

# Computing a Nash Equilibrium of a Congestion Game: PLS-completeness

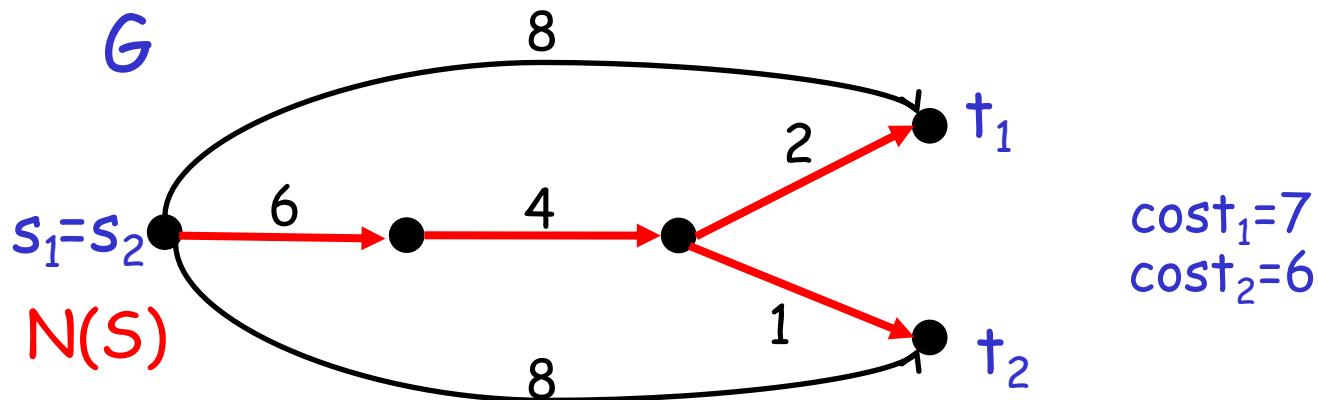
based on Chapters 19 & 20 of  
Twenty Lectures on Algorithmic Game Theory,  
Tim Roughgarden

# Global Connection Game

- $G=(V,E)$ : directed graph
- $c_e$ : non-negative cost of the edge  $e \in E$
- player  $i$  has a source node  $s_i$  and a sink node  $t_i$
- Strategy for player  $i$ : a path  $P_i$  from  $s_i$  to  $t_i$
- Given a strategy vector  $S$ , the cost of player  $i$

$$\text{cost}_i(S) = \sum_{e \in P_i} c_e / k_e(S)$$

$k_e(S)$ : number of players whose path contains  $e$



## ■ Global Connection Game

- potential game
- a NE always exists
- better-response dynamics always converge to a NE

## ■ Facts

- no one knows how to define a dynamic converging to a NE in poly-time
- no one knows how to compute a NE in poly-time

## ■ question:

- can we derive an evidence that the problem is hard?

## ■ (tricky) answer:

- theory of PLS-completeness

# Congestion Game

- $E$ : set of resources
- $k$  players
- player  $i$  picks a strategy  $S_i$  from an explicit set of strategies  $\mathcal{S}_i \subseteq 2^E$
- each resource  $e \in E$  has possible costs  $c_e(1), c_e(2), \dots, c_e(k)$
- Given a strategy vector  $S$ , the cost of player  $i$  is:

$$\text{cost}_i(S) = \sum_{e \in S_i} c_e(k_e(S))$$

$k_e(S)$ : number of players whose chosen strategy contains  $e$

# properties of CG

- Congestion Game is a potential game
- Rosenthal potential function:

$$\Phi(\mathbf{s}) = \sum_{e \in E} \sum_{i=1}^{k_e(\mathbf{s})} c_e(i)$$

- ➡ a NE always exists (any local minimum of  $\Phi$  is a NE)
- ➡ better response dynamic converges to a NE

## CG-NE problem

Given an instance of Congestion Game, find any NE

can we prove that CG-NE is NP-hard?

...if yes, this would yield to quite surprising consequences.

# Addressing a typechecking error

- an NP problem is a **decision problem** admitting short (polynomial size) **witnesses** for YES-instances and poly-time **verifier**
  - inputs accepted by the verifier are called **witnesses**
- CG-NE is not a decision problem
- **class FNP** (**Functional** NP): problem just like NP problems except that, for YES-instances, a witness must be provided
  - also called **search problems**
- An algorithm for an FNP problem:
  - takes as input an instance
  - outputs a witness for a YES-instance or say "No".

## Reduction from one search problem $L_1$ to another one $L_2$

Two polynomial-time algorithms:

- $A_1$  mapping instances  $x \in L_1$  to instances  $A_1(x)$  of  $L_2$
- $A_2$  mapping witnesses of  $A_1(x)$  to witnesses of  $x$   
(and "no" to "no")

**Notice:** if  $L_2$  is solvable in poly-time then  $L_1$  is solvable in poly-time as well.



# Theorem

CG-NE is not FNP-complete unless  $NP=coNP$

## proof

Assume we have two poly-time algs

- $A_1$  that maps every SAT formula  $\phi$  to instances of CG  $A_1(\phi)$
- $A_2$  that maps every NE  $S$  of  $A_1(\phi)$  to a satisfying assignment  $A_2(S)$  of  $\phi$ , if one exists, or to the string "no" otherwise.

Then  $NP=CoNP$ .

Let  $\phi$  be unsatisfiable SAT formula,  $S$  be a NE of  $A_1(\phi)$ .

$S$  is a short, efficiently verifiable proof of the unsatisfiability of  $\phi$

A poly-time verifier:

- compute  $A_1(\phi)$
- verify that  $S$  is a NE of  $A_1(\phi)$
- verify that  $A_2(S)$  returns "no"

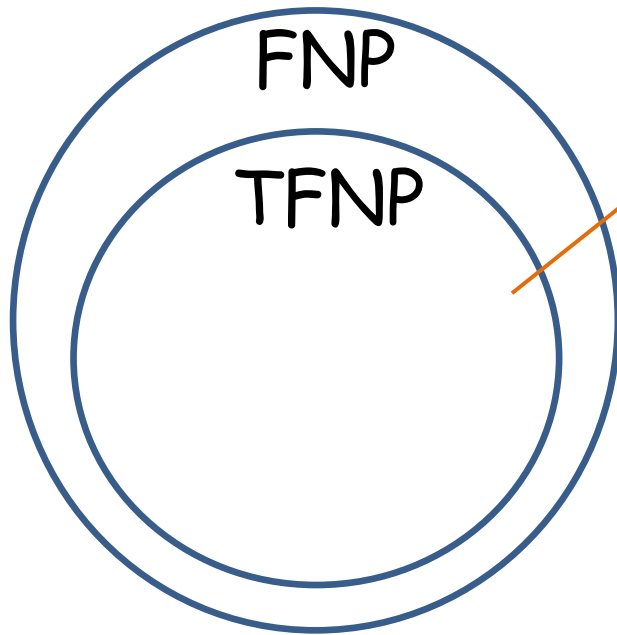


**Note:** we're using only the fact that every instance of CG has a NE

**TFNP** (total FNP): problems in FNP for which every instance has at least one witness.

## Theorem

If a TFNP problem is FNP-complete then  $NP=coNP$ .



-CG-NE

-problem of finding a mixed-strategy NE  
for a finite game

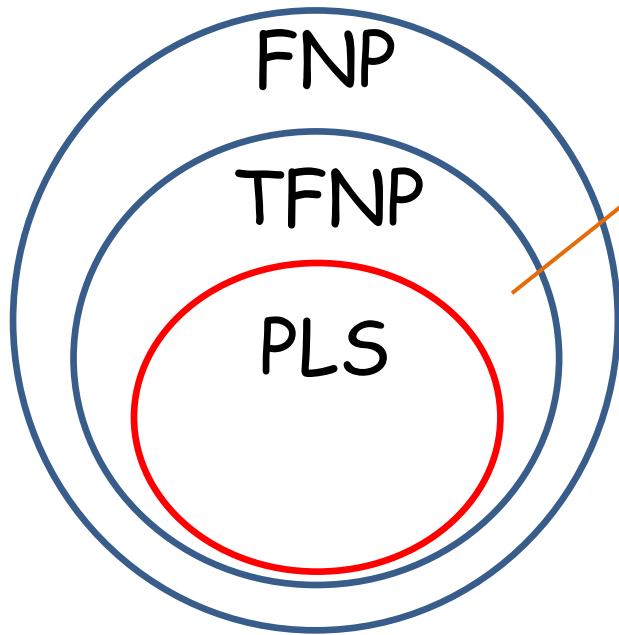
-factoring

-...

can we prove that CG-NE is TFNP-complete?

no: no complete problem is known for TFNP  
(and people think no one can exist)

Syntactic classes vs Semantic classes



-CG-NE

-problem of finding a mixed-strategy NE  
for a finite game

-factoring

-...

can we prove that CG-NE is TFNP-complete?

no: no complete problem is known for TFNP  
(and people think no one can exist)

which is the right class for CG-NE?

**PLS**: abstract local search problems

# Maximum Cut problem

## ■ Input:

- an undirected graph  $G=(V,E,w)$  with non-negative edge weights

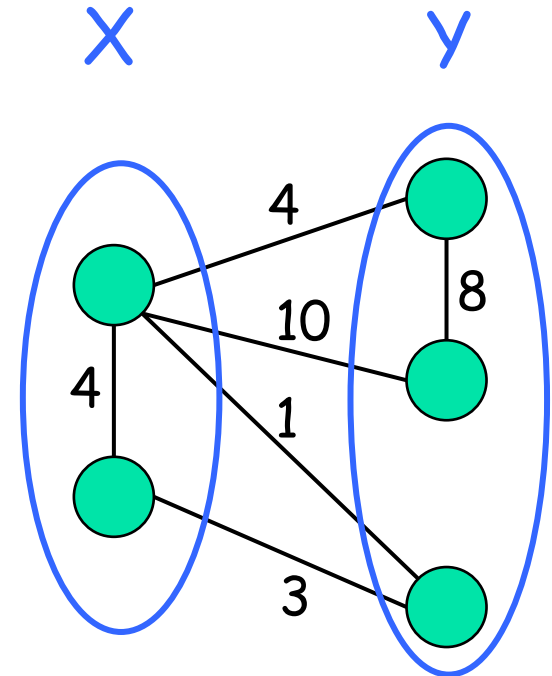
## ■ Solution:

- a cut  $(X,Y)$ , where  $X$  and  $Y$  are a partition of  $V$

## ■ Measure (to maximize):

- the weight of the cut,

$$\sum_{\substack{(x,y) \in E: \\ x \in X, y \in Y}} w(x,y)$$



It is NP-hard

# A natural heuristic: Local search algorithm

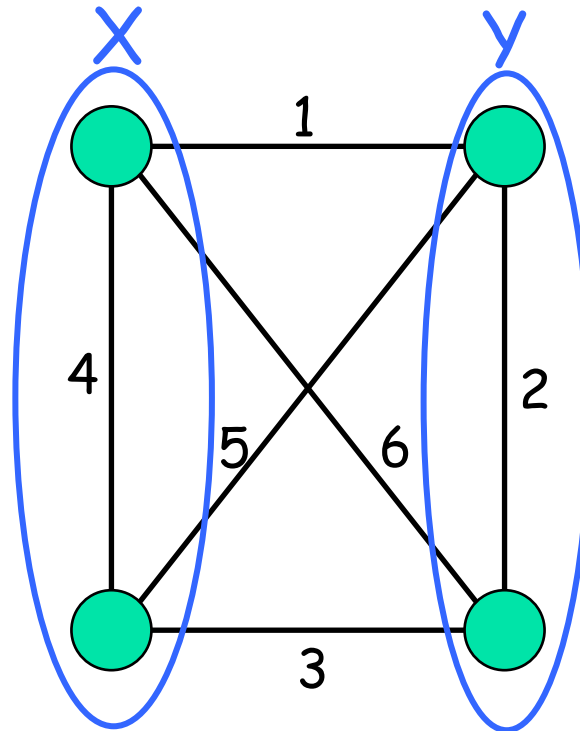
- initialize with an arbitrary cut  $(X,Y)$
- **while** there is an improving local move **do**  
take an arbitrary such move

improving local move:

move a single vertex  $v$  from one side of the cut to the other side, if this improves the weight of the current cut.

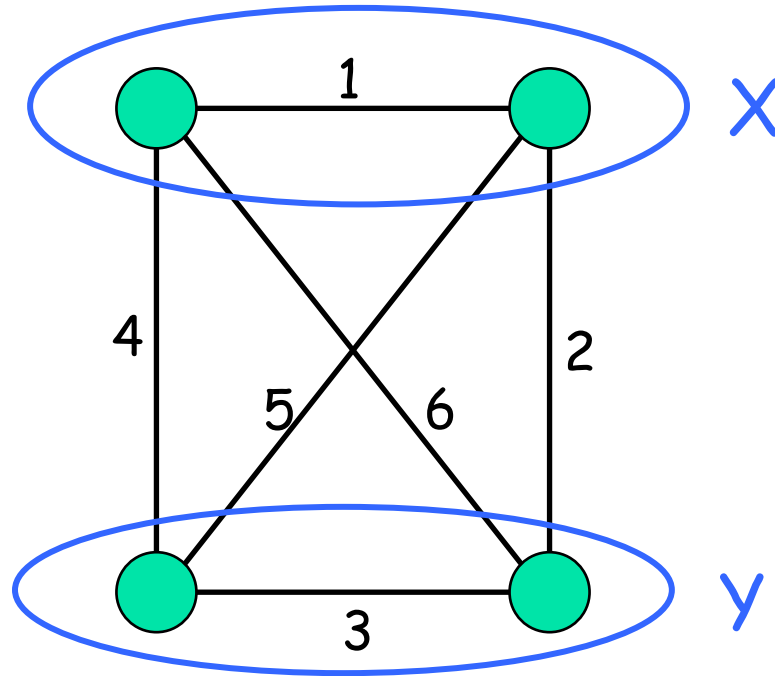
local optimum: cut with no improving local move available.

# local optimum vs global optimum



local opt of weight 15

# local optimum vs global optimum



global opt of weight 17



# local optimum vs global optimum

is finding a local opt easier than finding a global opt?

sometimes strictly easier: unweighted graphs

- max cut is still NP-hard for unweighted graphs
- local search algorithm converges in poly-time

facts:

- no known poly-time local search alg for finding local opt  
for general weights
- no known poly-time alg for computing a local opt for  
general weights

## local Max-Cut problem

Given an instance of Max Cut, find any local opt.

....this problem is PLS-complete.

## Ingredients of an Abstract Local Search Problem

1. The first polynomial-time algorithm takes as input an instance and outputs an arbitrary feasible solution.
2. The second polynomial-time algorithm takes as input an instance and a feasible solution, and returns the objective function value of the solution.
3. The third polynomial-time algorithm takes as input an instance and a feasible solution, and either reports "locally optimal" or produces a solution with better objective function value.

## A PLS reduction from $L_1$ to $L_2$

Two polynomial-time algorithms:

- $A_1$  mapping instances  $x \in L_1$  to instances  $A_1(x)$  of  $L_2$
- $A_2$  mapping every local optimum of  $A_1(x)$  to local optimum of  $x$

**Notice:** if  $L_2$  is solvable in poly-time then  $L_1$  is solvable in poly-time as well.

## Definition.

A problem  $L$  is **PLS-complete** if  $L \in \text{PLS}$  and every problem in PLS reduces to it.

## Theorem (Johnson, Papadimitriou, Yannakakis '85, Schaffer, Yannakakis 91)

Computing a local maximum of a maximum cut instance with general non-negative edge weights is a PLS-complete problem.

## Theorem (Johnson, Papadimitriou, Yannakakis, '85, Schaffer, Yannakakis 91)

Computing a local maximum of a maximum cut instance with general non-negative edge weights using local search can require an exponential (in  $|V|$ ) number of iterations, no matter how an improving local move is chosen in each iteration.

# Theorem (Fabrikant, Papadimitriou, Talwar 2004)

CG-NE is PLS-complete.

proof

CG-NE  $\in$  PLS

3 algorithms of the formal definition:

Alg 1: given the instance, returns any strategy profile  $S$

Alg 2: given a strategy profile  $S$ , compute  $\Phi(S)$

Alg 3: given a strategy profile  $S$ , computes a better response for any player, if any, or report “ $S$  is a NE”.

completeness: reduction from MaxCut

## proof

a player for each vertex  $v$

two resources  $r_e$  and  $\bar{r}_e$  for each edge  $e$

two strategies for player  $v$ :  $S_v = \{r_e : e \in \delta(v)\}$   
 $\bar{S}_v = \{\bar{r}_e : e \in \delta(v)\}$

cost of a resource  $r \in \{r_e, \bar{r}_e\}$ :  
 $c_r(1)=0$  and  $c_r(2)=w(e)$

bijection between  $2^{|V|}$  strategy profiles and cuts of the graph  
cut corresponding to strategy profile  $S$ :

$$(X_S := \{v : v \text{ plays } S_v \text{ in } S\}, Y_S := V \setminus X_S)$$

$$\Phi(S) = \sum_{r \in R} \sum_{i=1}^{k_r(S)} c_r(i) = W - W(X_S, Y_S)$$

$$W = \sum_{e \in E} w(e)$$

➡  $(X_S, Y_S)$  is a local maximum cut iff  $S$  local minimum for  $\Phi(S)$



what about the problem  
of computing mixed Nash Equilibria?



# MNE problem

Given an instance of a 2-player game in normal form (**bimatrix game**), find any mixed NE

Nash's theorem guarantees that a mixed NE always exists

$MNE \in TFNP$

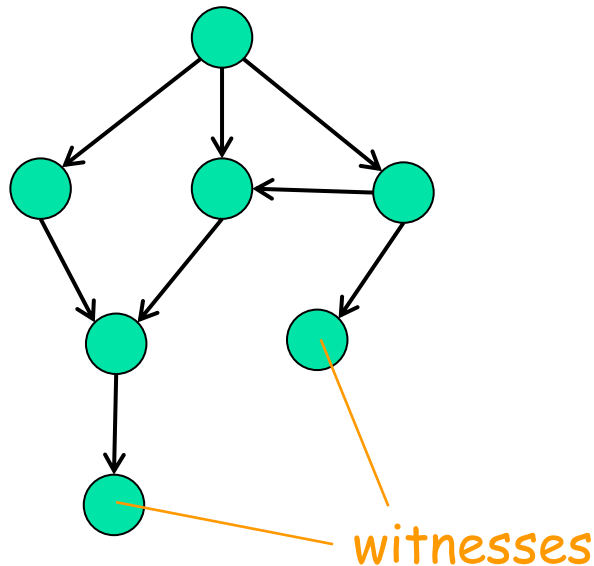
no polynomial time algorithm is known for MNE

what is the right class for MNE problem?

# PLS: abstract local search problems

nodes: feasible solutions

edges: improving moves

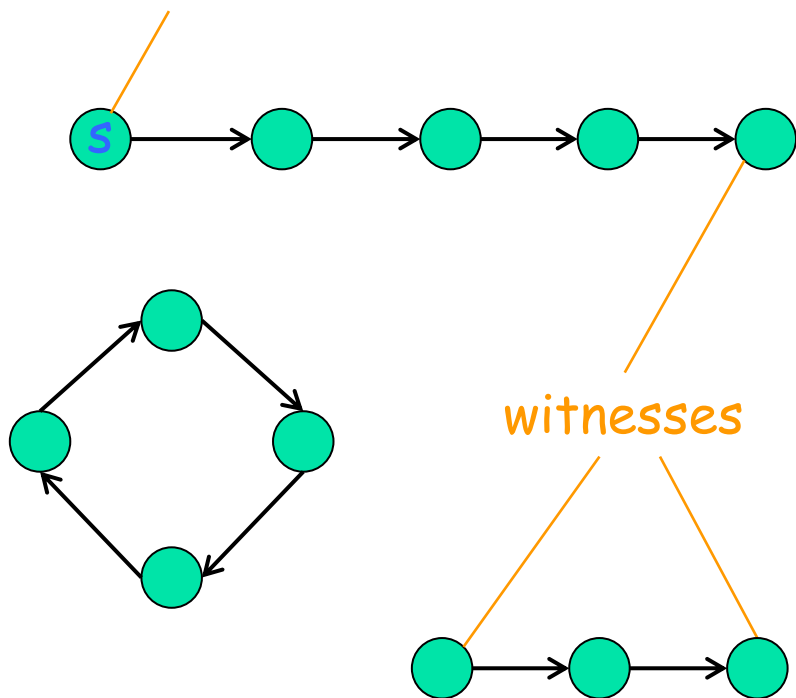


generic reason of membership:  
solvable by local search, i.e. by  
following a directed path to a  
sink vertex.

# PPAD

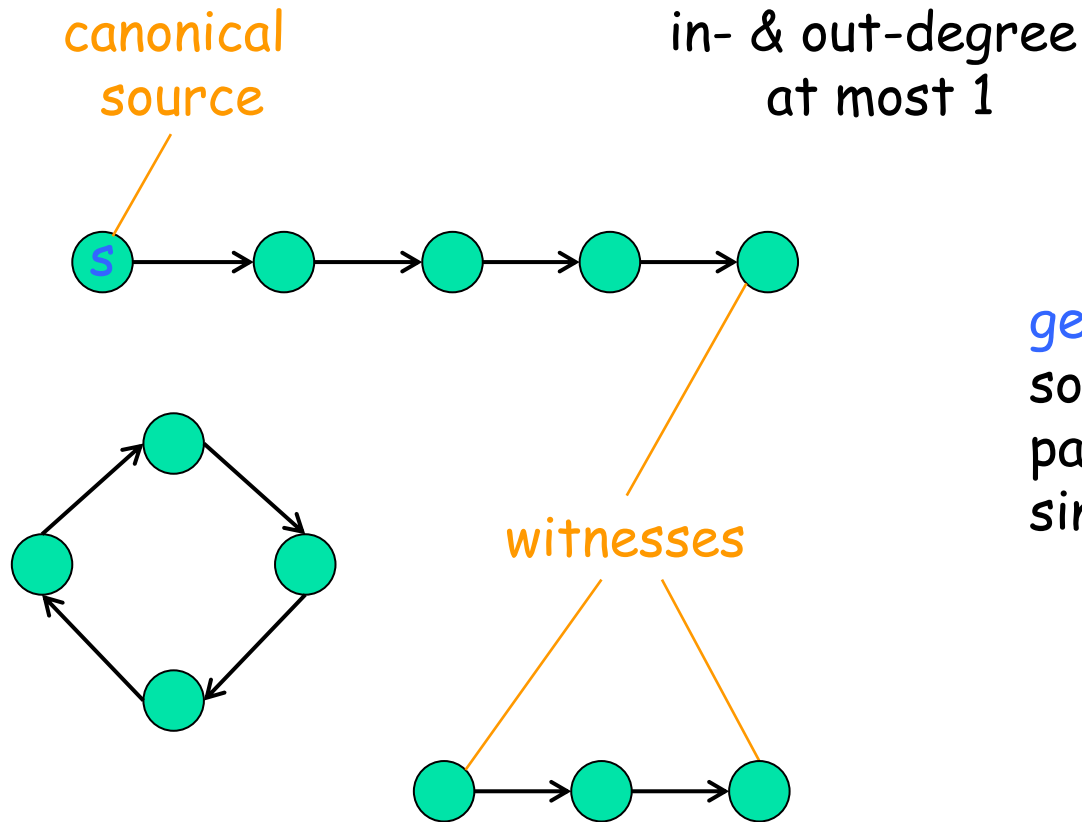
canonical  
source

in- & out-degree  
at most 1



generic reason of membership:  
solvable by following a directed  
path from the source to the  
sink vertex.

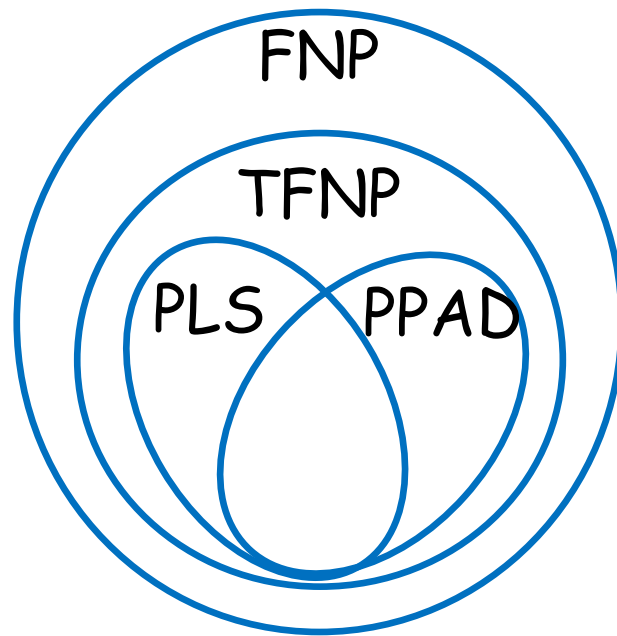
PPAD: polynomial parity argument in a directed graph



generic reason of membership:  
solvable by following a directed  
path from the source to the  
sink vertex.

class **PPAD** introduced in 94  
by Christos H. **Papadimitriou**





**Theorem** (Daskalakis, Goldberg, Papadimitriou 06, Chen, Deng, Teng 06)

Computing any MNE of a bimatrix game is PPAD-complete