (\forall a \in l^\infty). \end{aligned} \tag{3.8}$$

Now we introduce a set $B = \bigcup_{q=1}^{\infty} [c_{2q-1}, c_{2q}] \subset \mathbb{N}$ and a sequence $\chi = \{\chi_i\}$ where $\chi_i = 1$ for $i \in B$ and 0 otherwise. By (3.8),

$$\begin{aligned} \mu_{1,n}(B) &= {}^{\circ}\left(\frac{1}{{}^*c_{2p+1}}\sum_{i=1}^{{}^*c_{2p+1}} {}^*\chi_i\right) \\ &= {}^{\circ}\left(\frac{{}^*c_{2p}}{{}^*c_{2p+1}}\right) {}^{\circ}\left(\frac{1}{{}^*c_{2p}}\sum_{i=1}^{{}^*c_{2p}} {}^*\chi_i + \frac{1}{{}^*c_{2p}}\sum_{i={}^*c_{2p}+1}^{{}^*c_{2p+1}} {}^*\chi_i\right) \\ &= {}^{\circ}\left(\frac{{}^*c_{2p}}{{}^*c_{2p+1}}\right)\mu_{1,m}(B) \leqslant \frac{1}{2}\mu_{1,m}(B). \end{aligned}$$

It remains to prove that $\mu_{1,m}(B) \neq 0$. In order to see this, let us observe that ${}^*\chi_i = 1$ for $i \in [{}^*c_{2p-1}, {}^*c_{2p}]$ and that ${}^*c_{2p} - {}^*c_{2p-1} \geqslant \frac{1}{2}{}^*c_{2p}$. Hence, $\sharp\{i \in [1, {}^*c_{2p}] \mid {}^*\chi_i = 1\} \geqslant \frac{1}{2}{}^*c_{2p}$ (where $\sharp$ means cardinality) and

$$\mu_{1,m}(B) = {}^{\circ}\left(\frac{1}{{}^*c_{2p}}\sum_{i=1}^{{}^*c_{2p}} {}^*\chi_i\right) \geqslant {}^{\circ}\left(\frac{1}{{}^*c_{2p}}\left(\frac{1}{2}\,{}^*c_{2p}\right)\right) = \frac{1}{2},$$

so that $\mu_{1,n}$ is not extremal.

We continue the proof assuming $k \in {}^*\mathbb{N}_\infty$ and $n/k \not\approx 0$ or, equivalently, ${}^{\circ}(k/n) < \infty$. We have to show that $\mu_{k,n}$ is not extremal. First we notice that

$$0 \leqslant \frac{-k + [\sqrt{k}+1]^2}{n} \leqslant \frac{2\sqrt{k}+1}{n} \approx \frac{2\sqrt{k}}{n} = \frac{2k}{n\sqrt{k}} \approx 0$$

and analogously,

$$\frac{-k+[\sqrt{k}]^2}{n}\approx 0, \qquad \frac{k+n-[\sqrt{k+n}]^2}{n}\approx 0, \qquad \frac{k+n-[\sqrt{k+n}+1]^2}{n}\approx 0.$$

Applying now Proposition 3.10, one can assume that $k=(2r)^2$, $k+n=(2s)^2$.

Let us introduce a set $C\equiv\bigcup_{i=1}^{\infty}[(2i-1)^2,(2i)^2]$ and observe that

$$\mu_{k,n}(C\triangle TC)\leqslant {}^{\circ}\!\left(\frac{\sharp\{i\mid i^2\in[k,k+n]\}}{n}\right)={}^{\circ}\!\left(\frac{\sharp\{i\mid 2r\leqslant i\leqslant 2s\}}{(2s)^2-(2r)^2}\right)=0.$$

Extremality of μ would imply, therefore, that $\mu_{k,n}(C)$ should have been equal to 0 or 1. On the other hand,

$$\mu_{k,n}(C)={}^{\circ}\!\left(\frac{1}{n}\sum_{i=r+1}^{s}((2i)^2-(2i-1)^2)\right)={}^{\circ}\!\left(\frac{(4s+1+4r+3)(r-s)}{8r^2-8s^2}\right)=\frac{1}{2}.$$

This contradiction implies the result. ∎

3.12. *Remark.* The necessary conditions in Theorem 3.11 are not sufficient. To see this, we again recall Corollary 3.8 and consider a state $\mu\equiv\mu_{4^p-n,2n}$ for arbitrary $p,n\in{}^*\mathbb{N}_\infty$. If $n\geqslant 2^{2p-1}$, then non extremality of μ follows from the theorem. Otherwise, we may introduce the set $A\equiv\bigcup_{q=1}^{\infty}(2^{2q-1},2^{2q}]$ which is easily seen to be μ-a.e. T-invariant, but $\mu(A)=\frac{1}{2}$.

Now we give a corollary which relates the results of this section with the description of the generalized Dixmier traces (for the proof cf. [AGPS1]).

3.13. Corollary. *Let τ be a generalized Dixmier trace on the ideal $\mathcal{I}_m(T)$ (see Theorem 2.12). Then τ is in the closure of the convex hull of the family*

$$\left\{\tau_{k,m,n}\mid m\in{}^*\mathbb{N},\, k,n\in{}^*\mathbb{N}_\infty,\, \left(\frac{n}{k}\right)\approx 0\right\},$$

where $\tau_{k,m,n}=\tau_{\varphi_{k,m,n}}$ is the trace associated with the state $\varphi_{k,m,n}$ given by (3.5) via formula (2.5) on the same domain $\mathcal{I}_m(T)$.

3.14. *Remark.* The states $\mu_{k,n}$ can be looked upon intuitively as averages on intervals of the set ${}^*\mathbb{N}$. This suggests to call *ergodic* all the intervals associated with ergodic states.

Then it is easy to show that if the interval I is ergodic and a subinterval J is such that $|J|/|I|\not\approx 0$ (where $|I|$ denotes the length of I), then $\mu_I=\mu_J$.

A sketch of the proof is the following: let $I = I_0 \cup J \cup I_1$ be a partition of I into subintervals. It turns out that

$$^{\circ}\left(\frac{|I_0|}{|I|}\right)\mu_{I_0} + {}^{\circ}\left(\frac{|J|}{|I|}\right)\mu_J + {}^{\circ}\left(\frac{|I_1|}{|I|}\right)\mu_{I_1} = \mu_I,$$

hence, by the ergodicity of I, $\mu_I = \mu_J$.

4. A Computational Example

We shall now discuss some advantages of representing singular traces by means of NSA.

A remarkable advantage lies, in our opinion, in the increased computability of the value of a singular trace on a given operator when such a trace is parametrized by some infinite number.

In what follows, we shall work out an example in which we explicitly calculate the value of the Dixmier trace of an operator, even though it depends on the non-standard parameter.

To this aim we shall make use of formula (3.3) choosing a compact operator T such that $S_n(T) = \log n$. The choice of "summing" logarithmic divergences has extensively been used by Connes in some applications to non-commutative geometry [C].

Let $q \geqslant 1$ be a fixed natural number, we consider a positive compact operator A_q whose sequence of eigenvalues $(\lambda_n \mid n = 3, 4...)$ is defined in the following way: let $(n_k \mid k = 0, 1, ...)$ be an unbounded increasing sequence of natural numbers (with $n_0 \equiv 1$) whose explicit dependence on q will be given below. For $n \in (2^{n_k}, 2^{n_{k+1}}]$, we define

$$\lambda_n := \frac{n_{k+1} - n_k}{2^{n_{k+1}} - 2^{n_k}} \tag{4.1}$$

For $m \geqslant 2$ we consider the sum $\sigma_{2^m} := \sum_{j=3}^{2^m} \lambda_j$.

Let $n_k < m \leqslant n_{k+1}$, then we have

$$\sigma_{2^m} = n_k + \frac{2^m - 2^{n_k}}{2^{n_{k+1}} - 2^{n_k}} \cdot (n_{k+1} - n_k) - 1 \tag{4.2}$$

since

$$\sigma_{2^m} = \sum_{r=0}^{k-1} \sum_{j=2^{n_r}+1}^{2^{n_{r+1}}} \lambda_j + \sum_{j=2^{n_k}+1}^{2^m} \lambda_j.$$

Now let $p>1$ and hence $n_s<p\leqslant n_{s+1}$ for some s, we have

$$\frac{1}{p}\sum_{m=1}^{p}\frac{\sigma_{2^m}}{\log 2^m}=\frac{1}{\log 2}\cdot\frac{1}{p}\left(\sum_{k=0}^{s-1}\sum_{m=n_k+1}^{n_{k+1}}\frac{\sigma_{2^m}}{m}+\sum_{m=n_s+1}^{p}\frac{\sigma_{2^m}}{m}\right). \tag{4.3}$$

We now proceed to estimate the sums appearing on the r.h.s. of (4.3).

By means of (4.2) we have

$$\sum_{m=n_k+1}^{n_{k+1}}\frac{\sigma_{2^m}}{m}=\left[n_k-1-\frac{2^{n_k}}{2^{n_{k+1}}-2^{n_k}}(n_{k+1}-n_k)\right]\sum_{m=n_k+1}^{n_{k+1}}\frac{1}{m}+\frac{n_{k+1}-n_k}{2^{n_{k+1}}-2^{n_k}}\cdot\sum_{m=n_k+1}^{n_{k+1}}\frac{2^m}{m}. \tag{4.4}$$

We notice that the following equalities hold

$$\sum_{m=n_k+1}^{n_{k+1}}\frac{1}{m}=\log\frac{n_{k+1}}{n_k}+O\left(\frac{1}{n_k}-\frac{1}{n_{k-1}}\right) \tag{4.5a}$$

$$\sum_{m=n_k+1}^{n_{k+1}}\frac{2^m}{m}=\frac{1}{\log 2}\left(\frac{2^{n_{k+1}}}{n_{k+1}}-\frac{2^{n_k}}{n_k}\right)\left(1+O\left(\frac{1}{n_k}\right)\right) \tag{4.5b}$$

from which it follows

$$\sum_{m=n_k+1}^{n_{k+1}}\frac{\sigma_{2^m}}{m}=\left[n_k\log\frac{n_{k+1}}{n_k}+O\left(1-\frac{n_k}{n_{k+1}}\right)\right]\left[1+O\left(\frac{1}{n_k}\right)\right] \tag{4.6}$$

under the assumption $O(1/n_k)\geqslant O(2^{n_k}/2^{n_{k+1}})$.

To verify such a condition we fix the initial sequence $(n_k \mid k=0, 1, ...)$ to be of the form $n_k:=2^{kq}$, where $q\in\mathbb{N}$.

Formula (4.6) takes then the form:

$$\sum_{m=n_k+1}^{n_{k+1}}\frac{\sigma_{2^m}}{m}=[2^{kq}q\log 2+O(1)][1+O(2^{-kq})]. \tag{4.7}$$

Therefore we obtain

$$\frac{1}{p\log 2}\sum_{k=0}^{s-1}\sum_{m=n_k+1}^{n_{k+1}}\frac{\sigma_{2^m}}{m}=\frac{1}{q\log 2}\left(p\log 2\,\frac{2^{sq}-1}{2^q-1}+O(s)\right) \tag{4.8}$$

Now, by definition, taking $p \in {}^*\mathbb{N}_\infty$, we have

$$\begin{aligned}
\tau_p^{\text{Dix}}(A_q) &= {}^\circ\left(\frac{1}{p}\sum_{m-1}^{p}{}^*\left(\frac{\sigma_{2^m}}{\log 2^m}\right)\right) \\
&= {}^\circ\left(\frac{1}{p\log 2}\sum_{k=0}^{s-1}\sum_{m=n_k+1}^{n_{k+1}}{}^*\left(\frac{\sigma_{2^m}}{m}\right)\right) \\
&\quad + {}^\circ\left(\frac{1}{p\log 2}{}^*\left(\frac{\sigma_{2^m}}{m}\right)\right) \\
&= {}^\circ\left(\frac{2^{sq}}{p}\right)\frac{q}{2^q-1} + {}^\circ\left(\frac{1}{p\log 2}\sum_{m=n_s+1}^{p}{}^*\left(\frac{\sigma_{2^m}}{m}\right)\right) \qquad (4.9)
\end{aligned}$$

where, in the last equality, we have used (4.8) and the fact that ${}^\circ(s/p) = 0$.

To end the computation of the Dixmier trace of A_q we need to evaluate the second term on the r.h.s. of (4.9).

We have, for $p \in \mathbb{N}$,

$$\begin{aligned}
\frac{1}{p\log 2}\sum_{m=n_s+1}^{p}\frac{\sigma_{2^m}}{m} &= \frac{1}{\log 2}\left(\frac{n_s}{p}\right)\left(1 + O\left(\frac{1}{n_s}\right)\right)\left(\log\frac{P}{n_s} + O\left(\frac{1}{n_s}\right)\right) \\
&\quad + \left(\frac{1}{p(\log 2)^2}\left(\frac{n_{s+1}-n_s}{2^{n_{s+1}}-2^{n_s}}\right)\left(\frac{2^p}{p} - \frac{2^{n_s}}{n_s}\right)\left(1 + O\left(\frac{1}{n_s}\right)\right)\right). \qquad (4.10)
\end{aligned}$$

Hence, by estimates similar to the previous ones, and taking $p \in {}^*\mathbb{N}_\infty$ we obtain

$$ {}^\circ\left(\frac{1}{p\log 2}\sum_{m=n_s+1}^{p}{}^*\left(\frac{\sigma_{2^m}}{m}\right)\right) = \frac{1}{\log 2}\,{}^\circ\left(\frac{2^{sq}}{p}\right)\log\,{}^\circ\left(\frac{p}{2^{sq}}\right). \qquad (4.11)$$

From (4.9) and (4.11) it follows

$$\tau_p^{\text{Dix}}(A_q) = t\left(\frac{q}{2^q-1} - \log_2(t)\right) \qquad (4.12)$$

where $t := {}^\circ(2^{sq}/p)$.

In general t can take any value in the interval $[2^{-q}, 1]$. In particular, in the case $p = 2^{sq+r}$, $1 \leqslant r \leqslant q$, formula (4.12) becomes

$$\tau_p^{\text{Dix}}(A_q) = 2^{-r}\left(\frac{q}{2^q-1} + r\right) \qquad (4.13)$$

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