Network Formation Games
Network Formation Games

- NFGs model distinct ways in which *selfish* agents might create and evaluate networks.
- We'll see two models:
  - Global Connection Game
  - Local Connection Game
- Both models aim to capture two competing issues: players want
  - to minimize the cost they incur in building the network
  - to ensure that the network provides them with a high quality of service
Motivations

- NFGs can be used to model:
  - social network formation (edge represent social relations)
  - how subnetworks connect in computer networks
  - formation of networks connecting users to each other for downloading files (P2P networks)
Setting

- What is a stable network?
  - we use a NE as the solution concept
  - we refer to networks corresponding to Nash Equilibria as being stable

- How to evaluate the overall quality of a network?
  - we consider the social cost: the sum of players’ costs

- Our goal: to bound the efficiency loss resulting from stability
Global Connection Game

E. Anshelevich, A. Dasgupta, J. Kleinberg, E. Tardos, T. Wexler, T. Roughgarden,
The Price of Stability for Network Design with Fair Cost Allocation, FOCS’04
The model

- $G=(V,E)$: directed graph
- $c_e$: non-negative cost of the edge $e \in E$
- $k$ players
- player $i$ has a source node $s_i$ and a sink node $t_i$
- player $i$'s goal: to build a network in which $t_i$ is reachable from $s_i$ while paying as little as possible
- Strategy for player $i$: a path $P_i$ from $s_i$ to $t_i$
The model

- Given a strategy vector $S$, the constructed network will be $N(S) = \bigcup_i P_i$
- The cost of the constructed network will be shared among all players as follows:

$$\text{cost}_i(S) = \sum_{e \in P_i} c_e / k_e(S)$$

$k_e(S)$: number of players whose path contains $e$

sometimes we write $k_e$ instead of $k_e(S)$
when $S$ is clear from the context

this cost-sharing scheme is called *fair* or *Shapley cost-sharing mechanism*
Remind

- We use **Nash equilibrium (NE)** as the solution concept.
- A strategy vector $S$ is a NE if no player has convenience to change its strategy.
- Given a strategy vector $S$, $N(S)$ is **stable** if $S$ is a NE.
- To evaluate the overall quality of a network, we consider the **social cost**, i.e. the sum of all players' costs:
  \[\text{cost}(S) = \sum_i \text{cost}_i(S)\]
- A network is **optimal** or **socially optimal** if it minimizes the social cost.
We use Nash equilibrium (NE) as the solution concept.

A strategy vector $S$ is a NE if no player has convenience to change its strategy.

Given a strategy vector $S$, $N(S)$ is stable if $S$ is a NE.

To evaluate the overall quality of a network, we consider the social cost, i.e., the sum of all players' costs.

$$\text{cost}(S) = \sum_i \text{cost}_i(S)$$

A network is optimal or socially optimal if it minimizes the social cost.

Notice: $\text{cost}(S) = \sum_{e \in N(S)} c_e$

The optimal network is a cheapest subgraph of $G$ containing a path from $s_i$ to $t_i$, for each $i$.

Cost $1 = 7$, Cost $2 = 6$. 

$G$

$N(S)$

$s_1 = s_2$

$t_1$

$t_2$
an example

what is the socially optimal network?
an example

what is the socially optimal network?

is it stable?

...no!

cost of the social optimum: 13

cost$_1$ = 7

cost$_2$ = 6

social cost of the network 13
an example

what is the socially optimal network?

cost of the social optimum: 13

is it stable?
...no!

cost$_1$ = 6
cost$_2$ = 11

social cost of the network
17
an example

what is the socially optimal network?  

cost of the social optimum: 13

is it stable?  
...yes!

cost\(_1\) = 6  
cost\(_2\) = 10

social cost of the network 16

graph G
one more example
one more example

optimal network has cost 12

cost_1 = 7

cost_2 = 5

is it stable?
one more example

\[ \text{cost}_1 = 5 \]
\[ \text{cost}_2 = 8 \]

is it stable? ...yes!

the social cost is 13
one more example

...a better NE...

\[ \text{cost}_1 = 5 \]
\[ \text{cost}_2 = 7.5 \]

the social cost is 12.5
Addressed issues

- Does a stable network always exist?
- Can we bound the price of anarchy (PoA)?
- Can we bound the price of stability (PoS)?
- Does the repeated version of the game always converge to a stable network?
PoA and PoS
for a given network $G$, we define:

PoA of the game in $G$ = \[ \max_{S \text{ s.t. } S \text{ is a NE}} \frac{\text{cost}(S)}{\text{cost}(S^*_G)} \]

PoS of the game in $G$ = \[ \min_{S \text{ s.t. } S \text{ is a NE}} \frac{\text{cost}(S)}{\text{cost}(S^*_G)} \]

$S^*_G$ : socially optimum for $G$
we want to bound \( \text{PoA} \) and \( \text{PoS} \) in the worst case:

\[
\text{PoA of the game} = \max_G \text{PoA in } G \\
\text{PoS of the game} = \max_G \text{PoS in } G
\]
some notations

we use:
\[ x=(x_1,x_2,\ldots,x_k); \quad x_{-i}=(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k); \quad x=(x_{-i},x_i) \]

\( G \): a weighted directed network

cost or length of a path \( \pi \) in \( G \) from a node \( u \) to a node \( v \) : \[ \sum_{e \in \pi} c_e \]

d\(_G\)(\( u, v \)): distance in \( G \) from a node \( u \) to a node \( v \) : length of any shortest path in \( G \) from \( u \) to \( v \)
Price of Anarchy
Price of Anarchy: a lower bound

optimal network has cost 1

best NE: all players use the lower edge

worst NE: all players use the upper edge

PoA of the game is ≥ k

PoS in G is 1

PoA in G is k
The price of anarchy in the global connection game with \( k \) players is at most \( k \)

**Theorem**

**Proof**

\( S \): a NE

\( S^* \): a strategy profile minimizing the social cost

for each player \( i \),

let \( \pi_i \) be a shortest path in \( G \) from \( s_i \) to \( t_i \)

we have

\[
\text{cost}_i(S) \leq \text{cost}_i(S_{-i}, \pi_i) \leq d_G(s_i, t_i) \leq \text{cost}(S^*)
\]
The price of anarchy in the global connection game with $k$ players is at most $k$

**proof**

S: a NE  
S*: a strategy profile minimizing the social cost

For each player $i$,

let $\pi_i$ be a shortest path in $G$ from $s_i$ to $t_i$

we have

$$\text{cost}_i(S) \leq \text{cost}_i(S_{-i}, \pi_i) \leq d_G(s_i, t_i) \leq \text{cost}(S^*)$$

$N(S^*)$  
$S_i$  
$\pi$: any path in $N(S^*)$

from $s_i$ to $t_i$

$\pi_i$  
$d_G(s_i, t_i) \leq \text{cost of } \pi \leq \text{cost}(S^*)$

$$\text{cost}(S) = \sum_i \text{cost}_i(S) \leq k \text{ cost}(S^*)$$
Price of Stability & potential function method
Price of Stability: a lower bound

\[ \epsilon > 0: \text{small value} \]

\[ 1 + \epsilon \]

\[ \mathbf{s}_1, \ldots, \mathbf{s}_k \]

\[ t_1, \ldots, t_k \]

\[ 1 \quad 1/2 \quad 1/3 \quad 1/(k-1) \quad 1/k \]

\[ 0 \quad 0 \quad 0 \quad 0 \quad 0 \]
The optimal solution has a cost of $1+\varepsilon$

is it stable?

$\varepsilon > 0$: small value
Price of Stability: a lower bound

\[ \varepsilon > 0: \text{small value} \]

...no! player k can decrease its cost...

is it stable?
Price of Stability: a lower bound

\[ t_1, \ldots, t_k \]

...no! player k-1 can decrease its cost...

is it stable?
Price of Stability: a lower bound

\[ \sum_{j=1}^{k} \frac{1}{j} = H_k \leq \ln k + 1 \]

the only stable network

social cost: \[ \frac{1}{j} = H_k \leq \ln k + 1 \]

the optimal solution has a cost of \(1+\varepsilon\)

\(\varepsilon > 0\): small value

PoS of the game is \(\geq H_k\)
Any instance of the global connection game has a pure Nash equilibrium, and better response dynamic always converges.

Theorem

The price of stability in the global connection game with $k$ players is at most $H_k$, the $k$-th harmonic number.

To prove them we use the Potential function method.
Notation:
\[ x=(x_1, x_2, \ldots, x_k); \quad x_{-i}=(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k); \quad x=(x_{-i}, x_i) \]

**Definition**

For any finite game, an *exact potential function* \( \Phi \) is a function that maps every strategy vector \( S \) to some real value and satisfies the following condition:

\[
\forall S=(S_1, \ldots, S_k), \quad S'_i \neq S_i, \text{ let } S'=(S_{-i}, S'_i), \text{ then }
\]

\[
\Phi(S) - \Phi(S') = \text{cost}_i(S) - \text{cost}_i(S')
\]

A game that possesses an exact potential function is called *potential game*
Theorem

Every potential game has at least one pure Nash equilibrium, namely the strategy vector $S$ that minimizes $\Phi(S)$

proof

consider any move by a player $i$ that results in a new strategy vector $S'$. We have:

$$\Phi(S) - \Phi(S') = \text{cost}_i(S) - \text{cost}_i(S')$$

$$\leq 0$$

$\text{cost}_i(S) \leq \text{cost}_i(S')$

player $i$ cannot decrease its cost, thus $S$ is a NE
In any finite potential game, better response dynamics always converge to a Nash equilibrium.

**Theorem**

**proof**

better response dynamics simulate local search on $\Phi$:
1. each move strictly decreases $\Phi$
2. finite number of solutions

**Note**: in our game, a best response can be computed in polynomial time.
Suppose that we have a potential game with potential function $\Phi$, and assume that for any outcome $S$ we have

$$\frac{\text{cost}(S)}{A} \leq \Phi(S) \leq B \cdot \text{cost}(S)$$

for some $A, B > 0$. Then the price of stability is at most $AB$.

**proof**

Let $S'$ be the strategy vector minimizing $\Phi$.

Let $S^*$ be the strategy vector minimizing the social cost.

We have:

$$\frac{\text{cost}(S')}{A} \leq \Phi(S') \leq \Phi(S^*) \leq B \cdot \text{cost}(S^*)$$
...turning our attention to the global connection game...

Let $\Phi$ be the following function mapping any strategy vector $S$ to a real value:

$$
\Phi(S) = \sum_{e \in E} \Phi_e(S)
$$

where

$$
\Phi_e(S) = c_e \, H_{k_e(S)}
$$

$$
H_k = \sum_{j=1}^{k} \frac{1}{j} \quad \text{k-th harmonic number}
$$

[we define $H_0 = 0$]
Let \( S=(P_1,\ldots,P_k) \), let \( P'_i \) be an alternate path for some player \( i \), and define a new strategy vector \( S'=(S_{-i},P'_i) \). Then:

\[
\Phi(S) - \Phi(S') = \text{cost}_i(S) - \text{cost}_i(S')
\]

**Lemma 1**

For any strategy vector \( S \), we have:

\[
\text{cost}(S) \leq \Phi(S) \leq H_k \text{cost}(S)
\]

...from which we have:

PoS of the game is \( \leq H_k \)
Lemma 2

For any strategy vector $S$, we have:

$$\text{cost}(S) \leq \Phi(S) \leq H_k \text{cost}(S)$$

**proof**

$$\text{cost}(S) \leq \Phi(S) = \sum_{e \in E} c_e H_{k_e(S)}$$

$$= \sum_{e \in N(S)} c_e H_{k_e(S)} \leq \sum_{e \in N(S)} c_e H_k = H_k \text{cost}(S)$$

$$1 \leq k_e(S) \leq k \quad \text{for } e \in N(S)$$
Theorem

**Theorem**

Given an instance of a GC Game and a value $C$, it is NP-complete to determine if a game has a Nash equilibrium of cost at most $C$.

**proof**

Reduction from 3-dimensional matching problem
3-dimensional matching problem

- **Input:**
  - disjoint sets $X$, $Y$, $Z$, each of size $n$
  - a set $T \subseteq X \times Y \times Z$ of ordered triples

- **Question:**
  - does there exist a set of $n$ triples in $T$ so that each element of $X \cup Y \cup Z$ is contained in exactly one of these triples?
3-dimensional matching problem

- **Input:**
  - disjoint sets $X, Y, Z$, each of size $n$
  - a set $T \subseteq X \times Y \times Z$ of ordered triples

- **Question:**
  - does there exist a set of $n$ triples in $T$ so that each element of $X \cup Y \cup Z$ is contained in exactly one of these triples?
There is a 3D matching if and only if there is a NE of cost at most $C=3n$
Assume there is a 3D matching.

**S**: strategy profile in which each player choose a path passing through the triple of the matching it belongs to
Assume there is a 3D matching.

$S$: strategy profile in which each player choose a path passing through the triple of the matching it belongs to

$\text{cost}(S) = 3n$

$S$ is a NE
Assume there is a NE of cost $\leq 3n$

$N(S)$ uses at most $n$ edges of cost 3

each edge of cost 3 can “serve” at most 3 players

then, the edge of cost 3 are exactly $n$

...and they define a set of triples that must be a 3D-matching
Max-cut game

- $G=(V,E)$: undirected graph
- Nodes are (selfish) players
- Strategy $S_u$ of $u$ is a color \{red, green\}
- player $u$’s payoff in $S$ (to maximize):
  - $p_u(S) = |\{(u,v) \in E : S_u \neq S_v\}|$

social welfare of strategy vector $S$

$$\sum_u p_u(S) = 2 \# \text{edges crossing the red-green cut}$$
Max-cut game

does a Nash Equilibrium always exist?

how bad a Nash Equilibrium Can be?

does the repeated game always converge to a Nash Equilibrium?
...let's play Max-cut game on Petersen Graph...is it a NE?
...let’s play *Max-cut game* on Petersen Graph

...is it a NE?
...let's play *Max-cut game* on Petersen Graph

...is it a NE?
...let’s play Max-cut game on Petersen Graph

...is it a NE?
...let's play Max-cut game on Petersen Graph

...is it a NE?
...let's play Max-cut game on Petersen Graph

...is it a NE?

...yes!

# of edges crossing the cut is 12
Exercise

Show that:
(i) Max-cut game is a potential game
(ii) PoS is 1
(iii) PoA $\geq \frac{1}{2}$
(iv) there is an instance of the game having a NE with social welfare of $\frac{1}{2}$ the social optimum