The Stackelberg Minimum Spanning Tree Game

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Stackelberg Game

- 2 players: leader and follower
- The leader moves first, then the follower moves
- The follower optimizes his objective function
  - …knowing the leader's move
- The leader optimizes his objective function
  - …by anticipating the optimal response of the follower
- **Our goal**: to find a good strategy for the leader

Setting

- We have a graph $G=(V,E)$, with $E=R\cup B$
- each $e\in R$, has a fixed positive cost $c(e)$
- Leader owns B, and has to set a price $p(e)$ for each $e\in B$
- function c and function p define a weight function $w: E \rightarrow \mathbf{R}^+$
- the follower buys an MST T of G (w.r.t. to w)
- Leader's revenue of T is:

$$\sum_{e\in E(T)\cap B} p(e)$$

goal: find prices in order to maximize the revenue

There is a trade-off:

- Leader should not put too a high price on the edges
  - otherwise the follower will not buy them
- But the leader needs to put sufficiently high prices to optimize revenue

Minimum Spanning Tree problem

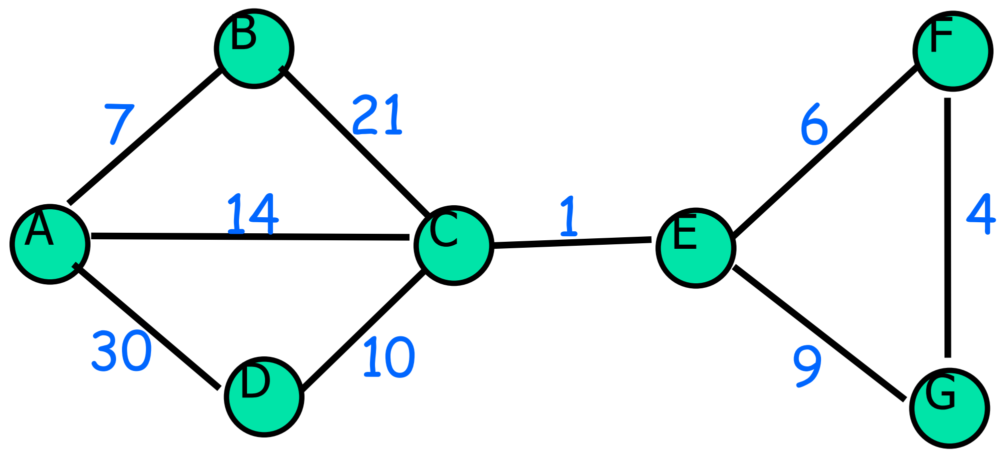
Minimum Spanning Tree (MST) problem

- Input:
  - undirected weighted graph $G=(V,E,w)$
- Solution:
  - a spanning tree of G, i.e. a tree $T=(V,F)$ with $F \subseteq E$
- Measure (to minimize):
  - Total weight of T: $\sum_{e \in F} w(e)$

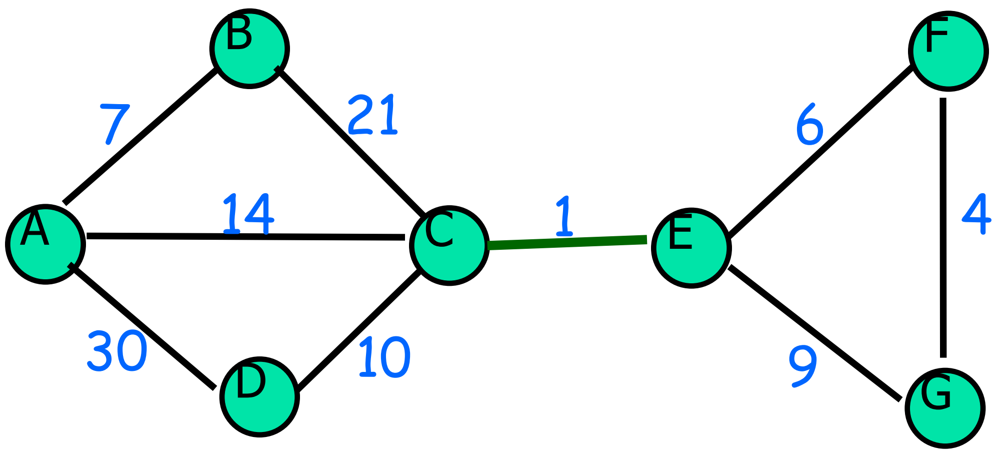
A famous algorithm: Kruskal's algorithm (1956)

- Start with an empty tree T
- consider the edges of G in non-decreasing order:
  - add the current edge e to T iff e does not form a cycle with the previous selected edges

Example



Example



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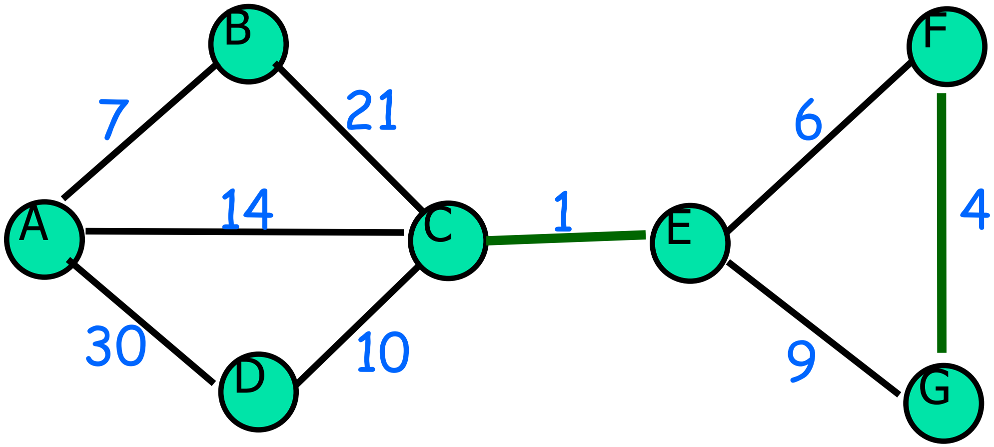

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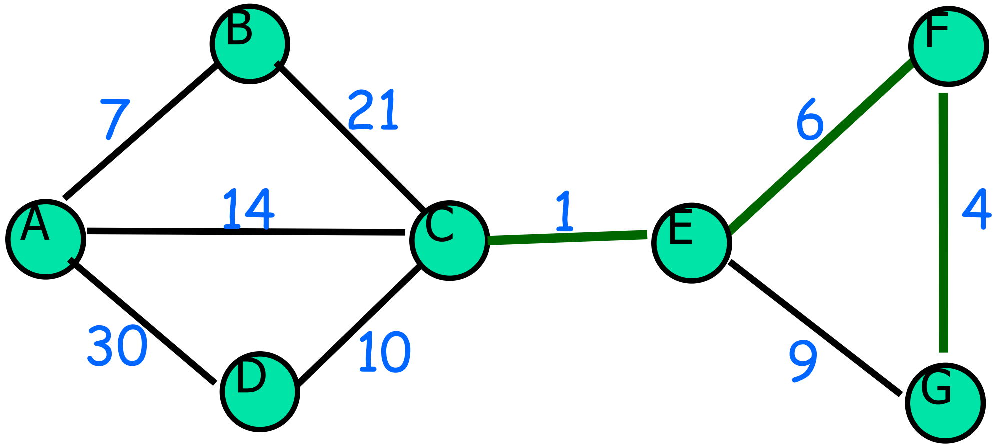

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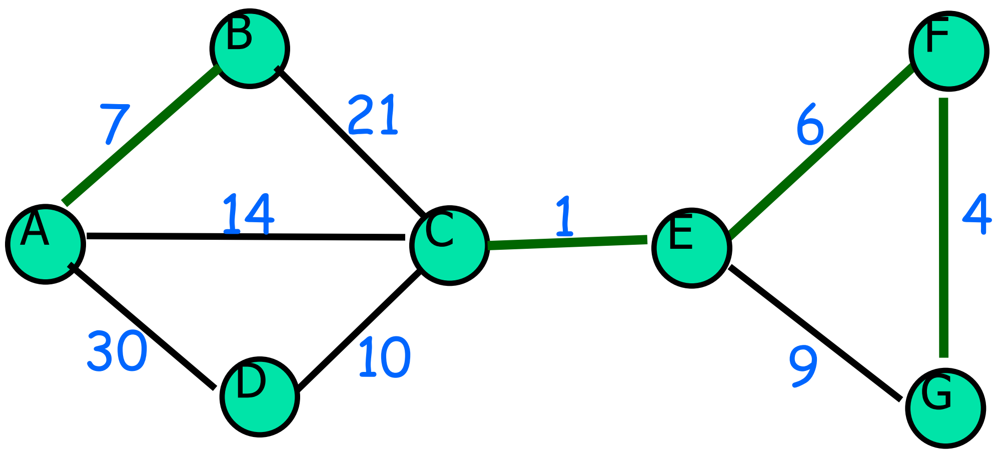

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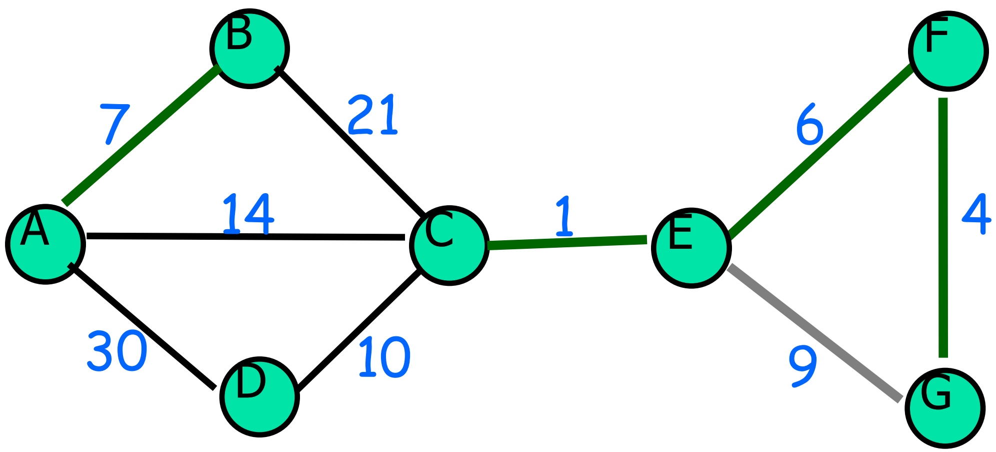

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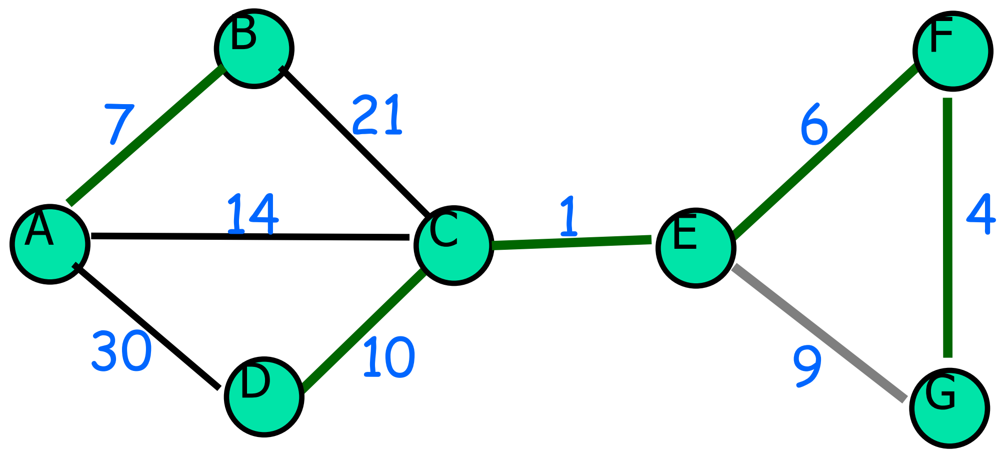

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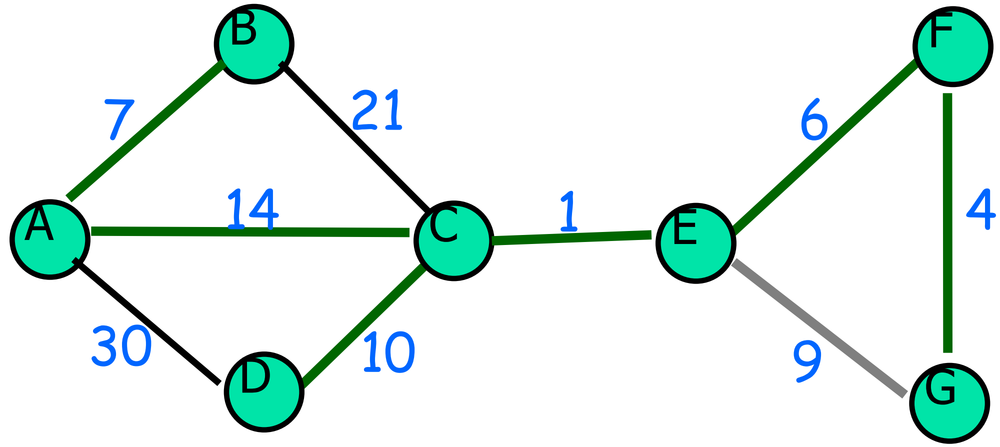

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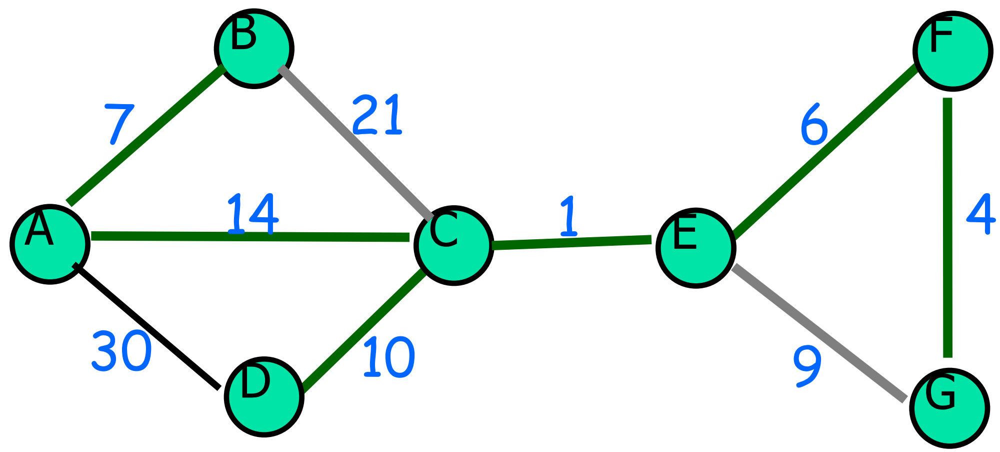

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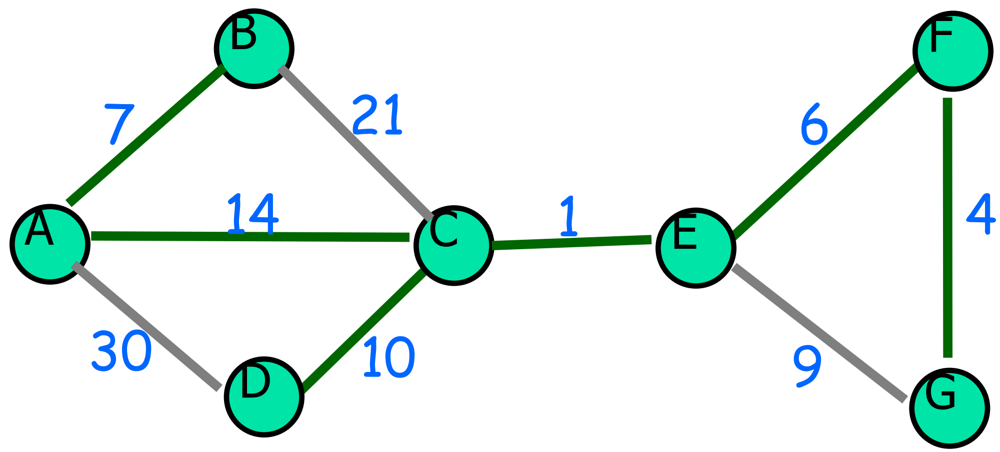

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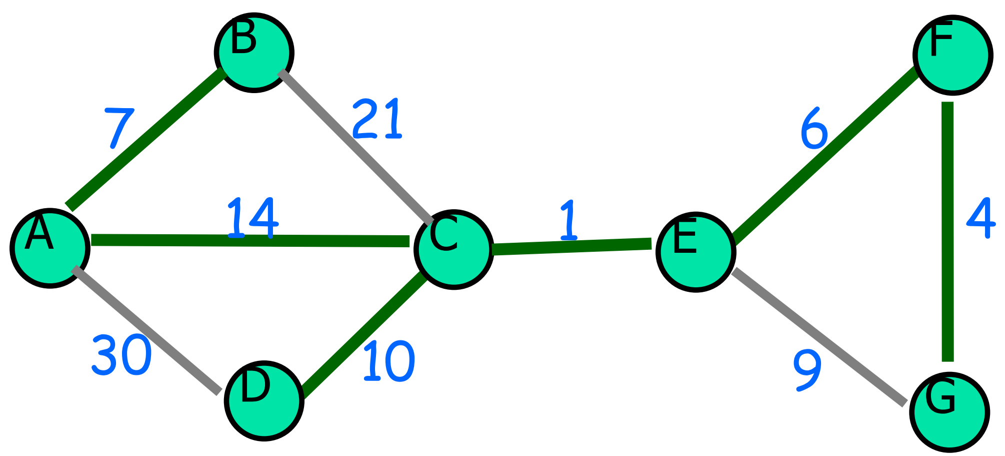
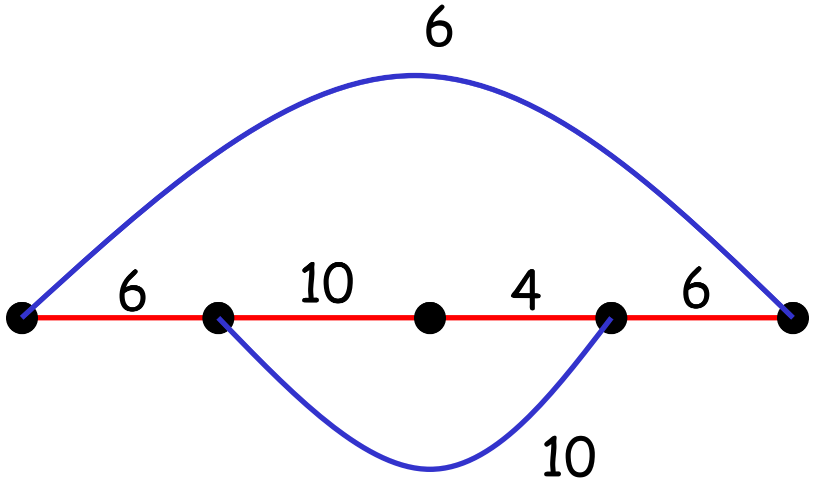


…turning to the Stackelberg MST Game

Example



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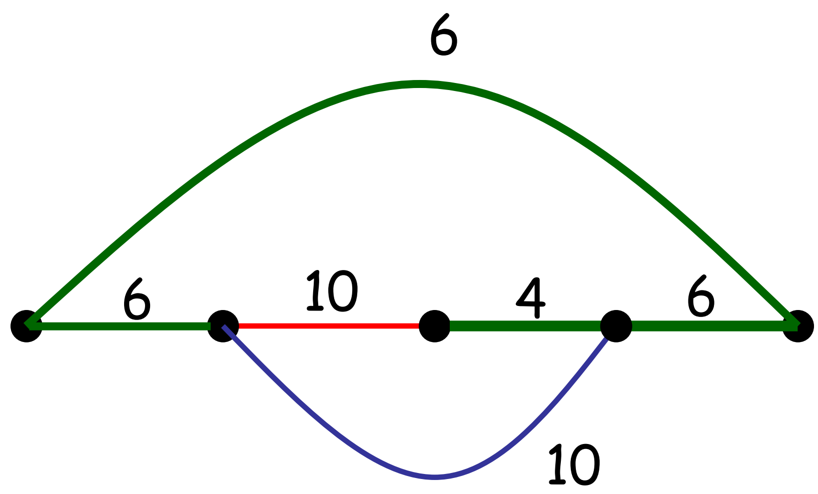
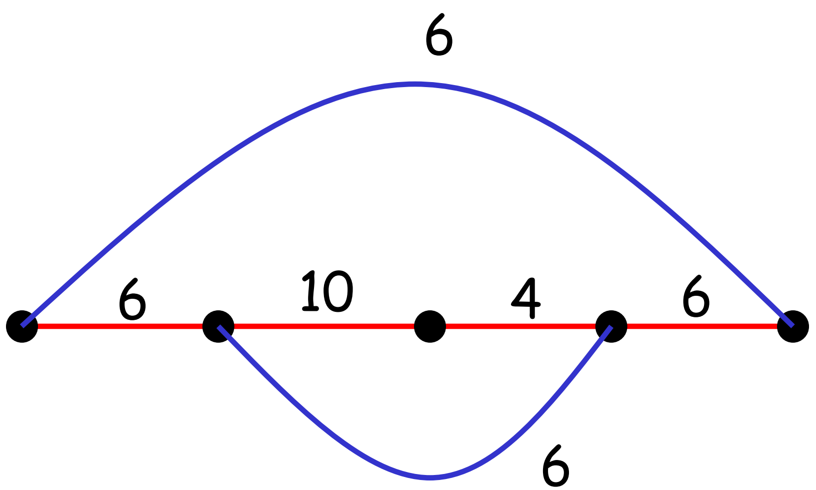


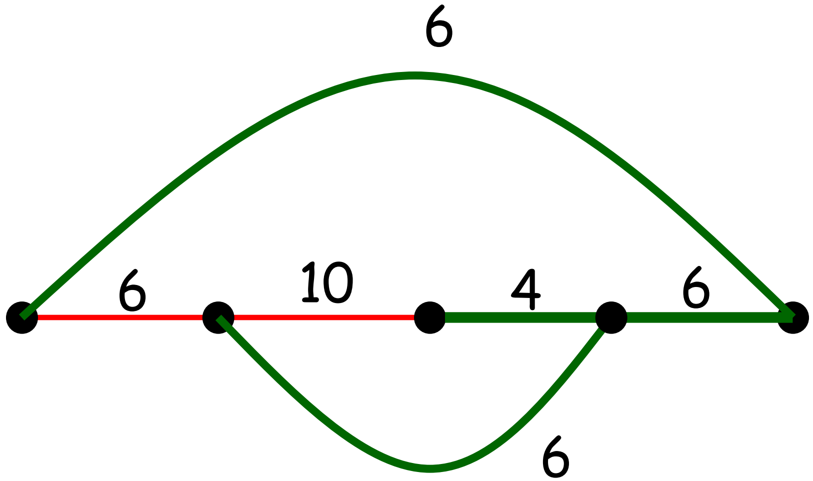
The revenue is 6

Example



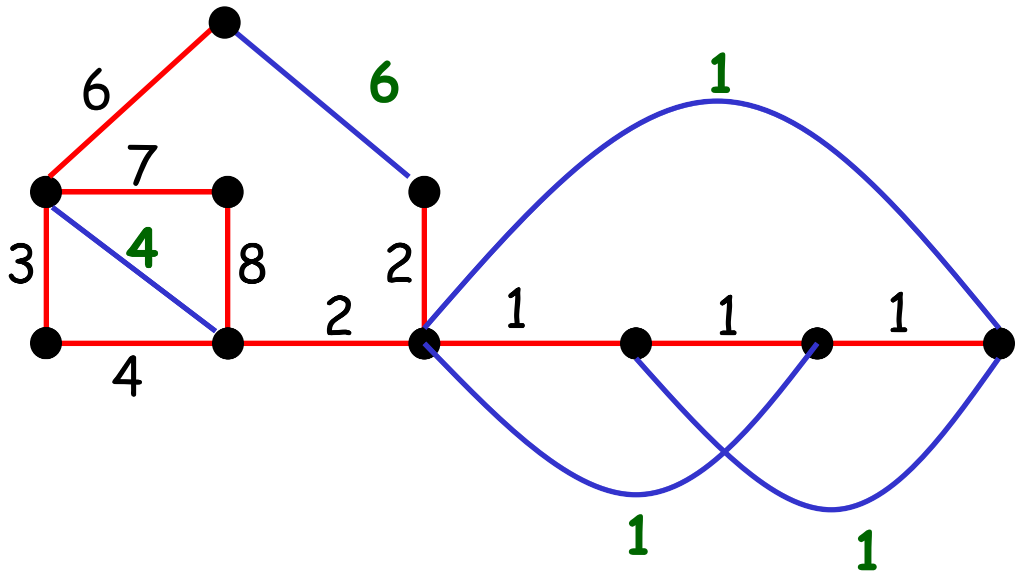
A better pricing…

Example

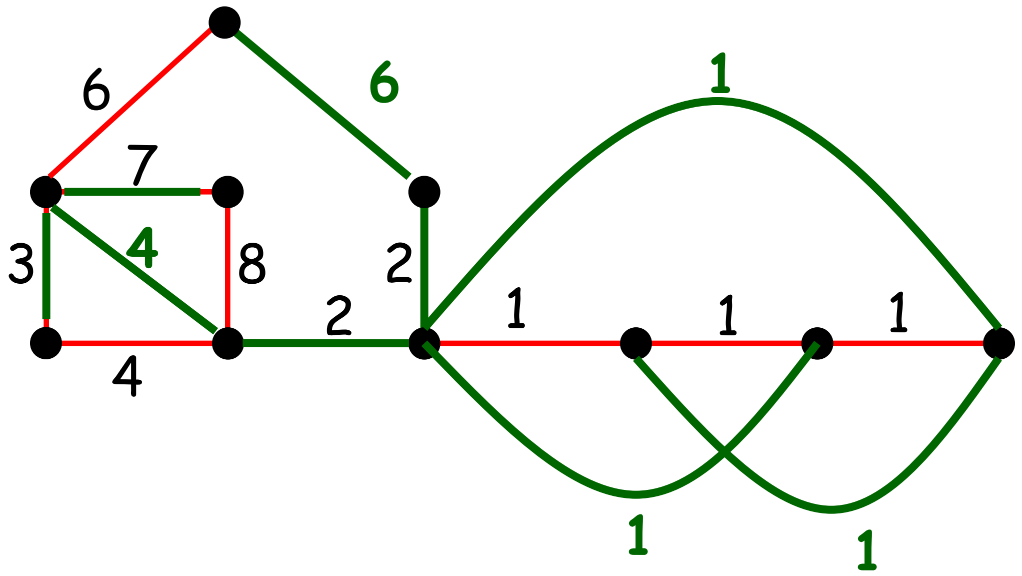


...with revenue 12

One more example

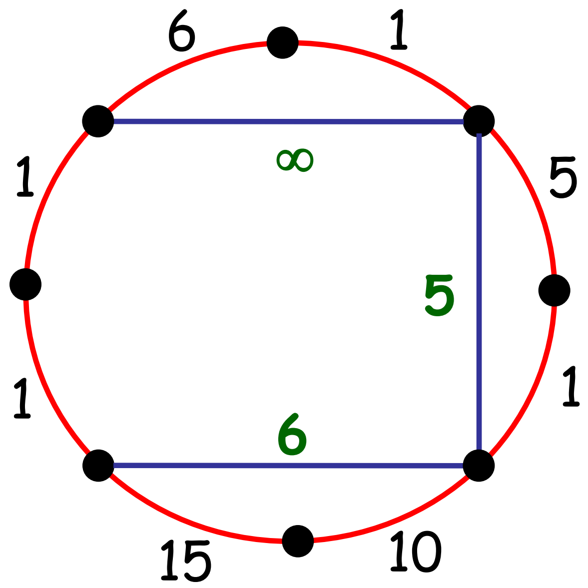


One more example



The revenue is 13

One more example



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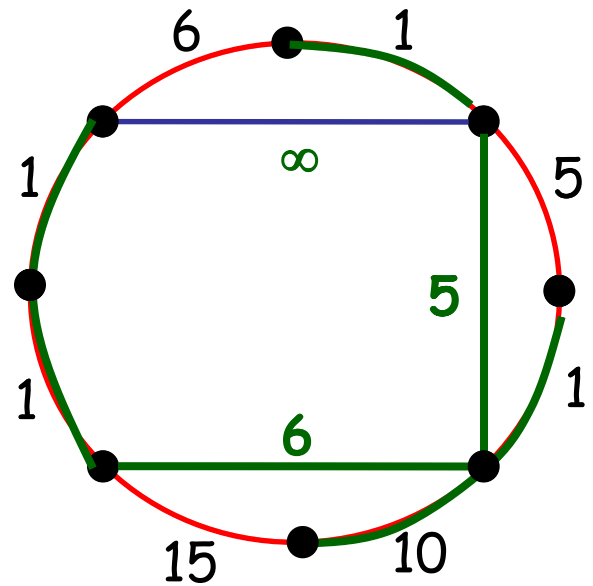


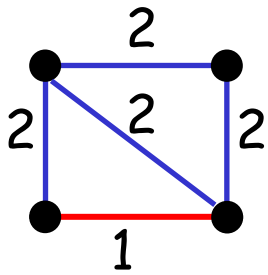
The revenue is 11

Assumptions

- G contains a spanning tree whose edges are all red
  - Otherwise the optimal revenue is unbounded
- Among all edges of the same weight, blue edges are always preferred to red edges
  - If we can get revenue r with this assumption, then we can get revenue $r-\varepsilon$, for any $\varepsilon>0$
  - by decreasing prices suitably

The revenue of the leader depends on the price function p and not on the particular MST picked by the follower

- Let $w_1 < w_2 < \ldots < w_h$ be the different edge weights
- The greedy(Kruskal's) algorithm works in h phases
- In its phase i, it considers:
  - all blue edges of weight w_i (if any)
  - Then, all red edges of weight w_i (if any)



- Number of selected blue edges of weight w_i does not depend on the order on which red and blue edges are considered!
- This implies…

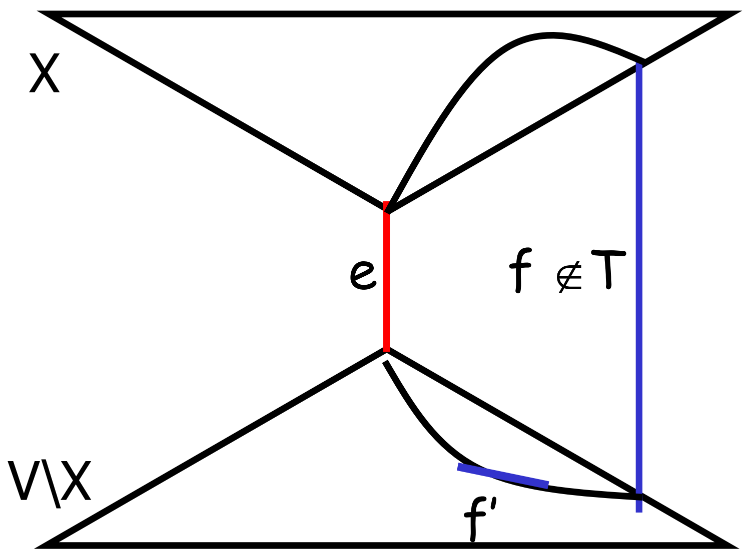
Lemma 1

In every optimal price function, the prices assigned to blue edges appearing in some MST belong to the set $\{c(e): e \in R\}$

Lemma 2

Let p be an optimal price function and T be the corresponding MST. Suppose that there exists a red edge e in T and a blue edge f not in T such that e belongs to the unique cycle C in T+f. Then there exists a blue edge f' distinct to f in C such that $c(e) < p(f') \leq p(f)$

proof



$c(e) < p(f)$

f': the heaviest blue edge in C (different to f)

$p(f') \leq p(f)$

if $p(f') \leq c(e)$...

...$p(f)=c(e)$ will imply a greater revenue

Theorem

The Stackelberg MST game is NP-hard, even when $c(e) \in \{1,2\}$ for all $e \in R$

reduction from Set cover problem

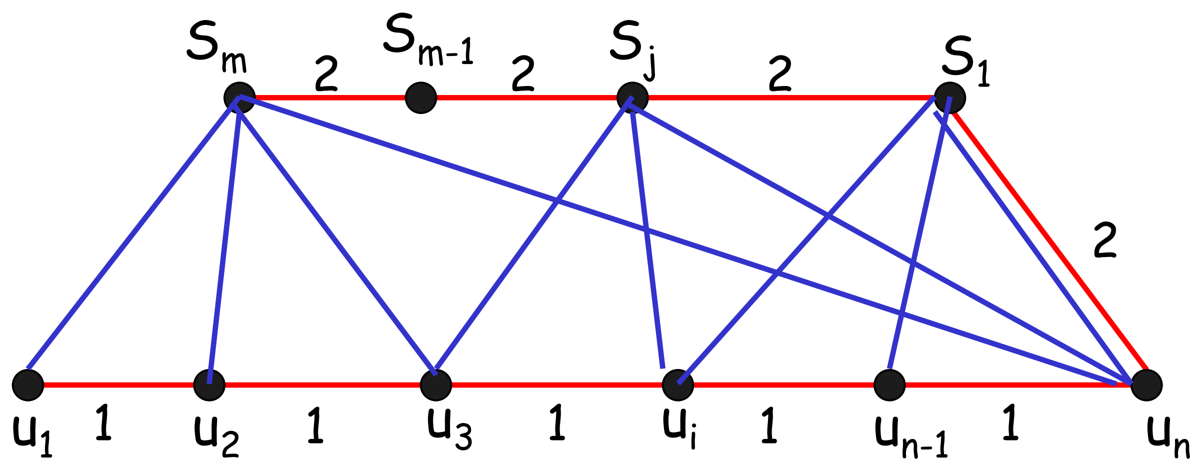
minimum Set Cover Problem

- INPUT:
  - Set of objects $U=\{u_1,\ldots,u_n\}$
  - $\mathcal{S}=\{S_1,\ldots,S_m\}$, $S_j \subseteq U$
- OUTPUT:
  - A cover $C \subseteq \mathcal{S}$ whose union is U and $|C|$ is minimized

$U=\{u_1,\dots,u_n\}$ $\mathcal{S}=\{S_1,\dots,S_m\}$

w.l.o.g. we assume:
$u_n \in S_j$, for every j

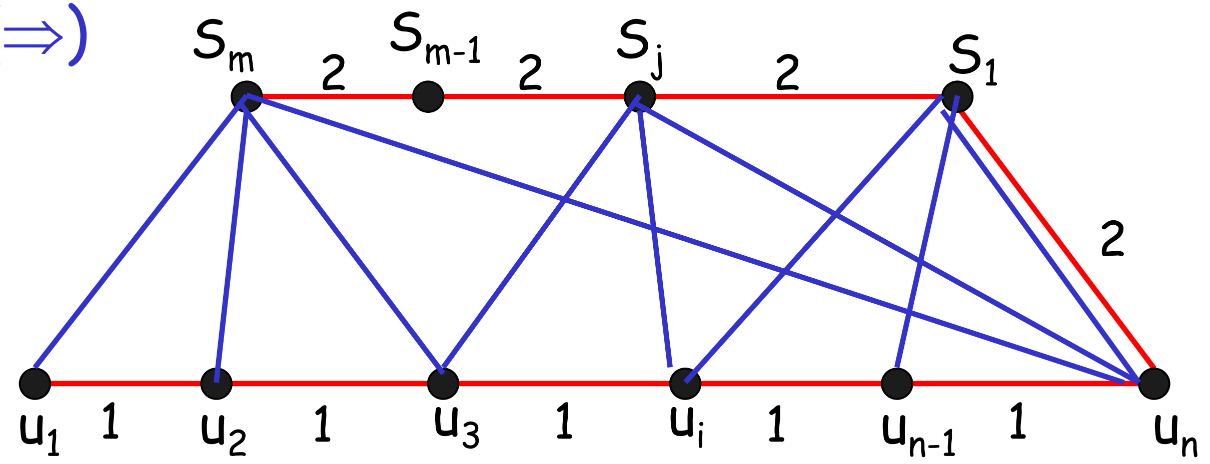
We define the following graph:



a blue edge
(u_i,S_j) iff $u_i \in S_j$

Claim:
$(U,\mathcal{S})$ has a cover of size at most t $\Leftrightarrow$
maximum revenue $r^* \geq n+t-1+2(m-t)= n+2m-t-1$

(⇒)



a blue edge (u_i,S_j) iff $u_i \in S_j$

We define the price function as follows:

For every blue edge $e=(u_i,S_j)$,
$p(e)=1$ if S_j is in the cover, 2 otherwise

➡ revenue $r= n+t-1+2(m-t)$

(⇐)

p: optimal price function p:B→{1,2,∞} such that the corresponding MST T minimizes the number of red edges

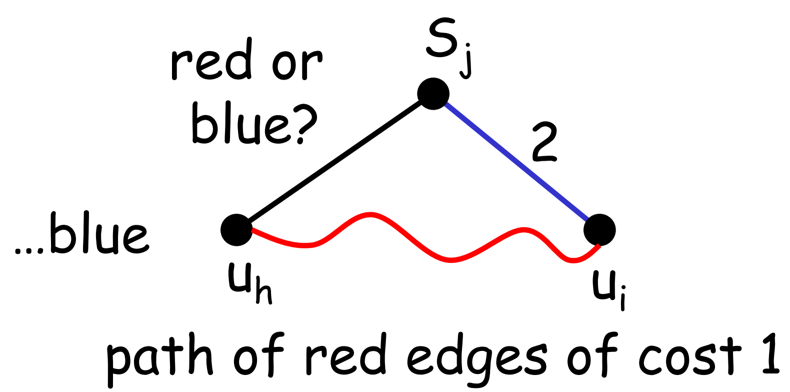
We'll show that:

1. T has blue edges only
2. There exists a cover of size at most t

Remark:

If all red edges in T have cost 1, then for every blue edge $e=(u_i,S_j)$ in T with price 2, we have that S_j is a leaf in T

by contradiction…



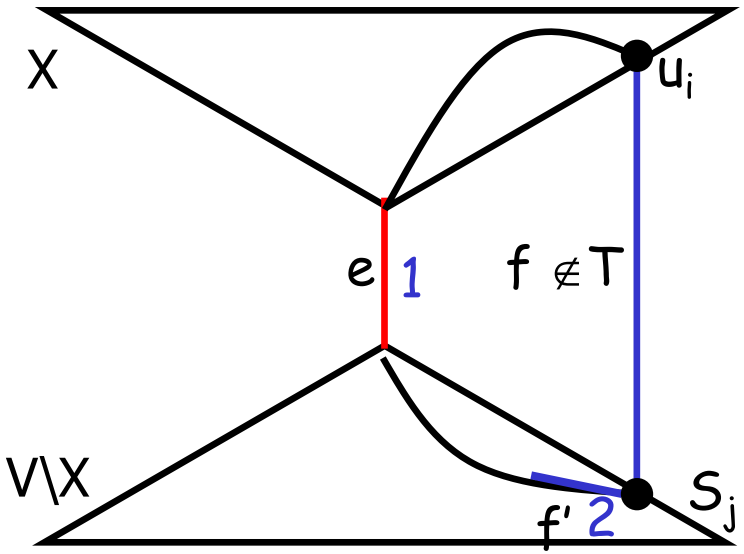
⇒ e cannot belong to T

(⇐), (1)

e: heaviest red edge in T

since (V,B) is connected, there exists blue edge $f \notin T$...

Lemma 2: $\exists$ $f' \neq f$ such that $c(e) < p(f') \leq p(f)$

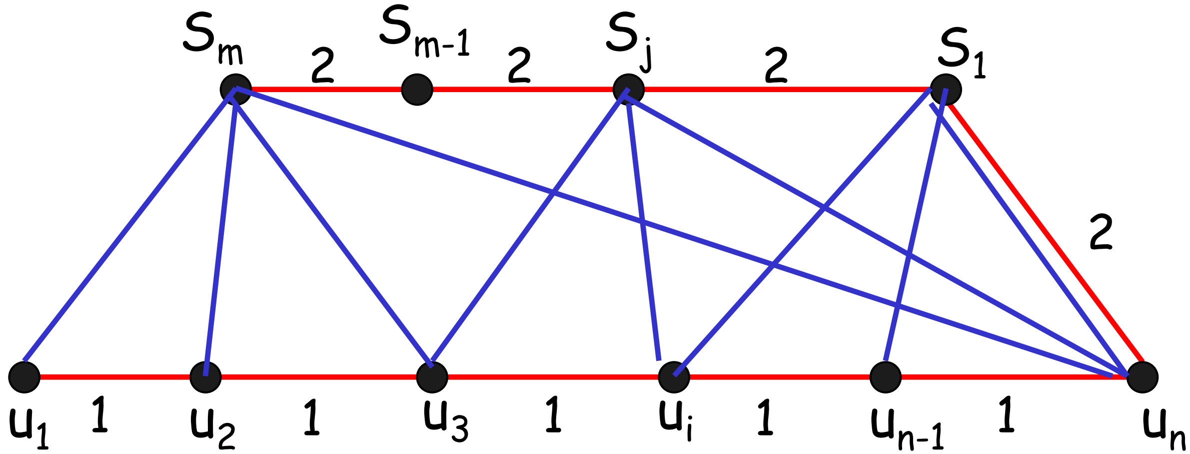


➡ $c(e)=1$ and $p(f')=2$

By previous remark...
all blue edges in C-{f,f'}
have price 1

➡ $p(f)=1$ and $p(f')=1$ leads to a new MST with same revenue and less red edges.
A contradiction.

(⇐), (2)

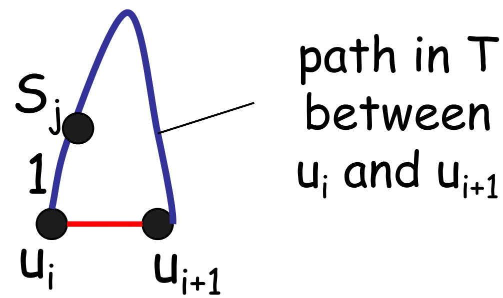


Assume T contains no red edge

We define:

C $=\{S_j$: S_j is linked to some blue edge in T with price 1$\}$

every u_i must be incident in T to some blue edge of price 1



⇒ **C** is a cover

any $S_j \notin$ **C** must be a leaf in T ⇒

revenue $= n+|\mathbf{C}|-1+2(m-|\mathbf{C}|)=n+2m-|C|-1 \geq n+2m-t-1$

⇒ $|\mathbf{C}| \leq t$

∎

The single price algorithm

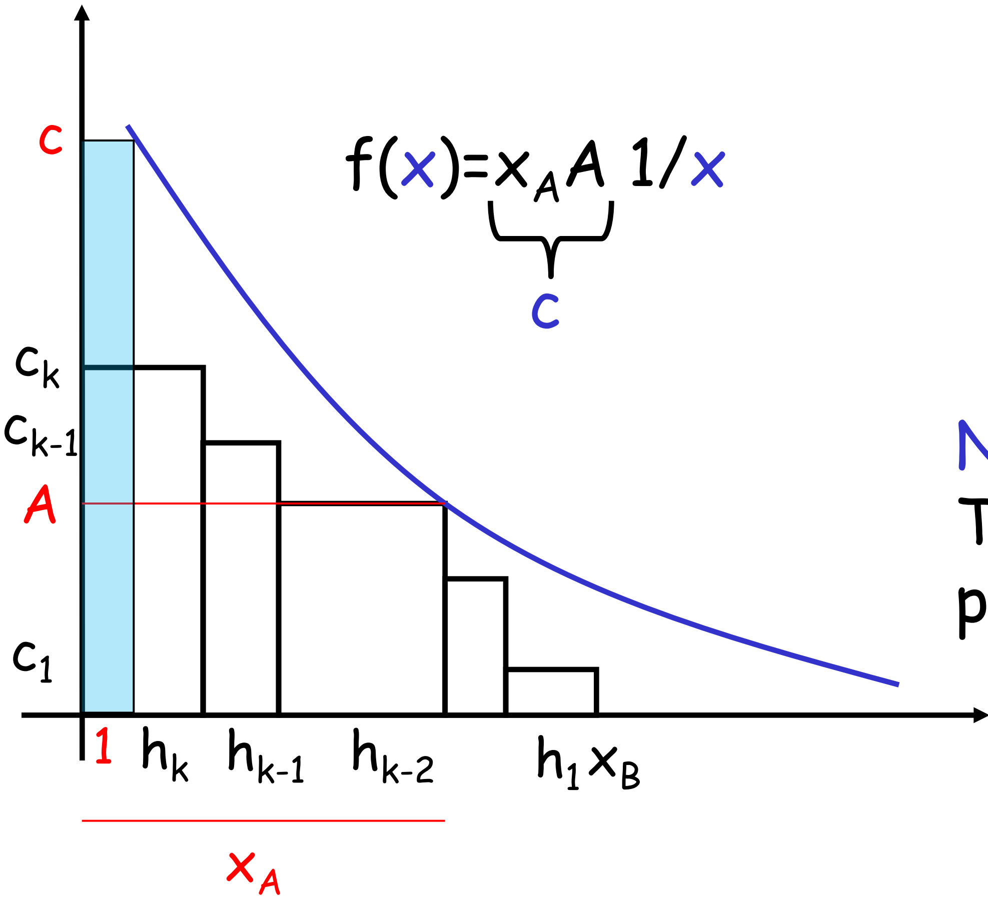
- Let $c_1<c_2<\ldots<c_k$ be the different fixed costs
- For i = 1,…,k
  - set $p(e)=c_i$ for every $e\in B$
  - Look at the revenue obtained
- return the solution which gives the best revenue

Theorem

Let r be the revenue of the single price algorithm; and let r* be the optimal revenue. Then, $r^*/r \leq \rho$, where $\rho = 1 + \min\{\log|B|, \log(n-1), \log(c_k/c_1)\}$

T: MST corresponding to the optimal price function

h_i: number of blue edges in T with price c_i



$f(x)=\underbrace{x_A A}_{c}\, 1/x$

$c \geq c_k$

$x_B=\Sigma_j h_j \leq \min\{n-1, |B|\}$

Notice:

The revenue r of the single price algorithm is at least c

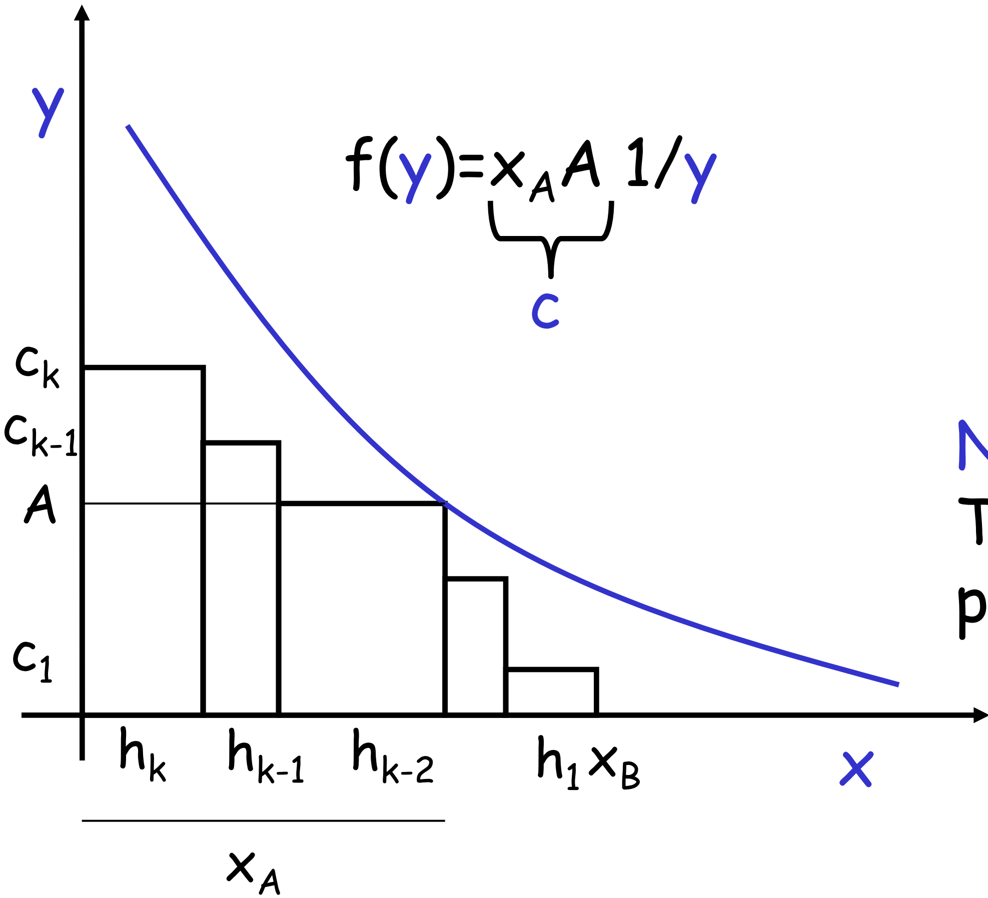
hence:

$r^*/r \leq 1+\log x_B$

$$r^* \leq c + \int_1^{x_B} c\, 1/x\, dx = c(1+ \log x_B - \log 1) = c(1+\log x_B)$$

T: MST corresponding to the optimal price function

k_i: number of blue edges in T with price c_i



$c \geq c_k$

$x_B=\Sigma_j h_j \leq \min\{n-1,|B|\}$

Notice:

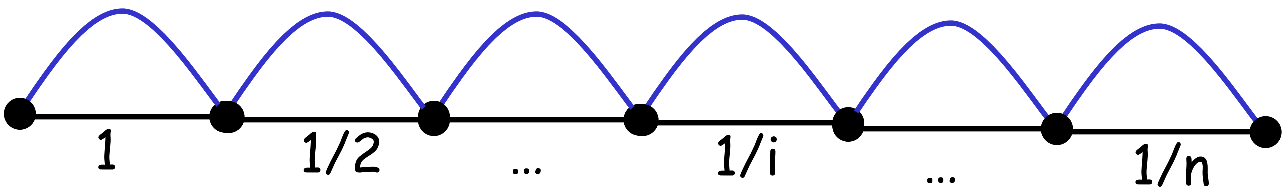
The revenue r of the single price algorithm is at least c

hence:

$r^*/r \leq 1+\log(c_k/c_1)$

$$r^* \leq c + \int_{c_1}^{c_k} c\, 1/y\, dy = c(1+\log c_k - \log c_1) = c(1+\log(c_k/c_1))$$

An asymptotically tight example



The single price algorithm obtains revenue r=1

The optimal solution obtains revenue

$$r^* = \sum_{j=1}^{n} 1/j = H_n = \Theta(\log n)$$

Exercise: prove the following

Let r be the revenue of the single price algorithm; and let r* be the optimal revenue. Then, $r^*/r \leq k$, where k is the number of distinct red costs

Exercise:

Give a polynomial time algorithm that, given an acyclic subset $F \subseteq B$, find a pricing p such that:

(i) The corresponding MST T of p contains exactly F as set o blue edges, i.e. $E(T) \cap B = F$

(ii) The revenue is maximized