Approximation Algorithms

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Approximation Ratio (Error)

Definition

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Optimization Problem

Given an optimization problem
$\mathcal{P} = (X, Y(x), m(x, Y(x)), MIN/MAX)$, we say $A(x)$ is an
r-approximation for $\mathcal{P}$ if
for any instance $x \in X$, the algorithm $A(x)$ returns a feasible
solution y^A such that

$$\frac{m(x, y^A)}{\mathbf{opt}(x)} \geq r \quad \text{(in the case of MAX)}$$

Vertex Cover (VC)

A lower bound for optimal VC

Matching and Covering

FACT 1: Given any graph $G(V, E)$, consider any **Maximal Matching** $M \subseteq E$. Then, any VC for G must contain at least 1 vertex for every edge in M.

Proof.

immediate consequence of def.s of Matching and VC.
- Remind: Matching = any subset of **disjoint** edges

Lower Bound for the Optimum

FACT 2:

$$\mathbf{opt}(G) \;\geqslant\; |M|$$

Vertex Cover (VC)

An apx algorithm for Min VC

Matching Algorithm $M - ALG$

- Input: $G(V, E)$;
- Find (any) Maximal Matching M;
- Return $C = \{$ all nodes touched by $M\}$;

Vertex Cover (VC)

An apx algorithm for Min VC

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THM 3.

M-ALG is a 2-apx algorithm for Min-VC.

Vertex Cover (VC)

Proof of THM 3

M-ALG

FACT 4. The returned solution C (i) is always a Vertex Cover for G AND (ii) $|C| = 2|M|$.

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$$\frac{|C|}{\mathbf{opt}(G)} \leqslant \frac{2|M|}{|M|} \leqslant 2$$