A CHARACTERIZATION OF BOUNDED SYMMETRIC DOMAINS OF TYPE IV

L. GEATTI, A. IANNUZZI, AND J.-J. LOEB

Abstract. Let Ω be a bounded symmetric domain of type IV and dimension bigger than four. We show that a Stein manifold of the same dimension as Ω and with the same automorphism group is biholomorphic to Ω.

1. Introduction

Let X be a complex manifold. The group $Aut(X)$ of holomorphic automorphisms of X endowed with the compact-open topology is a topological group. We say that X is characterized by its automorphism group if any complex manifold Y of the same dimension as X and such that $Aut(Y)$ is topologically isomorphic to $Aut(X)$ is biholomorphic to X.

Most manifolds are not characterized by their automorphism group. For instance two annuli $\{0 < r < |z| < R\}$ and $\{0 < r' < |z| < R'\}$ in the complex plane have isomorphic automorphism groups, but they are biholomorphic if and only if the ratio R/r is equal to R'/r'. In higher dimension similar examples are given by domains of the form $\{z \in \mathbb{C}^n \ : \ r < \|z\| < R\}$ (cf. Isaev [Is1]). By considering small deformations of the unit ball, one can construct infinitely many non-biholomorphic strictly pseudoconvex domains in $\mathbb{C}^n$ with trivial automorphism group (see [BSW]).

This suggests that connected manifolds which are characterized by their automorphism group should have a large automorphism group, e.g. a transitive one. The space $\mathbb{C}^n$ is an example of such manifolds ([Is1]). As it was shown by Isaev (see [Is2] and correction [Is3]), there exist exactly two non-biholomorphic complex manifolds of dimension n with the same automorphism group as the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$, namely $\mathbb{B}^n$ and the complement of its closure in the complex projective space $\mathbb{P}^n(\mathbb{C})$. Hence the unit ball $\mathbb{B}^n$ is characterized by its automorphism group in the class of Stein manifolds.

Kodama and Shimizu proved that the polydisc Δ^n is characterized by its automorphism group in the class of domains in Stein manifolds ([KdSh]). Furthermore, Isaev showed that Δ^n is characterized by its automorphism group among complex manifolds all of whose isotropy subgroups in the automorphism group are compact ([Is4]). He also asked whether such a characterization holds true in the class of all complex manifolds. In Section 1 we give a negative answer to this question by exhibiting an n-dimensional complex manifold Y which is not biholomorphic to Δ^n (not even homotopic) but has the same automorphism group as Δ^n (Example 2.1).

In this framework, it is natural to ask whether a bounded symmetric domain of dimension n is characterized by its automorphism group in the class of Stein manifolds. Let $\Omega = G/K$ be an irreducible bounded symmetric domain of dimension

Mathematics Subject Classification (2000): 32M15, 32Q28, 32Q45.

Key words: Bounded symmetric domain, automorphism group, Stein and hyperbolic manifolds.

Research partially supported by Istituto Nazionale di Alta Matematica.

n, where G is the connected component of its automorphism group and K is a maximal compact subgroup of G. As a first result we obtain the following partial characterization.

Proposition 3.1. *Let X be a Stein manifold of dimension n. Assume that G acts effectively on X and that K has a fixed point in X. Then X is biholomorphic to $\Omega = G/K$.*

The main result of the paper is the following characterization of bounded symmetric domains of type IV in the class of Stein manifolds.

Theorem 5.5. *Let Ω be the n-dimensional bounded symmetric domain of type IV and let X be a Stein manifold of the same dimension as Ω. If $n > 4$ and the connected component of $Aut(X)$ is isomorphic to G, then X is biholomorphic to Ω.*

In order to prove the above theorems, we embed X as a K-invariant domain in its Stein universal $K^{\mathbb{C}}$-globalization X^* (see Heinzner [He], Thm.6.6).

If K has a fixed point in X, then X^* is biholomorphic to a domain in $\mathbb{C}^n$ and X is realized as a complete circular domain invariant under a suitable linear K-action. By applying Cartan's Theorem on linear equivalence of complete circular domains in $\mathbb{C}^n$, we show that X contains the Harish-Chandra realization of Ω as an open G-orbit. Furthermore, by the analytic continuation principle the Harish-Chandra embedding $\mathbb{C}^n \to G^{\mathbb{C}}/Q$ is G-equivariant when restricted to X. Here $G^{\mathbb{C}}/Q$ denotes the compact dual symmetric space of Ω. As a consequence X is biholomorphic to a Stein, G-invariant domain in $G^{\mathbb{C}}/Q$. Then, from the G-invariant complex geometry of $G^{\mathbb{C}}/Q$, it follows that X is biholomorphic to Ω.

When K has no fixed points in X, we only consider symmetric pairs (G, K) of type IV. In this case we are able to determine the universal globalization X^* of X. This is done by classifying minimal K-orbits in X^* and the corresponding slice representations. As a result, X is realized as a K-invariant domain in a K-equivariant line bundle L over the affine complex quadric of dimension $n-1$. It turns out that for $n > 4$ such a bundle is necessarily trivial. In this case we show that no Stein, K-invariant subdomain of L can have G as an automorphism group. Indeed, by applying the criterion given in Lemma 5.1, we show that many of these domains have infinite dimensional automorphism group.

The results of this paper suggest that a characterization of a larger class of bounded symmetric domains by their automorphism groups may be possible. In this framework we also mention the following characterization of bounded symmetric domains in the class of Kobayashi hyperbolic manifolds, a result which we often use in the proof of the main theorem.

Proposition 3.2. *Let X be a hyperbolic manifold of dimension n. Assume that $Aut(X)$ is isomorphic to the automorphism group of a bounded symmetric domain Ω of the same dimension. Then X is biholomorphic to Ω.*

The paper is organized as follows. In Section 2 we collect some preliminary results and fix the notation. We also exhibit a complex manifold Y which is not biholomorphic to Δ^n and whose automorphism group is the same as $Aut(\Delta^n)$. In Section 3 we prove Proposition 3.1. and Proposition 3.2. In Section 4 we prove those preliminary results which are needed in the proof of Theorem 5.5. In Section 5 we conclude the proof of Theorem 5.5.

After this paper was completed, we became aware of the article [HuIs] by Huckleberry and Isaev, whose results have intersection with ours. Their goal is the classification of complex manifolds on which a classical real Lie group acts almost effectively by holomorphic transformations, without requiring that the group coincides with the automorphism group of the manifold. In addition, they assume a certain relation to hold between the complex dimension of the manifold and the dimension of the complex representation defining the group. Under that assumption the dimension of the group is “large” with respect to the dimension of the manifold. Bounded symmetric domains of type IV fulfill such condition and Theorem 5.5 can also be deduced from their classification. However, our methods are different from theirs. It should be remarked that among bounded symmetric domains only those of type IV fulfill the condition of [HuIs], and the question of a possible characterization of bounded symmetric domains by their automorphisms remains open.

2. Preliminaries

We begin this section by exhibiting two complex manifolds which are not biholomorphic and have the same automorphism group. Denote by Δ^n the polydisc in $\mathbb{C}^n$ and by Y the complement of its closure in $(\mathbb{P}^1(\mathbb{C}))^n$. The manifold Y is homotopic to $(\mathbb{P}^1(\mathbb{C}))^n \setminus \mathbb{C}^n$ which is not contractible. One can show that Y is neither holomorphically convex, nor holomorphically separable. In particular it is not biholomorphic to Δ^n.

Example 2.1. *The polydisc Δ^n and Y have the same automorphism group.*

Proof. It is well known that the group $Aut(\Delta^n)$ consists of the automorphisms of $(\mathbb{P}^1(\mathbb{C}))^n$ leaving Δ^n invariant. Therefore the restriction map defines an inclusion of $Aut(\Delta^n)$ into $Aut(Y)$.

Conversely let $g \in Aut(Y)$ and set $L = (\mathbb{P}^1(\mathbb{C}))^n \setminus \mathbb{C}^n$. Note that L is a compact subset of Y and that $Y \setminus L$ is contained in $\mathbb{C}^n$. In particular the restriction of g to $Y \setminus g^{-1}(L)$ is a holomorphic map into $\mathbb{C}^n$. By Hartog's theorem such restriction extends to a holomorphic map $\widetilde{g} : \mathbb{C}^n \setminus (g^{-1}(L) \cap \mathbb{C}^n) \to \mathbb{C}^n$. Gluing g and $\widetilde{g}$ together one obtains a holomorphic map $G : (\mathbb{P}^1(\mathbb{C}))^n \to (\mathbb{P}^1(\mathbb{C}))^n$. A similar argument applied to g^{-1} yields a holomorphic map F which, by the analytic continuation principle, is the inverse of G. It follows that every g in $Aut(Y)$ extends to an automorphism of $(\mathbb{P}^1(\mathbb{C}))^n$. This implies that $Aut(\Delta^n)$ and $Aut(Y)$ coincide.

A similar argument shows that the unit ball in $\mathbb{C}^n$ and the complement of its closure in $\mathbb{P}^n(\mathbb{C})$ have the same automorphism group. In the next lemma we recall some well known facts about the groups $SU(2) \times SU(2)$ and $SO(4)$, which are used in the sequel. For the sake of completeness, we include a proof.

Lemma 2.2. (i) *The only non-trivial, proper, connected, normal subgroups of $SU(2) \times SU(2)$ are*

$$\{I_2\} \times SU(2), \quad SU(2) \times \{I_2\}.$$

(ii) *The only 4-dimensional connected subgroups of $SU(2) \times SU(2)$ containing a non-trivial, proper normal subgroup are of the form*

$$S^1 \times SU(2), \quad SU(2) \times S^1.$$

(iii) *$SU(2) \times SU(2)$ is a double covering of $SO(4) \cong SU(2) \times SU(2)/\{\pm(I_2, I_2)\}$.*

(iv) *The restriction of the above covering map to $SU(2) \times \{I_2\}$ and to $\{I_2\} \times SU(2)$ is an embedding.*

Proof. (i) Let H be a proper connected normal subgroup of $SU(2) \times SU(2)$ and denote by $p_1 : SU(2) \times SU(2) \to SU(2)$ the projection onto the first component. Then $p_1(H)$ is normal and connected in $SU(2)$ and, since $\{\pm I_2\}$ is the unique non-trivial, proper normal subgroup of $SU(2)$, either $p_1(H)$ is trivial or $p_1(H) = SU(2)$.

In the first case H must coincide with $\{I_2\} \times SU(2)$, since it is a non-trivial, connected, normal subgroup of $\{I_2\} \times SU(2)$.

In the case when $p_1(H) = SU(2)$, denote by K the kernel of the restriction $p_1|_H$ of p_1 to H. Then K is a proper normal subgroup of $\{I_2\} \times SU(2)$, otherwise the dimension of H would be six, i.e. H would coincide with $SU(2) \times SU(2)$. It follows that K is a subgroup of in $\{I_2\} \times \{\pm I_2\}$.

We claim that H is contained in $SU(2) \times \{\pm I_2\}$. Assume by contradiction that there exists (g_1, g_2) in H with g_2 different from $\pm I_2$. Since H is invariant under conjugation by elements of the form $(I_2, g) \in SU(2) \times SU(2)$, it necessarily contains all elements of the form $\{g_1\} \times \gamma$ with $\gamma \in Int_{SU(2)}(g_2)$. Note that the conjugacy class $Int_{SU(2)}(g_2)$ of g_2 in $SU(2)$ contains a real 1-dimensional torus. As a consequence

$$(g_1^{-1}, g_2^{-1}) \cdot \{g_1\} \times Int_{SU(2)} g_2 = \{I_2\} \times g_2^{-1} Int_{SU(2)}(g_2)$$

is contained in K, contradicting the fact that K is a subgroup of $\{I_2\} \times \{\pm I_2\}$. Thus H is contained in $SU(2) \times \{\pm I_2\}$ as claimed.

Since H is connected and the restriction $p_1|_H$ is surjective, the statement follows.

(ii) Let H be a 4-dimensional connected subgroup of $SU(2) \times SU(2)$ containing e.g. $SU(2) \times \{I_2\}$ (see (i)). The kernel of $p_1|_H$ is a closed, 1-dimensional subgroup of $\{I_2\} \times SU(2)$ which is connected, being $p_1(H) = SU(2)$ simply connected. Thus it is of the form $\{I_2\} \times S^1$. It follows that H contains the product $(SU(2) \times \{I_2\})(\{I_2\} \times S^1) = SU(2) \times S^1$ and the statement is a consequence of the fact that H has dimension 4 and it is connected.

For (iii) and (vi) we briefly recall the standard construction of the universal covering of $SO(4)$. Consider the usual identification of $SU(2)$ with the group of units U in the ring of quaternions $\mathbb{H}$. Observe that the action of $U \times U$ on $\mathbb{H} \cong \mathbb{R}^4$ defined by $(g, h) \cdot v = gvh^{-1}$ is given by isometries, i.e. elements in $SO(4)$. Then the induced map $U \times U \to SO(4)$ realizes the desired covering map. For further details we refer to [BtD], p.292. □

Throughout the paper a Lie group is denoted by a capital latin letter and its Lie algebra by the corresponding gothic letter. The connected component of the identity of a Lie group L is denoted by L^0. For a Lie group L acting on a complex manifold X the algebra of L-invariant holomorphic functions on X is denoted by $\mathcal{O}(X)^L$.

3. The case when K has a fixed point in X.

Let Ω be an irreducible bounded symmetric domain of dimension n. In this section we show that Ω is characterized by its automorphism group among Stein manifolds X of the same dimension, under the assumptions that a maximal compact subgroup of $Aut(X)^0$ has a fixed point in X. Write $\Omega \cong G/K$, where $G = Aut(\Omega)^0$ is a centerless, connected, simple Lie group and K is a maximal compact subgroup of G (see [Wo2], p.293). Denote by $G^{\mathbb{C}}$ the universal complexification of G and by U the compact real form of $G^{\mathbb{C}}$ containing K. As G/K is a Hermitian symmetric space, $rank(G) = rank(K) = rank(U)$ and the center of K is one-dimensional. Write $K = Z \cdot K_s$, where $Z \cong S^1$ denotes the connected center of K and $K_s = [K, K]$. Fix $\mathfrak{t}$ a compact Cartan subalgebra of $\mathfrak{k}$, $\mathfrak{g}$ and $\mathfrak{u}$. Then $\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{t}_s$, where $\mathfrak{z}$ is the Lie algebra of Z and $\mathfrak{t}_s$ is a Cartan subalgebra of $\mathfrak{k}_s$. Set $r = \dim_{\mathbb{R}} \mathfrak{t} = rk(\mathfrak{g})$. The adjoint action of $\mathfrak{t}^{\mathbb{C}}$ decomposes the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ into root spaces

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{Z \in \mathfrak{g}^{\mathbb{C}} \mid [H, Z] = \alpha(H)Z,\ \forall H \in \mathfrak{t}^{\mathbb{C}}\}$ and α is a complex linear functional on $\mathfrak{t}^{\mathbb{C}}$. The root system Δ of $\mathfrak{g}^{\mathbb{C}}$ consists of those α for which $\mathfrak{g}_\alpha \neq \{0\}$. For $\alpha \in \Delta$, denote by $h_\alpha \in i\mathfrak{t}$ the associated coroot, defined by the condition $\alpha(h_\alpha) = 2$. For a fixed positive system Δ^+, denote by $\{\alpha_1, \ldots, \alpha_r\}$, the corresponding set of simple roots. The set of fundamental weights $\{\omega_1, \ldots, \omega_r\}$ is by definition the dual basis of the coroots $\{h_{\alpha_1}, \ldots, h_{\alpha_r}\}$. All irreducible representations of U are finite dimensional and stand in one-to-one correspondence with dominant, analytically integral linear functionals $\Lambda = \sum_i \Lambda_i \omega_i$, where $\Lambda_i = \Lambda(h_{\alpha_i})$ are non-negative integer coefficients. The correspondence is that Λ is the highest weight of the associated irreducible representation ρ_Λ (see [BtD], p.253). By definition, the fundamental representations $\phi_1, \ldots, \phi_r$ are those corresponding to the highest weights $\omega_1, \ldots, \omega_r$, respectively. The dimension of the irreducible representation ρ_Λ with highest weight Λ is given by Weyl's dimension formula

$$\dim(\rho_\Lambda) = \prod_{\alpha > 0} \frac{\langle \Lambda + \delta, \alpha \rangle}{\langle \delta, \alpha \rangle} = \prod_{\alpha > 0} \frac{\sum_{i=1}^r k_\alpha^i (\Lambda_i + 1) \langle \alpha_i, \alpha_i \rangle}{\sum_{i=1}^r k_\alpha^i \langle \alpha_i, \alpha_i \rangle},$$

where $\alpha = \sum_i k_\alpha^i \alpha_i$ and δ denotes the half-sum of all positive roots. Observe that $\langle \Lambda, \alpha \rangle \geq 0$ and $\langle \delta, \alpha \rangle > 0$, for every $\alpha \in \Delta^+$ (see [Ka], (XII.6), p.101). Hence each factor $\frac{\langle \Lambda + \delta, \alpha \rangle}{\langle \alpha, \delta \rangle}$ in the above formula is greater or equal than 1. It follows that given irreducible representations ρ_Λ and $\rho_{\Lambda'}$ with highest weights $\Lambda = \sum_i \Lambda_i \omega_i$ and $\Lambda' = \sum_i \Lambda'_i \omega_i$, respectively, one has

$$\dim(\rho_\Lambda) \geq \dim(\rho_{\Lambda'}),$$

provided that $\Lambda_i \geq \Lambda'_i$ for all i and $\Lambda_j > \Lambda'_j$, for at least one index j. In particular, a fundamental representation of the lowest dimension has also the lowest dimension among all irreducible representations.

Bounded symmetric domains fall into four infinite families and two exceptional cases (see [Hl], p.518). For each bounded symmetric domain $\Omega = G/K$, the following table contains the type, the group G (up to finite center) , its maximal compact subgroup K, the isotropy representation and the complex dimension of Ω.

AIII	$SU(p,q)$	$S(U(p) \times U(q))$	$\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}$	$n = pq, \quad 1 \leq p \leq q$
BDI	$SO_0(n,2)$	$SO(n)SO(2)$	$\mathbb{C}^n \otimes \mathbb{C}$	$n \geq 3$
DIII	$SO(2p)^*$	$U(p)$	$\Lambda^2\mathbb{C}^p \otimes \mathbb{C}$	$n = \frac{p(p-1)}{2}, \quad p \geq 3$
CI	$Sp(p,\mathbb{R})$	$U(p)$	$S^2\mathbb{C}^p \otimes \mathbb{C}$	$n = \frac{p(p+1)}{2}, \quad p \geq 3$
EIII	$E_{6(-14)}$	$Spin(10)SO(2)$	$\mathbb{C}^{16} \otimes \mathbb{C}$	$n = 16$
EVII	$E_{7(-25)}$	$E_6 S^1$	$\mathbb{C}^{27} \otimes \mathbb{C}$	$n = 27$

The isotropy representation of the domain $\overline{\Omega}$, with the opposite complex structure, is equivalent to the corresponding dual representation (see [Wo2], p.287).

Proposition 3.1. *Let X be a Stein manifold of dimension n. Assume that G acts effectively on X and that K has a fixed point in X. Then X is biholomorphic to $\Omega = G/K$.*

Proof. Let $x_0 \in X$ be a fixed point of K. We claim that x_0 cannot be a fixed point of G. Suppose by contradiction that $G \cdot x_0 = x_0$ and denote by

$$\iota_0 \colon G \to GL(n, \mathbb{C})$$

the isotropy representation at x_0. Since G is a centerless, simple Lie group (i.e. it has no non-trivial normal subgroups) and the G-action on X is effective, ι_0 is an injective homomorphism. The claim follows from the fact that G has no effective representation of dimension n. Indeed a complex linear action of G on a finite dimensional complex vector space extends to the universal complexification

$G^{\mathbb{C}}$ of G and, by restriction, to its compact real form U. One can check that all fundamental (and therefore all irreducible) representations of $SO(n+2)$, E_6 and E_7 have dimension strictly larger than n (see [BtD], Ch.6, Sect.5, for the dimensions of the fundamental representations of $SO(n+2)$, and [LiE], for the dimensions of the fundamental representations of the exceptional Lie groups). This settles cases BDI, $EIII$ and $EVII$.

For the remaining cases $AIII$, CI and $DIII$, it is sufficient to check (see [BtD], Ch.6, Sect.5) that the fundamental representations of the groups $U = SU(p+q)$, $Sp(p)$ and $SO(2p)$ satisfy the inequalities

$$\dim(\phi_1) = \dim(\rho_{\omega_1}) < n < \dim(\rho_{2\omega_1}) \quad \text{and} \quad n < \dim(\phi_j), \quad \text{for } 2 \leq j \leq rk(U)\,.$$

This concludes the proof of the claim.

Denote by G_0 the isotropy subgroup of x_0 in G. By the above claim, G_0 is a proper subgroup of G containing K. Since K is *maximal* in G, one has $G_0 = K$. It follows that $G \cdot x_0$ is an open orbit in X, diffeomorphic to Ω. Recall that the complex structures of Ω and of $\overline{\Omega}$ (opposite to each other) are the only possible complex structures on the quotient G/K (see [Wo1]). Likewise, the isotropy representation of K at x_0 is an irreducible n-dimensional representation equivalent either to a representation in the above table, or to its dual representation. Observe that in each case the center $Z \cong S^1$ of K acts on $T_{x_0}X \cong \mathbb{C}^n$ by scalar multiplication $z \mapsto e^{im\theta}z$, with $m = \pm 1$.

Since the group K is compact, there exist an open K-invariant neighbourhood W of x_0 and a K-equivariant holomorphic open embedding $\phi\colon W \to \mathbb{C}^n$ such that $\phi(x_0) = 0$ and K acts linearly on $\mathbb{C}^n$ via the isotropy representation. It follows that $\mathcal{O}(U)^{S^1} = \mathcal{O}(U)^K = \mathbb{C}$ and, by the analytic continuation principle,

$$\mathcal{O}(X)^K = \mathbb{C}.$$

By a result of P. Heinzner (cf. [He], Appl.(a), p.660) the manifold X admits a global linearly K-equivariant holomorphic open embedding in $\mathbb{C}^n$. In this way it can be identified with a Stein K-invariant circular domain in $\mathbb{C}^n$ which, by S^1-orbit-convexity, is in fact complete circular (cf. [He], Sect. 3).

Observe that $G \cdot 0$ itself is a Stein K-invariant complete circular domain in $\mathbb{C}^n$ containing the origin. Then by a theorem of Cartan ([Ca]), there exists a linear biholomorphism $L\colon \mathbb{C}^n \to \mathbb{C}^n$ mapping $G \cdot 0$ into the bounded symmetric domain Ω in its Harrish-Chandra realization and X into a domain $L(X)$ containing Ω. Consider now the Harish-Chandra embedding $\xi\colon \mathbb{C}^n \to G^{\mathbb{C}}/Q$, mapping $\mathbb{C}^n$ into an open dense subset of the compact dual hermitian symmetric space $G^{\mathbb{C}}/Q$. The image of Ω under ξ is the open orbit of $[Q]$ under left translations by G. By the analytic continuation principle G acts by left-translations on the whole $\xi(L(X))$. In this way $\xi(L(X))$ is a Stein G-invariant domain in $G^{\mathbb{C}}/Q$ properly containing $\xi(\Omega)$. From the analysis of the G-invariant complex geometry of $G^{\mathbb{C}}/Q$ (see [FHW], Sect.3.2, or [Wo1]) it follows that $\xi(L(X)) = \xi(\Omega)$ and X is biholomorphic to Ω, as desired.

□

The next proposition shows that an arbitrary bounded symmetric domain is characterized by its automorphism group in the class of hyperbolic manifolds.

Proposition 3.2. *Let X be a hyperbolic manifold of dimension n. Assume that $Aut(X)$ contains the automorphism group G of an n-dimensional, bounded symmetric domain Ω as a closed subgroup. Then X is biholomorphic to Ω.*

Proof. Since X is hyperbolic the G-action on X is proper. In particular all isotropy subgroups in G are compact. Moreover, by dimensional reasons, such subgroups are maximal compact in G and all G-orbits are open. Since X is connected, it consists of a single G-orbit and therefore it is biholomorphic to $\Omega \cong G/K$, as whished. $\square$

Note that under the assumptions of the above proposition it turns out that X is necessarily Stein and K has a fixed point in X.

4. The case when K has no fixed points in X

In the remaining part of the paper we restrict our attention to bounded symmetric domains of type IV. Let $\Omega = G/K$ denote the n-dimensional bounded symmetric domain of type IV. Then the real Lie group

$$G = SO_0(n,2), \quad n \text{ odd}, \qquad G = SO_0(n,2)/\{\pm I_{n+2}\}, \quad n \text{ even},$$

acts transitively on Ω and coincides with the connected group of holomorphic automorphisms of Ω. A maximal compact subgroup K of G is given by

$$SO(n) \times SO(2), \quad n \text{ odd}, \qquad SO(n) \times SO(2)/\{\pm(I_n, I_2)\}, \quad n \text{ even}.$$

Observe that the maps $s \mapsto (s, I_2)$ and $t \mapsto (I_n, t)$ define embeddings of $SO(n)$ and $SO(2)$ into K, respectively.

In this section we assume that the maximal subgroup K of G acts on X without fixed points. Since X is Stein, it can be realized as a K-invariant domain in its universal globalization X^* ([He], Thm.6.6). By classifiying all possible minimal K-orbits in X^* we explicitly determine X^*. In Section 5 we will show that for $n > 4$ no K-invariant Stein domain in X^* has automorphism group isomorphic to G.

Remark 4.1. (i) Since K is a real form of $K^{\mathbb{C}}$, by the analytic continuation principle a K-fixed point in X^* is also a $K^{\mathbb{C}}$-fixed point. Moreover since $X^* = K^{\mathbb{C}} \cdot X$, therefore all K-fixed points in X^* lie in X. In particular, if K has no fixed points in X then it has no fixed points in X^*.
(ii) If K acts effectively on X, then it acts effectively on X^*.

Since K is compact, it acts on X^* by isometries with respect to a suitable K-invariant metric. We fix one such metric.

Lemma 4.2. *If K has no fixed points in X, then $SO(n)$ has no fixed points in X^*.*

Proof. By Remark 4.1(i) there are no K-fixed points in X^*. Assume by contradiction that there exists an $SO(n)$-fixed point $x \in X^*$. Since $SO(2)$ is central in K, one has

$$stx = tsx = tx, \qquad \text{for all } t \in SO(2),\ s \in SO(n).$$

So $SO(n)$ fixes the entire $SO(2)$-orbit of x. On the other hand, the restriction of the isotropy representation at x to $SO(n)$ is the standard action on $\mathbb{C}^n$, since the only complex n-dimensional representation of $SO(n)$ is the standard representation (see [BtD], Ch.6, Sect.5). As a consequence x is an isolated $SO(n)$-fixed point. This implies that x is fixed by all elements of $SO(2)$ and therefore of K, giving a contradiction. $\square$

Definition 4.3. *Given a K-invariant, strictly plurisubharmonic exhaustion function ρ of X^*, a minimal orbit is any K-orbit in the minimum set of ρ.*

A minimal K-orbit is totally real (see [HaWe]) and, as a consequence of [AzLo2], Sect. 3, one has

Lemma 4.4. *Let $x \in X^*$ be an element on a minimal K-orbit. Then the $K^{\mathbb{C}}$-orbit through x is closed.*

Proposition 4.5. *Assume that K has no fixed points in X^*. If $n \neq 4$, then there exists a minimal K-orbit in X^* of dimension either equal to n or to $n-1$. If $n = 4$, then there exists a minimal K-orbit in X^* of dimension greater than 1.*

Proof. Let $\rho\colon X^* \to \mathbb{R}$ be a K-invariant strictly plurisubharmonic exhaustion function of X^*. Then the minimum set $min(\rho)$ is not empty and consists of minimal K-orbits. Since such orbits are totally real, their dimension is at most n. Let M be one such orbit. Observe that $SO(n)$ acts on M almost effectively. This follows from the fact that for $n \neq 4$ the group $SO(n)$ is simple (hence all normal subgroups are discrete) and by Lemma 4.2 it has no fixed points in X^*. Hence its ineffectivity on M is necessarily discrete. Since $\dim SO(n) = \frac{n(n-1)}{2}$, by the bound on the dimension of an almost effective group of isometries of a compact manifold (see [Ko1], Thm.3.1, p.46), it follows that $\dim M \geq n-1$, as claimed.

If $n = 4$, then the group $SO(4)$ is semisimple (covered by $SU(2) \times SU(2)$). We need to show that a K-orbit M in X^* cannot be one-dimensional. Indeed $SO(4)$ acts non-trivially on M by Lemma 4.2. If M were one-dimensional, its isometry group would be one-dimensional and $SO(4)$ would contain a five-dimensional ineffective normal subgroup. But this is impossible by Lemma 2.2. □

4.1. **The case of a minimal orbit of dimension n.**

In this subsection we obtain an explicit classification of X^* under the assumption that it contains a minimal K-orbit M of dimension n. Since M is a totally real submanifold of maximal dimension in X^*, one has $\mathcal{O}(X^*)^K = \mathbb{C}$. The $K^{\mathbb{C}}$-orbit through an element $x \in M$ is closed by Lemma 4.4. It is also open since it is n-dimensional. Therefore one has

$$X^* = K^{\mathbb{C}} \cdot x.$$

In other words, X^* is completely determined by M: if $M = K/L$, then $X^* = K^{\mathbb{C}}/L^{\mathbb{C}}$.

Lemma 4.6. (i) *The subgroup $SO(n)$ acts effectively on each of its orbits in M.*
(ii) *The subgroup $SO(2)$ acts freely on M.*

Proof. (i) Assume that there exists an element $s \in SO(n)$ acting trivially on some $SO(n)$-orbit S in M. Since $SO(2)$ is central in K, one has $stx = tsx = tx$, for all $x \in S$ and $t \in SO(2)$. Then the element s acts trivially on $M = SO(2) \cdot S$ and therefore on X^*, contradicting the effectivity of K.
(ii) Assume that there exist $t \in SO(2)$ and $x \in M$ such that $tx = x$. Since $SO(2)$ is central in K, one has $tkx = ktx = kx$, for all $k \in K$. Hence t acts trivially on M and on X^*, contradicting again the effectivity of K. □

Proposition 4.7. *If M is $SO(n)$-homogeneous, then $n = 3$ and $M = SO(3)/F$, for some finite subgroup F of $SO(3)$. In particular $X^* = SO(3, \mathbb{C})/F$.*

Proof. By Lemma 4.6(ii), the $SO(2)$-action on M defines an $SO(2)$-principal bundle

$$M \to M/SO(2). \tag{1}$$

Since M is $SO(n)$-homogeneous and the above projection is $SO(n)$-equivariant the base $N := M/SO(2)$ is an $SO(n)$-homogeneous manifold of dimension $n-1$.

Claim. $SO(n)$ acts on N almost effectively, with isotropy subgroup $SO(n-1)$ or $O(n-1)$.

Proof of the Claim. Since N is positive dimensional and $SO(n)$-homogeneous, the $SO(n)$-action on N is non-trivial. If $n \neq 4$, then the claim follows directly from the simplicity of $SO(n)$. If $n = 4$, assume by contradiction that $SO(4)$ does not act almost effectively on N and denote by T the 3-dimensional ineffectivity normal subgroup. Then the T-action on the 1-dimensional fibers of the fibration (1) is necessarily trivial, otherwise the points on those fibers would have a 2-dimensional stabilizer in T. But this is impossible by Lemma 2.2. Hence the T-action is trivial on M and therefore on X^*, contradicting the effectivity of K. Finally by [Ko1], Thm.3.1, p.46, one has $N = S^{n-1}$ or $N = \mathbb{P}^{n-1}(\mathbb{R})$, and the claim follows.

Since M is $SO(n)$-homogeneous and the projection (1) is $SO(n)$-equivariant, the isotropy subgroup in $SO(n)$ of a point in N acts transitively on its fiber. Let $\chi\colon SO(n-1) \to Diff(S^1)$ (respectively $\chi\colon O(n-1) \to Diff(S^1)$) be such an action. We claim that χ is a character, i.e. it is given by linear transformations. Fix $t_0 \in S^1$ and write

$$\chi(s)t_0 = b(s)t_0,$$

where $b(s)$ denotes an element in $SO(2)$ depending on $s \in SO(n-1)$ (respectively $O(n-1)$). Recall that the central subgroup $SO(2)$ acts freely on S^1, and therefore it may be identified with S^1. In particular, an element $t \in S^1$ can be written as $t = ut_0$, for some $u \in SO(2)$. Moreover for $t = ut_0 \in S^1$ one has

$$\chi(s)t = \chi(s)ut_0 = u\chi(s)t_0 = ub(s)t_0 = b(s)ut_0 = b(s)t\,,$$

showing that χ is a character, as claimed.

Observe that for $n > 3$ the connected semisimple group $SO(n-1)$ has no non-trivial characters, and anyway the character $\chi\colon O(n-1) \to \mathbb{C}^*$ defined by $g \mapsto \det(g)$ does not act transitively on the fibers of (1). It follows that $n = 3$. In this case $M \cong SO(3)/F$, for some finite subgroup F of $SO(3)$ and $X^* = SO(3,\mathbb{C})/F$. This concludes the proof of the lemma. $\square$

Proposition 4.8. *Assume that X^* contains a minimal orbit $M = K/L$ of dimension n. If M is not $SO(n)$-homogeneous, then M and X^* are given by the following table:*

$n \geq 3$	M	L	X^*
odd	$S^{n-1} \times S^1$	$SO(n-1) \times \{I_2\}$	$Q^{n-1} \times \mathbb{C}^*$
odd	$\mathbb{P}^{n-1}(\mathbb{R}) \times S^1$	$O(n-1) \times \{I_2\}$	$Q^{n-1}/\mathbb{Z}_2 \times \mathbb{C}^*$
odd	$S^{n-1} \times_{\mathbb{Z}_2} S^1$	$\Gamma \cdot SO(n-1) \times \{I_2\}$	$Q^{n-1} \times_{\mathbb{Z}_2} \mathbb{C}^*$
even	$S^{n-1} \times_{\mathbb{Z}_2} S^1$	$SO(n-1) \times \{I_2\}$	$Q^{n-1} \times_{\mathbb{Z}_2} \mathbb{C}^*$

where $\Gamma = \{(I_n, I_2), (\left(\begin{smallmatrix} -I_2 & 0 \\ 0 & I_{n-2} \end{smallmatrix}\right), I_2)\}$ *and* $Q^{n-1} \cong SO(n,\mathbb{C})/SO(n-1,\mathbb{C})$ *is the affine complex quadric of dimension* $n-1$.

Proof. Consider the quotient map $M \to M/SO(n)$. Since M is K-homogenous, the $SO(2)$-action is free on M by Lemma 4.6(ii), and the orbit map is $SO(2)$-equivariant, one has that the orbit space $M/SO(n)$ is a 1-dimensional $SO(2)$-homogeneous space. It follows that the $SO(n)$-orbits in M have dimension $n-1$ and are all of the same type. More precisely by Lemma 4.6(i) and [Ko1], Thm.3.1, p.46, they are diffeomorphic either to the sphere S^{n-1} or, for n odd, to the real projective space $\mathbb{P}^{n-1}(\mathbb{R})$.

In order to obtain the above table, we explicitly determine the isotropy subgroup L in K of an element $x \in M$. If $L_1 := SO(n) \cap L$ denotes the isotropy subgroup of x in $SO(n)$, then there are two possibilities:

$$L_1 \;=\; O(n-1), \qquad L_1 \;=\; SO(n-1)\,.$$

Claim. Assume that n is odd. Then there are the following possibilities for L

$$O(n-1) \times \{I_2\}, \quad SO(n-1) \times \{I_2\}, \quad \Gamma \cdot SO(n-1) \times \{I_2\},$$

where Γ is the subgroup of K consisting of the elements $\{(I_n, I_2), (\left(\begin{smallmatrix} -I_2 & 0 \\ 0 & I_{n-2} \end{smallmatrix}\right), I_2)\}$.

Proof of the claim. When n is odd, $K = SO(n)\times SO(2)$. Denote by $p_1\colon K \to SO(n)$ the projection onto the first factor. Then the following inclusions hold

$$SO(n-1) \subset L_1 \subset p_1(L).$$

We need to distinguish some cases.

(1.a) If $L_1 = O(n-1)$, then by the maximality of $O(n-1)$ in $SO(n)$, one has $p_1(L) = L_1 = O(n-1)$. In this case $L = O(n-1) \times \{I_2\}$: indeed, if L contains an element (s,t) with $s \in p_1(L)$ and $t \neq I_2$, then by multiplying on the left by an element in L_1 it is easy to see that it also contains the central element (I_{n-1}, t). But this contradicts the effectivity of K, since an element of this form acts trivially on M and therefore on X^*.

(1.b) If $L_1 = SO(n-1)$, then either $p_1(L) = L_1 = SO(n-1)$ or $p_1(L) = O(n-1)$. In the first case, the same argument as the one used in (1.a) shows that $L = SO(n-1) \times \{e\}$.

(1.c) If $L_1 = SO(n-1)$ and $p_1(L) = O(n-1)$, let (s,t) be an element in L, with $s \in p_1(L)$. If $p_1(s) \in SO(n-1)$, then the same argument as the one used in (1.a) shows that $t = I_2$. If $p_1(s) \in O(n-1) \setminus SO(n-1)$, then L contains the element (γ, t) with $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix}$. In particular L also contains the element $(\gamma^2, t^2) = (I_{n-1}, t^2)$. Effectivity on the action forces $t^2 = I_2$ and $t = \pm I_2$. More precisely $t = -I_2$, otherwise it would be $L_1 \neq SO(n-1)$. In this case $L = SO(n-1) \times \{I_2\} \cup (\gamma, -I_2)SO(n-1) \times \{I_2\}$. This concludes the discussion for n odd.

Claim. Assume that n is even. Then $L = SO(n-1) \times \{e\}$.

Proof of the claim. In this case $K = SO(n) \times SO(2)/\{\pm(I_n, I_2)\}$ and $L_1 = SO(n-1)$, since n is even and by Lemma 4.6(ii) the action of $SO(n)$ is effective on each of its orbits. Consider the projection $p\colon SO(n) \to PSO(n)$. Observe that $O(n-1)$ is a maximal p-saturated subgroup of $SO(n)$. Moreover, $p(O(n-1)) = p(SO(n-1))$ is a maximal subgroup of $PSO(n)$. Consider now the projection

$$p_1 : K \to PSO(n), \qquad [g,t] \mapsto [g]\,.$$

One has $p_1(L) = p(SO(n-1))$. Let $[s,t]$ be an element in L, with $s \in p_1(L)$ and $t \in SO(2)$. If $s \in SO(n-1)$, then by multiplying on the left by an element in L_1, we see that the element $[I_n, t]$ lies in L as well. Effectivity of the action, then implies that $t = I_2$. If $s \in O(n-1)\setminus SO(n-1)$, then by multiplying on the left by an element in L_1, we see that the element $[\gamma, t]$, with $\gamma = -I_{n-1}$, lies in L. In particular, the element $[\gamma,t]^2 = [\gamma^2, t^2] = [I_{n-1}, t^2]$, lies in L as well. Effectivity of the action implies $t = \pm I_2$. More precisely $t = -I_2$, otherwise it would be $L_1 \neq SO(n-1)$. Since $[\gamma, -I_2] = [-I_{n-1}, -I_2] = [I_{n-1}, I_2]$, one has $L = SO(n-1) \times \{I_2\}$ as claimed. This concludes the discussion for n even and concludes the proof of the proposition. □

4.2. **The case of a minimal orbit of dimension $n-1$.** In this subsection we obtain an explicit classification of X^* under the assumption that it contains a minimal K-orbit M of dimension $n-1$.

Lemma 4.9. (i) *The subgroup $SO(n)$ acts almost effectively on M. As a consequence M is $SO(n)$-homogeneous and is either diffeomorphic to the sphere S^{n-1} or to the real projective space $\mathbb{P}^{n-1}(\mathbb{R})$.*

(ii) *The subgroup $SO(2)$ acts trivially on M.*

Proof. (i) If $n \neq 4$, then the group $SO(n)$ is simple, and therefore has no non-trivial characters. Note that $SO(n)$ acts non-trivially on M. Otherwise, by the effectivity of the action of $SO(n)$ on X^*, the slice representation at a point in M would define

a non-trivial character of $SO(n)$. It follows that $SO(n)$ acts almost effectively on M. If $n = 4$, then $SO(4)$ is semisimple and all of its normal subgroups contain a copy of $SU(2)$. If $SO(4)$ does not act almost effectively on M, then its ineffectivity is given by a 3-dimensional normal subgroup T (see Lemma 2.2). Observe that by the effectivity of the action on X^*, the group T necessarily acts effectively on every 1-dimensional local complex slice. But this is impossible, since T contains a subgroup isomorphic to $SU(2)$ (see Lemma 2.2) which has no non-trivial characters.
(ii) By [Ko1], Thm.3.1, p.46, there exists a 1-dimensional normal subgroup H of K acting trivially on M. Denote by $p_1(H)$ the projection of H into $SO(n)$ for n odd, or into $PSO(n)$ for n even. Since $p_1(H)$ is a normal subgroup of $SO(n)$ (resp. of $PSO(n)$), it is discrete. This implies (ii).

□

Proposition 4.10. *Assume that X^* contains a minimal orbit $M = K/L$ of dimension $n-1$. Then X^* is given by a twisted product $K^{\mathbb{C}} \times_{L^{\mathbb{C}}} \mathbb{C}$, determined by a character $\nu\colon L^{\mathbb{C}} \to \mathbb{C}^*$. All possible M and X^* are listed in the following tables:*

$n > 3$	M	L	$\nu\colon L^{\mathbb{C}} \to \mathbb{C}^*$	X^*
odd	S^{n-1}	$SO(n-1) \times SO(2)$	$\nu(s,t) = t^{\pm 1}$	$Q^{n-1} \times \mathbb{C}$
odd	$\mathbb{P}^{n-1}(\mathbb{R})$	$O(n-1) \times SO(2)$	$\nu(s,t) = t^{\pm 1}$	$Q^{n-1}/\mathbb{Z}_2 \times \mathbb{C}$
odd	S^{n-1}	$O(n-1) \times SO(2)$	$\nu(s,t) = \det(s)t^{\pm 1}$	$Q^{n-1} \times_{\mathbb{Z}_2} \mathbb{C}$
even	S^{n-1}	$O(n-1) \times SO(2)/\{\pm(I_{n-1}, I_2)\}$	$\nu(s,t) = \det(s)t^{\pm 1}$	$Q^{n-1} \times_{\mathbb{Z}_2} \mathbb{C}$

and, for $n = 3$ and $K = SO(3) \times SO(2)$,

M	L	$\nu\colon L^{\mathbb{C}} \to \mathbb{C}^*$	X^*
S^2	$SO(2) \times SO(2)$	$\nu(s,t) = s^m t^{\pm 1},\ m \in \mathbb{Z}$	$K^{\mathbb{C}} \times_\nu \mathbb{C}$
$\mathbb{P}^2(\mathbb{R})$	$O(2) \times SO(2)$	$\nu(s,t) = s^m t^{\pm 1},\ m \in \mathbb{Z}$	$K^{\mathbb{C}} \times_\nu \mathbb{C}$
$\mathbb{P}^2(\mathbb{R})$	$O(2) \times SO(2)$	$\nu(s,t) = \det(s) s^m t^{\pm 1},\ m \in \mathbb{Z}$	$K^{\mathbb{C}} \times_\nu \mathbb{C}$

Proof. Let $M = K/L$ be a minimal (n-1)-dimensional orbit in X^*. Note that from Lemma 4.9, the subgroup L contains $SO(2)$ and $L \cap SO(n)$ is either $SO(n-1)$ or $O(n-1)$. Let $x \in M$. By Lemma 4.4 the $K^{\mathbb{C}}$-orbit through x is closed in X^* and by Luna's slice theorem (cf. [Lu]), there exists a $K^{\mathbb{C}}$-invariant open neighbourhood of M in X^* biholomorphic to the twisted product $K^{\mathbb{C}} \times_{L^{\mathbb{C}}} \mathbb{C}$. Here the group $L^{\mathbb{C}}$ acts on the complex slice $\mathbb{C}$ via the slice representation $\nu\colon L^{\mathbb{C}} \to \mathbb{C}^*$. Since $SO(2) \subset L$ acts trivially on M, it necessarily acts effectively on the 1-dimensional complex slice at x. Otherwise an element $t \in SO(2)$ acting trivially on the slice would act trivially on $M \times S^1$ and therefore on X^*, contradicting the effectivity of K on X^*. Effectivity of the action also implies

$$\nu_{|SO(2)}(t) \ = \ t^{\pm 1}$$

and $\mathcal{O}(X^*)^K = \mathbb{C}$. Then by [He], Appl.(a), p.660, one has $X^* = K^{\mathbb{C}} \times_{L^{\mathbb{C}}} \mathbb{C}$, as claimed.

Now we determine the pairs $(L^{\mathbb{C}}, \nu)$ and the corresponding twisted products.

(a) Assume that n is odd and $L = SO(n-1) \times SO(2)$.
If $n > 3$, since $SO(n-1, \mathbb{C})$ is semisimple, effectivity of the action implies that a character $\nu\colon L^{\mathbb{C}} \to \mathbb{C}^*$ is necessarily of the form $\nu(s,t) = t^{\pm 1}$. In this case $K^{\mathbb{C}} \times_{L^{\mathbb{C}}} \mathbb{C} \cong Q^{n-1} \times \mathbb{C}$.
If $n = 3$, one has $\nu(s,t) = s^m t^{\pm 1}$, for $m \in \mathbb{Z}$, yielding the corresponding twisted products $K^{\mathbb{C}} \times_\nu \mathbb{C}$.

(b) Assume that n is odd and $L = O(n-1) \times SO(2)$.
If $n > 3$, there are two possibilities. If the character $\nu\colon L^{\mathbb{C}} \to \mathbb{C}^*$ is of the form $\nu(s,t) = t^{\pm 1}$, then

$$K^{\mathbb{C}} \times_{L^{\mathbb{C}}} \mathbb{C} \cong Q^{n-1}/\mathbb{Z}_2 \times \mathbb{C}\,.$$

If the character $\nu\colon L^{\mathbb{C}} \to \mathbb{C}^*$ is of the form $\nu(s,t) = \det(s)t^{\pm 1}$, then

$$K^{\mathbb{C}} \times_{L^{\mathbb{C}}} \mathbb{C} \cong Q^{n-1} \times_{\mathbb{Z}_2} \mathbb{C}.$$

Similarly, if $n = 3$ a character $\nu\colon L^{\mathbb{C}} \to \mathbb{C}^*$ is either of the form $\nu(s,t) = s^m t^{\pm 1}$ or $\nu(s,t) = \det(s)s^m t^{\pm 1}$, for $m \in \mathbb{Z}$ and one obtains the corresponding twisted products $K^{\mathbb{C}} \times_\nu \mathbb{C}$.

(c) Assume that n is even. In this case $K = SO(n) \times SO(2)/\{\pm(I_n, I_2)\}$ and L is a quotient $\widehat{L}/\{\pm(I_n, I_2)\}$ of a subgroup $\widehat{L}$ of $SO(n) \times SO(2)$ containing $\{\pm(I_n, I_2)\}$. As a consequence $L = O(n-1) \times SO(2)/\{\pm(I_{n-1}, I_2)\}$.
A character $\nu\colon L^{\mathbb{C}} \to \mathbb{C}^*$ is necessarily of the form $\nu(s,t) = \det(s)t^{\pm 1}$, since it has to satisfy $\nu(I_{n-1}, I_2) = \nu(-I_{n-1}, -I_2)$. The corresponding twisted product is given by

$$K^{\mathbb{C}} \times_{L^{\mathbb{C}}} \mathbb{C} \cong Q^{n-1} \times_{\mathbb{Z}_2} \mathbb{C}.$$

□

4.3. **The four dimensional case with a minimal orbit of dimension 2.** In this subsection we consider the missing four-dimensional case (cf. Prop. 4.5, 4.8 and 4.10) and we determine X^* under the assumption that it contains a minimal K-orbit M of dimension 2.

Lemma 4.11. (i) *The subgroup $SO(4)$ acts transitively on M with 3-dimensional ineffectivity. As a consequence $M = SO(4)/H$, for some 4-dimensional subgroup H of $SO(4)$.*

(ii) *The subgroup $SO(2)$ acts trivially on M.*

Proof. (i) Recall that $SO(4)$ has no 5-dimensional subgroups (cf. Lemma 2.2), therefore it cannot act on M with one-dimensional orbits. Moreover, since it acts on X^* without fixed points (cf. Lemma 4.2), it acts transitively on M with 3-dimensional ineffectivity (cf. [Ko1], Thm.3.1, p.46). In particular $M = SO(4)/H$, for some 4-dimensional subgroup H of $SO(4)$.
(ii) Consider the universal covering $p : SU(2) \times SU(2) \to SO(4)$ and set $\hat{H} = p^{-1}(H)$. The connected component of $\hat{H}$ is either $SO(2) \times SU(2)$ or $SU(2) \times SO(2)$ (see Lemma 2.2) and consequently $SO(4)/H = SU(2) \times SU(2)/\hat{H}$ is a finite quotient of the 2-dimensional sphere S^2. Since every smooth vector field on S^2 vanishes at some point, $SO(2)$ necessarily has a fixed point x in M. The fact that $SO(2)$ is central in K implies $tsx = stx = sx$, for all $t \in SO(2),\ s \in SO(4)$, i.e. $SO(2)$ acts trivially on M as claimed. □

Write $M = K/L$. By Luna's slice theorem (cf. [Lu]) there exists a $K^{\mathbb{C}}$-invariant open neighbourhood of M in X^* which is $K^{\mathbb{C}}$-equivariantly biholomorphic to a twisted product $K^{\mathbb{C}} \times_{L^{\mathbb{C}}} \mathbb{C}^2$, where $L^{\mathbb{C}}$ acts on $\mathbb{C}^2$ by the complexification $\nu\colon L^{\mathbb{C}} \to GL(2, \mathbb{C})$ of the slice representation at $eL \in M$. In fact we will show that $X^* \cong K^{\mathbb{C}} \times_{L^{\mathbb{C}}} \mathbb{C}^2$.

In order to determine such twisted products it is convenient to consider the universal covering $p : SU(2) \times SU(2) \to SO(4)$ of $SO(4)$ whose kernel is $\{\pm I_4\}$. Set $\widehat{K}$ to be $SU(2) \times SU(2) \times SO(2)$ and consider the four to one covering $\Pi : \widehat{K} \to K = (SO(4) \times SO(2))/\{\pm I_6\}$ given by $(v, u, t) \to [p(v, u), t]$. For $\widehat{L} = \Pi^{-1}(L)$ one has

$$L = \widehat{L}/\Gamma,$$

where Γ is the kernel of Π given by

$$\Gamma = \{I_6, (-I_2, -I_2, I_2), (I_2, -I_2, -I_2), (-I_2, I_2, -I_2)\}.$$

Then the representation $\nu\colon L^{\mathbb{C}} \to GL(2,\mathbb{C})$ is uniquely determined by a representation $\hat{\nu}\colon \widehat{L}^{\mathbb{C}} \to GL(2,\mathbb{C})$, which is trivial on Γ and

$$K^{\mathbb{C}} \times_{\nu} \mathbb{C}^2 \;\cong\; \widehat{K}^{\mathbb{C}} \times_{\hat{\nu}} \mathbb{C}^2 \,.$$

Proposition 4.12. *Assume that X has dimension 4 and that X^* contains a minimal orbit $M = K/L$ of dimension 2. Then X^* is given by a twisted product $\widehat{K}^{\mathbb{C}} \times_{\widehat{L}^{\mathbb{C}}} \mathbb{C}$ determined by a representation $\hat{\nu}\colon \widehat{L}^{\mathbb{C}} \to GL(2,\mathbb{C})$ containing the group Γ in its kernel. All possible M and X^* are listed in the following table.*

M	$\widehat{L}$	$\hat{\nu}\colon \widehat{L}^{\mathbb{C}} \to GL(2,\mathbb{C})$	X^*
S^2	$SO(2)\times SU(2)\times SO(2)$	$\hat{\nu}(v,u,t) = v^m u t^{\pm 1}$, m odd	$\widehat{K}^{\mathbb{C}} \times_{\hat{\nu}} \mathbb{C}^2$
$\mathbb{P}^2(\mathbb{R})$	$O(2)\times SU(2)\times SO(2)$	$\hat{\nu}(v,u,t) = v^m u t^{\pm 1}$, m odd	$\widehat{K}^{\mathbb{C}} \times_{\hat{\nu}} \mathbb{C}^2$
$\mathbb{P}^2(\mathbb{R})$	$O(2)\times SU(2)\times SO(2)$	$\hat{\nu}(v,u,t) = det(v) v^m u t^{\pm 1}$, m odd	$\widehat{K}^{\mathbb{C}} \times_{\hat{\nu}} \mathbb{C}^2$

Proof. We already observed that there exists a $K^{\mathbb{C}}$-invariant open neighborhood of M in X^* which is biholomorphic to $K^{\mathbb{C}} \times_{\nu} \mathbb{C}^2 \;\cong\; \widehat{K}^{\mathbb{C}} \times_{\hat{\nu}} \mathbb{C}^2$.

Without loss generality we may assume that the connected component of e in $\widehat{L}$ is given by $SO(2)\times SU(2)\times SO(2)$ (cf. Lemma 2.2). Recall that such component is normal in $\widehat{L}$. Then the only other possibility for $\widehat{L}$ is to be its normalizer in $\widehat{K}$, which is given by $O(2) \times SU(2) \times SO(2)$. This implies that $M \cong \widehat{K}/\widehat{L}$ is given either by S^2 or $\mathbb{P}^2(\mathbb{R})$.

First assume that $\widehat{L} = SO(2)\times SU(2)\times SO(2)$. Note that $I_2 \times SU(2)\times SO(2)$ acts trivially on M (cf. Lemma 4.11) and by assumption the K-action is effective on X^*. As a consequence the restriction of $\widehat{\nu}$ to $I_2 \times SU(2) \times I_2$ is faithful and therefore it coincides with the standard representation. Further, the restriction of $\widehat{\nu}$ to $I_2 \times I_2 \times SO(2)$ is given by a faithful character, since such subgroup is central. The subgroup $SO(2)\times I_2 \times I_2$ is also central, implying that $\hat{\nu}(v,u,t) = v^m u t^{\pm 1}$ for some integer m. Finally recall that that group Γ is forced to be in the kernel of $\hat{\nu}$. As a consequence the integer m is necessarily odd.

The other possibility is that $\widehat{L} = O(2)\times SU(2)\times SO(2)$. Analogous arguments show that in this case either $\hat{\nu}(v,u,t) = v^m u t^{\pm 1}$ or $\hat{\nu}(v,u,t) = det(v)v^m u t^{\pm 1}$, where in both cases m is an odd integer.

Finally note that in all cases one has $\mathcal{O}(K^{\mathbb{C}} \times_{L^{\mathbb{C}}} \mathbb{C}^2)^K \cong \mathbb{C}$, implying by the analytic continuation principle that $\mathcal{O}(X)^K = \mathbb{C}$. Then [He], Appl.(a), p.660 applies to show that the manifold X^* is biholomorphic to $K^{\mathbb{C}} \times_{L^{\mathbb{C}}} \mathbb{C}^2$. □

5. The main theorem

In this section we conclude the proof of our main result. We need some preliminary facts.

Lemma 5.1. *Let L be a complex Lie group and let Y be a complex L-manifold. If $\mathcal{O}(Y)^L$ is infinite-dimensional, then $Aut(Y)$ is infinite dimensional.*

Proof. For v in the Lie algebra of L and f in $\mathcal{O}(Y)^L$ define $A : Y \to \mathcal{O}(Y)^L$ by $A(y) := exp(f(y)v)$. Since A is holomorphic and L-invariant, the map $Y \to Y$ defined by $y \to A(y)\cdot y$ is an automorphism of Y whose inverse is given by $y \to A(y)^{-1}\cdot y$. This shows that if $\mathcal{O}(Y)^L$ is infinite dimensional, then $Aut(Y)$ is infinite dimensional. □

In the sequel we apply the above lemma in the following situation. Let Y be a Stein $\mathbb{C}^*$-manifold such that the categorical quotient $Y//\mathbb{C}^*$ is positive dimensional. Since $Y//\mathbb{C}^*$ is Stein by [He], the algebra $\mathcal{O}(Y)^L \cong \mathcal{O}(Y//\mathbb{C}^*)$ is infinite dimensional. Then the above lemma implies that $Aut(X)$ is infinite dimensional. By definition a holomorphic map $p : X \to Y$ between complex spaces is locally hyperbolic if Y admits a covering of open subsets whose preimages in X are hyperbolic. Recall the following classical result.

Lemma 5.2. ([Ko2], Thm.3.2.15, p.64) *Let $p : X \to Y$ be a locally hyperbolic map between complex spaces. If Y is hyperbolic, then X is hyperbolic.*

Remark 5.3. Let K/L be a compact symmetric space of rank one and let $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{p}$ be the corresponding Lie algebra decomposition. Let $\mathfrak{a}$ be a maximal abelian subalgebra in $\mathfrak{p}$. The Weyl group $W \cong \mathbb{Z}_2$ acts on $\mathfrak{a}$ by reflections and every K-orbit in the complexified space $K^{\mathbb{C}}/L^{\mathbb{C}}$ intersects the slice $\exp i\mathfrak{a}\, L^{\mathbb{C}}$ in an orbit of the corresponding W-action on $\exp i\mathfrak{a}\, L^{\mathbb{C}}$. So there is a homeomorphism of orbit spaces (cf. [La])

$$K \setminus K^{\mathbb{C}}/L^{\mathbb{C}} \cong \mathfrak{a}/W \cong \mathbb{R}^{\geq 0}\,.$$

Fix a generator H of $\mathfrak{a}$. By a result of Azad-Loeb [AzLo1], a K-invariant function f on $K^{\mathbb{C}}/L^{\mathbb{C}}$ is plurisubharmonic if and only if the corresponding W-invariant function $\tilde{f} : \mathfrak{a} \to \mathbb{R}$, defined by $tH \to f(\exp itHL^{\mathbb{C}})$, is convex on $\mathfrak{a} \cong \mathbb{R}$. Since $\tilde{f}$ is an even convex function, it is continuos on $\mathfrak{a}$ and it has a minimum at 0.

We use this result when the symmetric space is the sphere

$$K/L = SO(n)/SO(n-1) \cong S^{n-1}$$

and its complexification is the affine quadric $Q^{n-1} \cong SO(n,\mathbb{C})/SO(n-1,\mathbb{C})$ of dimension $n-1$. For every positive real number r we denote by Q^{n-1}_r the Stein, hyperbolic, $SO(n)$-invariant tube around the sphere S^{n-1} in Q^{n-1} defined by

$$Q^{n-1}_r := SO(n)\exp(i[0,r)H)SO(n-1,\mathbb{C})\,.$$

Recall that all Stein, $SO(n)$-invariant subdomains of Q^{n-1} are of this form.

In the setting of Sect. 4, consider the case when the globalization X^* of X is of the form $X^* = Q^{n-1} \times \mathbb{C}$ and $SO(n) \times SO(2)$ acts factorwise on X^*. Let $\Delta_s := \{\, z \in \mathbb{C} \;:\; |z| < s \,\}$.

Lemma 5.4. *If $X^* = Q^{n-1} \times \mathbb{C}$, then X is a disk bundle, i.e. if (q, z) lies in X then the disk $\{q\} \times \Delta_{|z|}$ is contained in X.*

Proof. By assumption the intersection $Q^{n-1} \times \{0\} \cap X$ is not empty. Such an intersection is Stein and K-invariant, thus it is of the form $Q^{n-1}_r \times \{0\}$, for some $r \in (0,\infty]$. Let $p_1 : Q^{n-1} \times \mathbb{C} \to Q^{n-1}$ denote the projection onto the first factor and assume $p_1(X) = Q^{n-1}_s$, for some $s \geq r$. In order to prove the statement it is sufficient to show that $s = r$. Indeed this implies that if $(q, z) \in X$, then $(q, 0) \in X$. The $SO(2)$-orbit convexity of X in X^* (cf. [He], Sect.3) in turn implies that $\{q\} \times \Delta_{|z|}$ is contained in X.

So assume by contradiction that $s > r$ and let $(q, z) \in X$ with $z \neq 0$ and $q = \exp(irH)\, SO(n-1,\mathbb{C})$. Then there exists an invariant neighborhood of (q, z) in X of the form $U \times A$, with $U = SO(n)\exp(i(r-\varepsilon, r+\varepsilon)H)SO(n-1,\mathbb{C})$ and A a small annulus in $\mathbb{C}^*$ of radii $|z| - \varepsilon$ and $|z| + \varepsilon$. Since the product $SO(n)\exp(i(r-\varepsilon, r)H)SO(n-1,\mathbb{C}) \times \{0\}$ is contained in X, by the orbit convexity of X the product $SO(n)\exp(i(r-\varepsilon, r)H)SO(n-1,\mathbb{C}) \times \Delta_{|z|+\varepsilon}$ is also contained in X. This determines a Hartogs figure in X around $(q, 0)$, implying that $(q, 0) \in X$. The fact that $q = \exp irH\, SO(n-1,\mathbb{C})$ contradicts the definition of r and the lemma follows. □

Theorem 5.5. *Let $\Omega \cong G/K$ be the bounded symmetric domain of type IV of dimension $n > 4$. Let X be a Stein manifold of dimension $n > 4$ such that $Aut(X)^0$ is isomorphic to G. Then X is biholomorphically equivalent to Ω.*

Proof. If K has a fixed point in X then the statement is proved in Proposition 3.1. If K has no fixed points in X, then X is a Stein K-invariant domain in one of the manifolds X^* listed in Proposition 4.8 and Proposition 4.10. We complete the proof of the theorem by showing that no such X has automorphism group isomorphic to G. Indeed from the discussion below it turns out that either $Aut(X)$ is infinite dimensional or X is hyperbolic or an $Aut(X)$-invariant open subset of X is hyperbolic. In the last two cases if $Aut(X)$ were isomorphic to G, then X would have a K-fixed point (cf. Proposition 3.2), contradicting the assumptions.

Assume first that $X^* = Q^{n-1} \times \mathbb{C}^*$ and let $p_1 : Q^{n-1} \times \mathbb{C}^* \to Q^{n-1}$ be the projection onto the first factor. Since X is Stein and K-invariant in X^*, it is of the form

$$X = \{(q, z) \ : \ q \in p_1(X) \text{ and } \alpha(q) < |z| < \beta(q)\}$$

where $\log \circ \alpha$ and $-\log \circ \beta$ are $SO(n)$-invariant, plurisubharmonic functions on $p_1(X)$ (cf. [**?**]). We distinguish several cases.

- (A) Assume $p_1(X) = Q_r^{n-1}$, for some $0 < r < \infty$, and α, β constant. Then $X = Q_r^{n-1} \times A$, with A an annulus in $\mathbb{C}^*$. If $A = \mathbb{C}^*$, consider the $\mathbb{C}^*$-action by multiplication on the second factor. Then Lemma 5.1 implies that X has an infinite dimensional automorphism group. If $A \neq \mathbb{C}^*$, then X is the product of two hyperbolic manifolds. Hence it is hyperbolic.
- (B) Assume $p_1(X) = Q_r^{n-1}$, for some $0 < r < \infty$, and one of the two defining functions, e.g. β, non-constant. In this case β takes a maximum m on the sphere S^{n-1} (cf. Remark 5.3) and X is contained in $Q_r^{n-1} \times \Delta_m$. In particular it is hyperbolic.
- (C) Assume $p_1(X) = Q^{n-1}$ and α, β constant. Then $X = Q^{n-1} \times A$, with A an annulus in $\mathbb{C}^*$. By choosing a complex subgroup of $SO(n, \mathbb{C})$ isomorphic to $\mathbb{C}^*$ one obtains a $\mathbb{C}^*$-action on the first factor of X. The corresponding categorical quotient $X//\mathbb{C}^*$ is Stein and positive dimensional. Then by Lemma 5.1 the automorphism group $Aut(X)$ is infinite dimensional.
- (D) Assume $p_1(X) = Q^{n-1}$ and one of the two defining functions, e.g. β, non-constant. In this case X is contained in $Q^{n-1} \times \Delta_m^*$, where m is the maximum of β on Q^{n-1} (cf. case (B)). Let $p_2 : X \to \Delta_m^*$ be the restriction to X of the projection onto the second factor. Let U be any relatively compact open subset of Δ_m^*. Recall that the $SO(n)$-invariant function β determines a continuous function $\tilde{\beta}$ on $Q^{n-1}/SO(n) \cong \{tH, t \geq 0\} \cong \mathbb{R}^{\geq 0}$ such that $-\log \circ \tilde{\beta}$ is convex and non-constant (cf. Remark 5.3). Note that the preimage $p_2^{-1}(U)$ of U is contained in $Q_s^{n-1} \times U$, where

  $$s := \max\{ \ \tilde{\beta}^{-1}(|w|) \ : \ w \text{ lies in the closure of } U \ \}.$$

  In particular it is hyperbolic, showing that p_2 is locally hyperbolic. Then X is hyperbolic by Lemma 5.2.

Assume now that $X^* = Q^{n-1} \times \mathbb{C}$. By Lemma 5.4, in this case X is of the form

$$X = \{(q, z) \ : \ q \in p_1(X) \text{ and } |z| < \beta(q)\}$$

where $-\log \circ \beta$ is an $SO(n)$-invariant, plurisubharmonic function on $p_1(X)$. In the cases

$(A1)$ $p_1(X) = Q_r^{n-1}$, for some $0 < r < \infty$, and β constant,

$(B1)$ $p_1(X) = Q_r^{n-1}$, for some $0 < r < \infty$, and β non-constant,

$(C1)$ $p_1(X) = Q^{n-1}$ and β constant,

analogous results as in cases (A), (B) and (C) follow by the same arguments.

$(D1)$ Assume $p_1(X) = Q^{n-1}$ and β non-constant. As in case (D) one sees that X is contained in $Q^{n-1} \times \Delta_m$, where m is the maximum of β on Q^{n-1}. However here X is not hyperbolic, since it contains the zero section $Q^{n-1} \times \{0\}$.

We claim that every automorphism of X leaves such a section invariant. First note that every non-constant holomorphic curve $f : \mathbb{C} \to X$ is contained in $Q^{n-1} \times \{0\}$. Indeed $p_2 \circ f$ is a bounded, holomorphic function on $\mathbb{C}$. Hence it is constant, i.e. $p_2 \circ f \equiv c$. If $c \neq 0$, let $s := \tilde{\beta}^{-1}(|c|)$. Then $p_2^{-1}(c) = Q_s^{n-1} \times \{c\}$ is hyperbolic and therefore it contains no non-constant holomorphic curves. As a result $c = 0$. Since $SO(n, \mathbb{C})$ acts transitively on $Q^{n-1} \times \{0\}$, there exists a non-constant holomorphic curve through every point of $Q^{n-1} \times \{0\}$. This property characterizes the set $Q^{n-1} \times \{0\}$ in X. Then the claim follows.

As a consequence, by removing the zero section from X one obtains a Stein, G-invariant domain Y contained in $Q^{n-1} \times \mathbb{C}^*$. Then an analogous argument as in case (D) implies that such a domain is hyperbolic and by Lemma 3.2, there exists a K-fixed point in $Y \subset X$.

Finally consider the remaining cases, i.e. when X^* is one of the spaces

$$Q^{n-1}/\mathbb{Z}_2 \times \mathbb{C}^*, \qquad Q^{n-1} \times_{\mathbb{Z}_2} \mathbb{C}^*, \qquad Q^{n-1}/\mathbb{Z}_2 \times \mathbb{C}^*, \qquad Q^{n-1} \times_{\mathbb{Z}_2} \mathbb{C}^*.$$

Each of the above spaces is a quotient of $Q^{n-1} \times \mathbb{C}^*$ with respect to a free $\mathbb{Z}_2$-action. Therefore there is a two-to-one covering map Π from $Q^{n-1} \times \mathbb{C}^*$ (or $Q^{n-1} \times \mathbb{C}$) onto X^*. This yields a one-to-one correspondence between the set of Stein, K-invariant domains in X^* and the set of Stein, $SO(n) \times SO(2)$-invariant, $\mathbb{Z}_2$-invariant domains in $Q^{n-1} \times \mathbb{C}^*$ (or $Q^{n-1} \times \mathbb{C}$). Also recall that hyperbolicity is preserved by finite coverings or finite quotients by holomorphic transformations. Then, by considering the preimage of X by Π, it is straightforward to check that arguments analogous to those used in the product cases apply to all the above cases. This concludes the proof of the theorem. □

References

[AzLo1] Azad H., Loeb J.J. *Plurisubharmonic functions and Kählerian metrics on complexification of symmetric spaces.* Indag. Math. N.S. **4** (1992) no. 3, 365-375.

[AzLo2] Azad H., Loeb J.J. *Plurisubharmonic functions and the Kempf-Ness theorem.* Bull. London Math. Soc. **25** (1993) no. 2, 162–168.

[BtD] Bröker T., tom Diek T. *Representations of Compact Lie Groups.* Graduate Texts in Mathematics, Springer-Verlag, New York, 1985.

[BSW] Burns D., Shnider S., Wells, R. O. *Deformations of strictly pseudoconvex domains.* Invent. Math. **46** (1978) no. 3, 237–253.

[Ca] Cartan H. *Les fonctions de deux variables complexes et le problème de la représentation analytique.* J. Math. Pure. Appl **10** (1931) 1–114.

[FHW] Fels G., Huckleberry A. T., Wolf J. A. *Cycle Spaces of Flag Domains: A Complex Geometric Viewpoint.* Progress in Mathematics **245**, Birkhäuser, Boston 2005.

[HaWe] Harvey F. R., Wells R. O. *Zero sets of non-negative strictly plurisubharmonic functions.* Math. Ann. **201** (1973) 165–170.

[He] Heinzner P. *Geometric invariant theory on Stein spaces.* Math. Ann. **289** (1991) 631–662.

[Hl] Helgason S. *Differential geometry, Lie groups and symmetric spaces.* Mathematics **80**, Academic Press Inc., London, 1978.

[HuIs] Huckleberry, A. T.; Isaev, A. V. *Classical Symmetries of Complex Manifolds.* J. Geom. Anal. **20**(1), 132–152 (2010) .

[Is1] Isaev A. V. *Effective actions of the unitary group on complex manifolds.* Canad. J. Math. **54** (2002) no. 6, 1254–1279.

[Is2] Isaev A. V. *Characterization of the unit ball in* $\mathbb{C}^n$ *among complex manifolds of dimension* n. J. Geom. Anal. **14** (2004) no. 4, 697–700.

[Is3] Isaev A. V. *Erratum.* J. Geom. Anal. **18** (2008) no. 3, 919.

[Is4] Isaev A. V. *A remark on a theorem by Kodama and Shimizu.* J. Geom. Anal. **18** (2008) no. 3, 795–799.

[Ka] Kahn R. H. *Semi-simple Lie Algebras and Their Representations.* Dover Books in Mathematics, Dover Publications Inc., Mineola, New York, 2006.

[Ko1] Kobayashi S. *Transformation Groups in Differential Geometry.* Springer-Verlag, Berlin Heidelberg New York, 1972.

[Ko2] Kobayashi S. *Hyperbolic complex spaces.* Grundlehren der Mathematischen Wissenschaften **318** Springer-Verlag, Berlin, 1998.

[KdSh] Kodama A., Shimizu S. *An intrinsic characterization of the unit polydisc.* Mich. Math. J. **56** (2008) no. 1, 173–181.

[La] Lassalle M. *Séries de Laurent des fonctions holomorphes dans la complexification d'un espace symétrique compact.* Ann. Sci. Éc. Norm. Supér., Sér.II IV (1973) 81-105.

[LiE] http://www-math.univ-poitiers.fr/~maavl/LiE/

[Lu] Luna D. *Slices étale.* Bull. Soc. Math. France **33** (1973) 81–105.

[Pa] Palais R. *A global formulation of the Lie theory of transformation groups.* Mem. Amer. Math. Soc. **22**, 1955.

[Vla] Vladimirov V. S. Methods of the theory of functions of many complex variables. Dover Publications Inc., Mineola, New York , 2007

[Wo1] Wolf J.A. *Fine structure of Hermitian symmetric spaces.* In Symmetric Spaces (Short Courses, Washington Univ., St. Louis, Mo. 1969-70), 271–357, Pure and Appl. Math., Vol. 8, Marcel Dekker, New York, 1972.

[Wo2] Wolf J.A. *Spaces of constant curvature (*5^{th} *edition).* Publish or Perish, Wilmington, Delaware, 1984.

Laura Geatti and Andrea Iannuzzi: Dip. di Matematica, II Università di Roma "Tor Vergata", Via della Ricerca Scientifica, I-00133 Roma, Italy.

E-mail address: `geatti@mat.uniroma2.it, iannuzzi@mat.uniroma2.it`

Jean-Jacques Loeb, Département de Mathématiques et Informatiques, Université d'Angers, 2 Boulevard Lavoisier, 49045 Angers cedex 01

E-mail address: `Jean-Jacques.Loeb@univ-angers.fr`