In this note we prove a weak form of the Prime Number Theorem due to P. Chebyshev:

Theorem. *(Chebyshev 1850) There exist constants $c, c' \in \mathbf{R}_{>0}$ for which*

$$c'\frac{x}{\ln x} \;<\; \pi(x) \;<\; c\frac{x}{\ln x}, \qquad \text{for all } x \gg 0.$$

This version of the Prime Number Theorem is sufficient for properly estimating the running times of several algorithms. The proof is based on arithmetical properties of the binomial coefficients $\binom{2n}{n}$. For any real number $x > 0$ let $\pi(x)$ denote the number of primes smaller than x.

Lemma 1. *Let $n \in \mathbf{Z}_{>0}$ and let $\binom{2n}{n} = \prod_p p^{e_p}$ the decomposition of $\binom{2n}{n}$ into prime factors. Then $p^{e_p} \le 2n$. In particular, for $p > \sqrt{2n}$ we have $e_p \le 1$;*

Proof. Let p be a prime and let $e_p = \mathrm{ord}_p\binom{2n}{n}$. For every $m \in \mathbf{Z}_{>0}$ we have $\mathrm{ord}_p m! = \sum_{i\ge 1}[\frac{m}{p^i}]$. It follows that

$$e_p \;=\; \mathrm{ord}_p\binom{2n}{n} \;=\; \sum_{i\ge 1}\left([\frac{2n}{p^i}] - 2[\frac{n}{p^i}]\right).$$

Since for every $t \in \mathbf{R}$ the integer $[2t] - 2[t]$ is either 0 or 1, the summands are at most 1. Therefore e_p is not larger than the number of non-zero summands, which is at most $\ln 2n/\ln p$. It follows that p^{e_p} is at most $2n$ as required.

Remark 2. . For the binomial coefficient $\binom{2n}{n}$ one has the estimates

$$\frac{2^{2n}}{2n+1} < \binom{2n}{n} < 2^{2n}, \quad \text{for all } n \ge 1.$$

Proof. One has $2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n}\binom{2n}{k}$. Hence $\binom{2n}{n} < 2^{2n}$. The above also shows that 2^{2n} is the sum of $2n+1$ positive terms, of which $\binom{2n}{n}$ is the largest one. Hence $\binom{2n}{n} > \frac{2^{2n}}{2n+1}$.

Proposition 2. *There exists $c > 0$ for which $\pi(x) \;<\; c\frac{x}{\ln x}$ for all $x \gg 0$.*

Proof. We begin by proving that

$$\sum_{p\le x}\ln p \le 4x\ln 2.$$

The binomial coefficient $\binom{2n}{n}$ is divisible by all primes p between n and $2n$. Then by Remark 2 we get the inequality

$$\prod_{n<p\leq 2n} p \;<\; 2^{2n}, \qquad \text{for all } n \geq 1. \tag{a}$$

Let $x \in \mathbf{R}_{>0}$ and 2^k be the smallest power of 2 for which $x \leq 2^k$. We have $2^k < 2x$. Relation (a) yields

$$\prod_{2^{i-1}<p<2^i} p < 2^{2i}.$$

Taking logarithms and summing up the inequalities for $n = 1, 2, 2^2, \ldots, 2^{k-1}$ we find

$$\sum_{p<x} \ln p \;\leq\; \sum_{p\leq 2^k} \ln p \;=\; \sum_{1<p\leq 2} \ln p + \sum_{2<p\leq 2^2} \ln p + \ldots + \sum_{2^{k-1}<p\leq 2^k} \ln p \;\leq$$

$$\leq\; 2\ln 2 + 2^2 \ln 2 + \ldots + 2^k \ln 2 \;=\; (2^k - 2)\ln 2 \;<\; 2^{k+2} \ln 2 \;<\; 4x \ln 2.$$

On the other hand we have trivially

$$\sum_{p<x} \ln p \;>\; \sum_{\sqrt{x}<p<x} \ln p \;>\; (\pi(x) - \sqrt{x}) \ln \sqrt{x}.$$

Combining the two inequalities gives

$$(\pi(x) - \sqrt{x}) \ln \sqrt{x} < \sum_{p<x} \ln p \;\leq 4x \ln 2$$

and

$$\pi(x) \;<\; 8 \ln 2 \frac{x}{\ln x} + \sqrt{x}.$$

Since for $x \gg 0$ the term $8 \ln 2 \frac{x}{\ln x}$ on the right hand side is dominant over $\sqrt{x}$, the proposition follows.

Proposition 3. *There exists $c' > 0$ for which $\pi(x) \;>\; c' \frac{x}{\ln x}$ for all $x \gg 0$.*

Proof. Let $n \in \mathbf{Z}_{>0}$. By Lemma 1 we have

$$\binom{2n}{n} \;=\; \prod_{p\leq 2n} p^{e_p} \;\leq\; 2n^{\sqrt{2n}} \prod_{\sqrt{2n}<p\leq 2n} p \;\leq\; 2n^{\sqrt{2n}} \prod_{p\leq 2n} p.$$

Combining the above inequality with Remark 2 we get

$$\frac{2^{2n}}{2n+1} < 2n^{\sqrt{2n}} \prod_{p\leq 2n} p,$$

and, taking logarithms,

$$\sum_{p\le 2n} \ln p \;\ge\; n\ln 4 - \ln(2n+1) - \sqrt{2n}\ln(2n).$$

Since $\sum_{p\le 2n} \ln p \le \pi(2n)\ln(2n)$, we obtain the inequality

$$\pi(2n) \ge \frac{2n\ln 2 - \ln(2n+1) - \sqrt{2n}\ln(2n)}{\ln 2n}. \tag{1}$$

Observe that $x \le 2n \le x+2$, for some integer n. This fact togeteher with (1) implies

$$\begin{aligned}\pi(x) \;\ge\; \pi(2n) - 1 &\ge \frac{2n\ln 2 - \ln(2n+1) - \sqrt{2n}\ln(2n)}{\ln 2n} - 1\\ &\ge \ln 2\frac{x}{\ln x} - \frac{\ln(x+1)}{\ln(x)} - \sqrt{x} - 1\\ &\ge c'\frac{x}{\ln x}, \qquad \text{for } x \gg 0.\end{aligned}$$

Hence the proposition follows.