Lemma. *Let $a \in \mathbf{R}_{>0}$ and let $\phi_a : \mathbf{R}_{>0} \longrightarrow \mathbf{R}$ be the function given by*

$$\phi_a(x) \;=\; x \log x + \frac{a}{x}.$$

Then ϕ_a has a unique minimum. For $a \gg 0$ it is approximately equal to $\sqrt{2a \log a}$ and is approximately assumed in $x = \sqrt{2a/\log a}$.

Proof. The derivative ϕ'_a is given by $1 + \log x - \frac{a}{x^2}$ and the second derivative $\frac{1}{x} + 2\frac{a}{x^3}$ is positive on $\mathbf{R}_{>0}$. Therefore ϕ'_a has a unique zero and hence there is a unique minimum.

To approximate the minimum for $a \gg 0$, we omit the constant term 1 and approximate the solution of the equation $\log x = \frac{a}{x^2}$. Since $x^2 \log x^2 = 2a$, it is approximately given by $x^2 = \frac{2a}{\log 2a}$ and hence by $x = \sqrt{2a/\log a}$. The value of $\phi_a(x)$ is easily seen to be $\sqrt{2a \log a} + \sqrt{2a/\log a}(\frac{1}{2}\log 2 + \frac{1}{2}\log\log a)$ which is $\sqrt{2a \log a}(1 + O(\frac{\log\log a}{\log a}))$.

To justify this calculation somewhat, we compute a second order approximation. Let $\varepsilon > 0$ and let $\sqrt{2a/\log a}(1+\varepsilon))$ denote a zero of ϕ'_a. We have

$$1 + \log\left(\sqrt{2a/\log a}(1+\varepsilon)\right) = \frac{a}{\sqrt{2a/\log a}(1+\varepsilon)^2}.$$

Ignoring contributions by higher powers of ε, we find that

$$\varepsilon \;\approx\; \frac{-1 - \frac{1}{2}\log 2 + \frac{1}{2}\log\log a}{1 + \log a} \;=\; O(\frac{\log\log a}{\log a})),$$

which tends to 0 when $a \gg 0$. Using this estimate for ε one checks that the minimum value of ϕ_a itself is equal to $\sqrt{2a \log a}(1 + O(\frac{\log\log a}{\log a}))$.

This proves the lemma.

Corollary. *Let $m \in \mathbf{R}_{>0}$ and let $f_m : \mathbf{R}_{>0} \longrightarrow \mathbf{R}$ be the function given by*

$$f_m(u) \;=\; u^u m^{\frac{1}{u}}.$$

Then the function f_m has a unique minimum. For very large m it is approximately equal to $\exp(\sqrt{2\log m \log\log m})$ and is approximately assumed in $u = \sqrt{2\log m/\log\log m}$.

Proof. It suffices to apply the lemma to the function $\phi_a(x) = \log f_m(x)$ with $a = \log m$. Of course, since we applied the exponential function, the word "approximately" is a much rougher notion now.

The corollary has applications in estimates of running times of subexponential algorithms that depend on the distribution of smooth numbers. In the applications $m = \sqrt{n}$ of a number n that is to be factored or $m = p$ is a large prime divisor of n. In the context of the index calculus algorithm for discrete logarithms modulo a prime p, we have $m = p$ or $m = \sqrt{p}$. In any case m is very large. Typically $m \approx 10^{50}$, so that $a \approx 100$ and $\log\log a/\log a$ has order of magnitude equal to 0.3. This is why we assume $a \gg 0$ in the proof of lemma 1. The relative error of 33% is rather large. Therefore the estimates should be taken with a grain of salt!