A remark on the orbit structure of the complexification of a semisimple symmetric space.

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Abstract: We consider the action of a real semisimple Lie group G on the complexification $G_{\mathbf{C}}/H_{\mathbf{C}}$ of a semisimple symmetric space G/H and we present a refinement of Matsuki's results [1] in this case. We exhibit a finite set of points in $G_{\mathbf{C}}/H_{\mathbf{C}}$, sitting on closed G-orbits of locally minimal dimension, whose slice representation determines the G-orbit structure of $G_{\mathbf{C}}/H_{\mathbf{C}}$. Every such point $\bar{p}$ lies on a compact torus and occurs at specific values of the restricted roots of the symmetric pair $(\mathfrak{g}, \mathfrak{h})$. The slice representation at $\bar{p}$ is equivalent to the isotropy representation of a real reductive symmetric space, namely $Z_G(p^4)/G_{\bar{p}}$. In principle, this gives the possibility to explicitly parametrize all G-orbits in $G_{\mathbf{C}}/H_{\mathbf{C}}$.

MSC: 22E15 (22E46)
Keywords: semisimple symmetric space, complexification, orbit structure, slice representation.

Introduction.

Let G be a real reductive Lie group and let L and H be the fixed point subgroups of two involutions of G, acting on G by left and right translations respectively,

$$(l, h) : G \to G, \quad x \mapsto lxh^{-1}, \quad x \in G,\ l \in L,\ h \in H.$$

The $L \times H$-orbit structure of G has been studied by Matsuki [1], Helminck-Schwartz [2] and recently again by Helminck-Schwartz [3] and Miebach [4] with moment map techniques.

In this note we consider the special case of two commuting involutions of a semisimple connected complex group $G_{\mathbf{C}}$, the first one being a conjugation σ with real form G, and the second a holomorphic involution τ with fixed point subgroup $H_{\mathbf{C}}$. This is equivalent to considering the G-action on the complexification $G_{\mathbf{C}}/H_{\mathbf{C}}$ of the semisimple symmetric space G/H, where $H = G \cap H_{\mathbf{C}}$.

The action of G on $G_{\mathbf{C}}/H_{\mathbf{C}}$ is the restriction of an algebraic action of $G_{\mathbf{C}}$ on the complex affine algebraic variety $G_{\mathbf{C}}/H_{\mathbf{C}} \times G_{\mathbf{C}}/H_{\mathbf{C}}$, with real structure $\Sigma(\bar{z}_1, \bar{z}_2) = (\sigma(\bar{z}_2), \sigma(\bar{z}_1))$, where σ here denotes the induced conjugation on $G_{\mathbf{C}}/H_{\mathbf{C}}$. In this framework there is a well defined map $\mathbf{p}\colon G_{\mathbf{C}}/H_{\mathbf{C}} \to (G_{\mathbf{C}}/H_{\mathbf{C}})\|G$ onto the parameter space of closed G-orbits in $G_{\mathbf{C}}/H_{\mathbf{C}}$, assigning to every $\bar{x} \in G_{\mathbf{C}}/H_{\mathbf{C}}$ the unique closed G-orbit in the closure of $G \cdot \bar{x}$. Each fiber of this map contains a unique closed orbit, which is also the unique orbit of minimum dimension in the fiber (cf. [5]).

Let $\bar{x} \in G_{\mathbf{C}}/H_{\mathbf{C}}$ be a point on a closed orbit $G \cdot \bar{x}$. Denote by $T(G \cdot \bar{x})_{\bar{x}}$ the tangent space to $G \cdot \bar{x}$ at $\bar{x}$ and by $N_{\bar{x}}$ a complementary subspace to $T(G \cdot \bar{x})_{\bar{x}}$ in $T(G_{\mathbf{C}}/H_{\mathbf{C}})_{\bar{x}}$. The isotropy subgroup $G_{\bar{x}}$ acts on $T(G \cdot \bar{x})_{\bar{x}}$ via the isotropy representation and on $N_{\bar{x}}$ via the slice representation. By Luna's slice Theorem ([5], Prop. 1.2), there exists an open $G_{\bar{x}}$-invariant neighborhood V of 0 in $N_{\bar{x}}$ such that the map

$$G \times_{G_{\bar{x}}} V \to G_{\mathbf{C}}/H_{\mathbf{C}}, \qquad [g, X] \to g \exp iX x H_{\mathbf{C}}/H_{\mathbf{C}} \tag{1}$$

is a G-equivariant diffeomorphism onto an open G-invariant saturated neighborhood of $G \cdot \bar{x}$ in $G_{\mathbf{C}}/H_{\mathbf{C}}$ (see Sect.4 for the definition of a twisted bundle $G \times_{G_{\bar{x}}} V$). This means that on a slice neighborhood (as above in (1)) of $G \cdot \bar{x}$ in $G_{\mathbf{C}}/H_{\mathbf{C}}$ the G-orbit structure is completely determined by the slice representation of $G_{\bar{x}}$ at $\bar{x}$. Slice neighborhoods of closed orbits of minimal or locally minimal dimension contain the maximal linearized information about the orbit structure of the G-action.

Matsuki's results [1] imply that there are finitely many tori T_i in $G_{\mathbf{C}}$ and points p_i in $G_{\mathbf{C}}$ such that the union of the $T_i p_i$ (Cartan subsets), covers the quotient $(G_{\mathbf{C}}/H_{\mathbf{C}})\|G$. By elaborating upon Matsuki's results, we obtain a refinement in our situation.

First we give a description of Cartan subsets in terms of the restricted root system of the real symmetric pair $(\mathfrak{g}, \mathfrak{h})$. Using this description we prove that every Cartan subset admits a base point p in a finite set S of distinguished points on a compact torus in $G_{\mathbf{C}}$. Every point $p \in S$ occurs at specific values of the restricted roots, which put severe restrictions on the G-orbit of $\bar{p}$ in $G_{\mathbf{C}}/H_{\mathbf{C}}$. The orbit $G \cdot \bar{p}$ is closed. If $Z_G(p^4) = G$, then $G \cdot \bar{p}$ has minimal dimension and is a symmetric space of the same rank, real rank and dimension as G/H. If $Z_G(p^4)$ is a proper subgroup of G, then $G \cdot \bar{p}$ is a Cauchy-Riemann submanifold of $G_{\mathbf{C}}/H_{\mathbf{C}}$ of locally minimal dimension. Moreover, the orbit of $\bar{p}$ under $Z_G(p^4)$ is a symmetric space of the same rank and real rank as G/H, but of smaller dimension. In both cases the orbit $Z_G(p^4) \cdot \bar{p}$ is totally real in $G_{\mathbf{C}}/H_{\mathbf{C}}$, the isotropy subgroups of $\bar{p}$ in G and $Z_G(p^4)$ coincide and the slice representation at $\bar{p}$ is equivalent to the isotropy representation of the real reductive symmetric space $Z_G(p^4)/G_{\bar{p}}$.

By taking suitable invariant neighbourhoods of the orbits $G \cdot \bar{p}_1, \ldots, G \cdot \bar{p}_m$ in $G_{\mathbf{C}}/H_{\mathbf{C}}$ we obtain an invariant covering of $G_{\mathbf{C}}/H_{\mathbf{C}}$ consisting of saturated sets whose G-orbit structure is modelled on the isotropy representation of the symmetric spaces $Z_G(p_1^4)/G_{\bar{p}_1}, \ldots, Z_G(p_m^4)/G_{\bar{p}_m}$, respectively.

Our main addition to Matsuki's results is the explicit determination of the set S. The fact that every Cartan subset admits a base point in S is not a priori obvious, but depends on the combinatorics of the restricted root system of a semisimple symmetric space. Knowing the set S is a first step towards a parametrization of all G-orbits in $G_{\mathbf{C}}/H_{\mathbf{C}}$. The results of this paper have been used extensively in [6], [7] and [8] to compute the Cauchy-Riemann structure of G-orbits in $G_{\mathbf{C}}/H_{\mathbf{C}}$. This was a motivation to write them up.

The paper is organized as follows. In Section 1, we fix the notation and recall some general facts about symmetric spaces. In Section 2, we introduce a set S of distinguished points in $G_{\mathbf{C}}$ and we analyze their G-orbits. In Section 3, we revisit Cartan subsets introduced by Matsuki as cross sections of closed $G \times H_{\mathbf{C}}$-orbits in $G_{\mathbf{C}}$. We show that every Cartan subset admits a base point p in the set S and determines a cross section of closed $G_{\bar{p}}$-orbits in the slice representation at $\bar{p} \in G_{\mathbf{C}}/H_{\mathbf{C}}$. In Section 4, we construct a finite covering of $G_{\mathbf{C}}/H_{\mathbf{C}}$ consisting of G-invariant sets, whose orbit structure is modelled on the orbit structure of the isotropy representation of the real reductive symmetric space $Z_G(p^4)/G_{\bar{p}}$, for $p \in S$. Finally, in Section 5 we discuss some applications.

1. Notation and preliminaries.

Throughout the paper, groups are denoted by capital roman letters and their Lie algebras by the corresponding gothic letters. For example, $\mathfrak{g}$ and $\mathfrak{g}_{\mathbf{C}}$ denote the Lie algebras of G and $G_{\mathbf{C}}$, respectively. An involution of a group, the derived involution of its Lie algebra and their holomorphic extensions to the corresponding complex objects are all denoted by the same symbol. If X is a set and ϕ is a selfmap of X, the fixed point set of ϕ in X is denoted by X^ϕ. If a group G acts on a manifold X and $x \in X$, the isotropy subgroup of x in G is denoted by G_x and the orbit of x in X by $G \cdot x \cong G/G_x$. If L is a subgroup of a group G and $x \in G$, the *centralizer* of x in L is denoted by $Z_L(x)$. Similarly, if $\mathfrak{l}$ is a subalgebra of $\mathfrak{g}$ and $X \in \mathfrak{g}$, the centralizer of X in $\mathfrak{l}$ is denoted by $Z_{\mathfrak{l}}(X)$.

A semisimple (resp. reductive) symmetric space is a coset space G/L, where G is a real semisimple (resp. reductive) Lie group and $L \subset G$ is an open subgroup of the fixed point subgroup of an involutive automorphism of G. Let $G_{\mathbf{C}}$ be a connected complex semisimple Lie group $G_{\mathbf{C}}$ endowed with a conjugation σ (not a Cartan involution) and a holomorphic involution τ commuting with σ. Then there exists a Cartan involution Θ of $G_{\mathbf{C}}$ such that the following commutativity relations hold

$$\sigma\tau = \tau\sigma, \quad \Theta\sigma = \sigma\Theta, \quad \Theta\tau = \tau\Theta. \tag{2}$$

Denote by $U = G_{\mathbf{C}}^{\Theta}$ and $G = G_{\mathbf{C}}^{\sigma}$ the compact and the non-compact real forms of $G_{\mathbf{C}}$ corresponding to Θ and σ respectively, and by $H_{\mathbf{C}} = G_{\mathbf{C}}^{\tau}$ the complex fixed point subgroup of τ.

By (2), the restriction $\theta := \Theta|G$ defines a Cartan involution of G and $K = G \cap U$ is a maximal compact subgroup of G. Similarly, the restriction $\tau|G$ defines an involution of G commuting with θ, whose fixed point subgroup is given by $H = G \cap H_{\mathbf{C}}$. The coset space G/H is a semisimple symmetric space and the complex manifold $G_{\mathbf{C}}/H_{\mathbf{C}}$ is its complexification. The commutativity relations (2) ensure that the decompositions induced by Θ, σ and τ on $\mathfrak{g}_{\mathbf{C}}$ and on $\mathfrak{g}$ are all compatible with each other. For example, if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is the one induced by τ, then both $\mathfrak{h}$ and $\mathfrak{q}$ are θ-stable and $\mathfrak{g}$ has a combined decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{h}_{\mathfrak{p}} \oplus \mathfrak{q}_{\mathfrak{k}} \oplus \mathfrak{q}_{\mathfrak{p}},$$

where subscripts stand for intersections. The product involution $\sigma^c := \sigma\tau$ defines a conjugation of $G_{\mathbf{C}}$. The corresponding real form G^c has Lie algebra $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$. Since $\sigma^c\tau = \tau\sigma^c$, the restriction $\tau|G^c$ defines an involution of G^c with fixed point subgroup $G^c \cap H_{\mathbf{C}} = H$, and the restriction $\theta^c = \Theta|G^c$ defines a Cartan involution of G^c commuting with τ. In this way, the G-orbit and G^c-orbit of the base point $\bar{e} \in G_{\mathbf{C}}/H_{\mathbf{C}}$ define transversal embeddings

$$G/H \hookrightarrow G_{\mathbf{C}}/H_{\mathbf{C}} \hookleftarrow G^c/H$$

of c-dual symmetric spaces ([9], Sect.1.2.1), as totally real submanifolds of maximal dimension. If $\tau = \theta$ is the Cartan involution of G, then G/K is a Riemannian symmetric space, the real form G^c is the compact real form U and the c-dual symmetric space is the compact dual symmetric space U/K.

If $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \tau)$ is a symmetric algebra, a *Cartan subspace* of $\mathfrak{q}$ is by definition a maximal abelian subspace $\mathfrak{c} \subset \mathfrak{q}$ consisting of semisimple elements. The *rank of a symmetric space* G/H is the dimension of an arbitrary Cartan subspace in $\mathfrak{q}$, and its *real rank* is the dimension of a maximal abelian subspace in $\mathfrak{q}_{\mathfrak{p}}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{q}_{\mathfrak{p}}$. Then the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$ determines a decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = Z_{\mathfrak{g}}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \Delta_{\mathfrak{a}}} \mathfrak{g}^{\alpha},$$

where $\Delta_{\mathfrak{a}} = \{\alpha \in \mathfrak{a}^* \setminus \{0\} \mid \mathfrak{g}^{\alpha} \neq \{0\}\}$ is the restricted root system of $\mathfrak{g}$ (cf. [10]) and $\mathfrak{g}^{\alpha} = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X,\ \forall H \in \mathfrak{a}\}$ is the α-root space. In general, $\mathfrak{a}$ may not be a maximal abelian subspace in $\mathfrak{p}$, so $\Delta_{\mathfrak{a}}$ may not coincide with the usual restricted root system of $\mathfrak{g}$. Throughout the paper the image of a subset S of $G_{\mathbf{C}}$ under the canonical projection $\pi\colon G_{\mathbf{C}} \to G_{\mathbf{C}}/H_{\mathbf{C}}$ is denoted by $\overline{S}$. Let $\mathbf{p}$ be the quotient map $\mathbf{p}\colon G_{\mathbf{C}}/H_{\mathbf{C}} \to (G_{\mathbf{C}}/H_{\mathbf{C}})\|G$. A subset D of $G_{\mathbf{C}}/H_{\mathbf{C}}$ is said to be *saturated* if $\mathbf{p}^{-1}(\mathbf{p}(D)) = D$.

2. A family of distinguished points in $G_{\mathbf{C}}/H_{\mathbf{C}}$.

In this section we introduce a finite set S of distinguished points in $G_{\mathbf{C}}$. Each point $p \in S$ lies on a compact torus in $G_{\mathbf{C}}$ and the G-orbit of $\bar{p}$ in $G_{\mathbf{C}}/H_{\mathbf{C}}$ is closed (cf. [1], Sect.4). We show that the isotropy subgroups of $\bar{p}$ in G and in $Z_G(p^4)$ coincide and that the orbit $Z_G(p^4) \cdot \bar{p}$ is a real reductive symmetric space contained in $G \cdot \bar{p}$. We also show that the isotropy representation of $Z_G(p^4)/G_{\bar{p}}$ at $\bar{p}$ is equivalent to the slice representation of $G_{\bar{p}}$ at $\bar{p}$.

Let $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \tau)$ be the symmetric algebra associated to G/H. Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{q}_{\mathfrak{p}}$. Then for all a in the compact torus $\mathcal{A} := \exp i\mathfrak{a}$ one has $\sigma(a) = \tau(a) = a^{-1}$ and $\Theta(a) = a$. For $p = e^{iX} \in \mathcal{A}$, define the holomorphic involution $\tau_p := Ad_p \tau Ad_{p^{-1}}$ of $G_{\mathbf{C}}$ (resp. on $\mathfrak{g}_{\mathbf{C}}$). Fix a set of simple roots $\Pi_{\mathfrak{a}} = \{\gamma_1, \ldots, \gamma_r\}$ in the restricted root system $\Delta_{\mathfrak{a}}$, where r denotes the real rank of G/H. The conditions

$$\gamma_1(X) \equiv 0, \ldots, \gamma_r(X) \equiv 0 \mod \pi/4, \qquad X \in \mathfrak{a}, \tag{3}$$

determine a finite set of points $S = \{p_1, \ldots, p_n\}$ in $\mathcal{A}$, due to the periodicity of the exponential map.

Lemma 2.1. *Let $p = e^{iA_0} \in \exp i\mathfrak{a}$. Assume that A_0 satisfies conditions (3). Then the following facts hold:*

(i) $2\alpha(A_0) \equiv 0 \bmod \pi/2$, *for all* $\alpha \in \Delta_{\mathfrak{a}}$.

(ii) The automorphism Ad_{p^8} is the identity on $G_{\mathbf{C}}$; in particular Ad_{p^4} is an involution of $G_{\mathbf{C}}$, and it commutes with σ.

(iii) The fixed point group $G^+_{\mathbf{C}} := Z_{G_{\mathbf{C}}}(p^4)$ is σ-stable with real form $G^+ := Z_G(p^4)$.

(iv) The G^+-orbit of $\bar{p}$ in $G_{\mathbf{C}}/H_{\mathbf{C}}$ is a reductive symmetric space with involution τ_p and symmetric algebra $\mathfrak{g}^+ = \mathfrak{g}^+ \cap Ad_p\mathfrak{h}_{\mathbf{C}} \oplus \mathfrak{g}^+ \cap Ad_p\mathfrak{q}_{\mathbf{C}}$; the fixed point subgroup $(G^+)^{\tau_p} = G^+ \cap Ad_p H_{\mathbf{C}}$ coincides with the isotropy subgroup $G^+_{\bar{p}}$ of $\bar{p}$ in G^+ and $\mathfrak{m} := \mathfrak{g}^+ \cap Ad_p\mathfrak{q}_{\mathbf{C}}$ can be identified with the tangent space to $G^+ \cdot \bar{p}$ at $\bar{p}$.

(v) The isotropy subgroups of $\bar{p}$ in G^+ and G coincide, that is $G^+_{\bar{p}} = G_{\bar{p}}$; the slice representation at $\bar{p}$ is equivalent to the isotropy representation of the symmetric space $G^+/G_{\bar{p}}$, which is given by the adjoint action of $G_{\bar{p}}$ on $\mathfrak{m}$.

(vi) The symmetric space $G^+ \cdot \bar{p}$ is embedded in $G \cdot \bar{p}$ and in $G_{\mathbf{C}}/H_{\mathbf{C}}$ as a totally real submanifold; in particular $\dim G^+ \cdot \bar{p} \leq \dim G/H$. Moreover $G^+ \cdot \bar{p}$ has the same rank and the same real rank as G/H.

(vii) The orbit $G\cdot\bar p$ has locally minimal dimension: for all $\bar x$ in a suitable G-invariant neighborhood of $\bar p$ in $G_{\mathbf C}/H_{\mathbf C}$, one has $\dim G\cdot\bar x\geq\dim G\cdot\bar p$.

Proof. (i) is obvious and (ii) follows from the corresponding statements at Lie algebra level proved in [6], Lemma 2.14(i)-(ii).

(iii) follows from (ii) and the fact that for every $\alpha\in\Delta_{\mathfrak a}$ and $Z\in\mathfrak g^\alpha$, one has $Ad_{p^8}Z=\exp\mathrm{ad}_{i8A_0}Z=e^{i8\alpha(A_0)}Z$. In particular, the fixed point subalgebra of Ad_{p^4} is given by

$$\mathfrak g^+_{\mathbf C}=Z_{\mathfrak g_{\mathbf C}}(p^4)=Z_{\mathfrak g_{\mathbf C}}(\mathfrak a)\oplus\bigoplus_{\substack{\alpha(A_0)\equiv 0\\ \mathrm{mod}\ \pi/2}}\mathfrak g^\alpha.$$

(iv) By [6] Lemma 2.14(iii), the involutions σ and τ_p commute on $\mathfrak g^+_{\mathbf C}$ and on $G^+_{\mathbf C}$. In this way, (G^+,τ_p) is a real reductive symmetric pair. The corresponding symmetric algebra $(\mathfrak g^+,\tau_p)$ decomposes as

$$\mathfrak g^+=\mathfrak g^+\cap Ad_p\mathfrak h_{\mathbf C}\oplus\mathfrak g^+\cap Ad_p\mathfrak q_{\mathbf C},$$

where $\mathfrak g^+\cap Ad_p\mathfrak h_{\mathbf C}$ coincides with the isotropy subalgebra of $\bar p$ in $\mathfrak g^+$. The rest of the statement now follows.

(v) The inclusion $G^+_{\bar p}\subset G_{\bar p}=G\cap Ad_pH_{\mathbf C}$ is obvious. Conversely, for $g\in G_{\bar p}$, we have $\sigma(g)=g=\tau_p(g)$ which implies $(\sigma\tau_p)^2(g)=Ad_{p^4}(g)=g$. Hence $G_{\bar p}\subset G^+_{\bar p}$ and $G_{\bar p}=G^+_{\bar p}$, as claimed.

Denote by $\mathfrak g^-_{\mathbf C}$ the (-1)-eigenspace of the involution Ad_{p^4} on $\mathfrak g_{\mathbf C}$. By the results of (ii)-(iv), the tangent space to $G_{\mathbf C}/H_{\mathbf C}$ at $\bar p$ admits a $G_{\bar p}$-stable decomposition as

$$T(G_{\mathbf C}/H_{\mathbf C})_{\bar p}=\ \mathfrak g^-_{\mathbf C}\ \oplus\ \mathfrak g^+\cap Ad_p\mathfrak q_{\mathbf C}\ \oplus\ i(\mathfrak g^+\cap Ad_p\mathfrak q_{\mathbf C}),\tag{4}$$

where

$$\mathfrak g^-_{\mathbf C}\ \oplus\ \mathfrak g^+\cap Ad_p\mathfrak q_{\mathbf C}$$

can be identified with the tangent space $T(G\cdot\bar p)_{\bar p}$ to the G-orbit of $\bar p$ at $\bar p$ and $i(\mathfrak g^+\cap Ad_p\mathfrak q_{\mathbf C})$ with a complementary subspace. The action of $G_{\bar p}$ on each component of (4) is the adjoint action. It corresponds to the isotropy representation at $\bar p$ of the symmetric space $G^+\cdot\bar p$ on the component $\mathfrak g^+\cap Ad_p\mathfrak q_{\mathbf C}\ \cong\ T(G^+\cdot\bar p)_{\bar p}$ and to the slice representation at $\bar p$ on $i(\mathfrak g^+\cap Ad_p\mathfrak q_{\mathbf C})$.

(vi) The quotient $G^+/(G^+)^{\tau_p}$ embeds as a totally real submanifold of maximal dimension in its complexification $G^+_{\mathbf C}/(G^+_{\mathbf C})^{\tau_p}$, which in turn is a complex submanifold of $G_{\mathbf C}/(G_{\mathbf C})^{\tau_p}$. It follows that $G^+/(G^+)^{\tau_p}$ is a totally real submanifold of $G_{\mathbf C}/(G_{\mathbf C})^{\tau_p}\cong G_{\mathbf C}/H_{\mathbf C}$. Since G/H is a totally real submanifold of maximal dimension of $G_{\mathbf C}/H_{\mathbf C}$, one has $\dim G^+\cdot p\leq\dim G/H$.

Note that the same argument used in (iv) shows that also $\mathfrak g^+\cap Ad_p\mathfrak q_{\mathbf C}=\mathfrak g\cap Ad_p\mathfrak q_{\mathbf C}$ holds. Let $\mathfrak t\oplus\mathfrak a$ be a maximally split Cartan subspace of $\mathfrak q$. Since $\mathfrak t\oplus\mathfrak a$ is contained in $\mathfrak g^+\cap Ad_p\mathfrak q_{\mathbf C}$, the symmetric space $G^+/G^+_{\bar p}$ has the same rank and the same real rank as G/H.

(vii) Remark 3.16 in [6] shows that G-orbits of lowest dimension intersect $\overline{\mathcal A}$ and that in some open neighbourhood U of $\bar p$ in $\overline{\mathcal A}$, the dimension of the isotropy subalgebra can only increase.

Minimal orbits. The set S defined by conditions (3) contains a subset consisting of points lying on symmetric orbits in $G_{\mathbf C}/H_{\mathbf C}$. They occur in the following special case of conditions (3)

$$\gamma_1(X)\equiv 0,\dots,\gamma_r(X)\equiv 0\quad \mathrm{mod}\ \pi/2,\qquad X\in\mathfrak a.\tag{5}$$

The next lemma collects their properties.

Lemma 2.2. *Let $p=e^{iA_0}\in\exp i\mathfrak a$. Assume that p satisfies conditions (5). Then the following facts hold:*

(i) $\alpha(A_0)\equiv 0\ \mathrm{mod}\ \pi/2,\ \ \forall\alpha\in\Delta_{\mathfrak a}$;

(ii) The automorphism Ad_{p^4} is the identity on $G_{\mathbf C}$ and the involutions σ and τ_p commute on $G_{\mathbf C}$. In this case $Z_{G_{\mathbf C}}(p^4)=G_{\mathbf C}$.

(iii) The G-orbit of $\bar{p}$ in $G_{\mathbf{C}}/H_{\mathbf{C}}$ is a semisimple symmetric space with involution τ_p, and symmetric algebra $\mathfrak{g} = \mathfrak{g} \cap Ad_p\mathfrak{h}_{\mathbf{C}} \oplus \mathfrak{g} \cap Ad_p\mathfrak{q}_{\mathbf{C}}$; the fixed point subgroup $(G)^{\tau_p} = G \cap Ad_pH_{\mathbf{C}} = G_{\bar{p}}$ coincides with the isotropy subgroup of $\bar{p}$ in G and $\mathfrak{g} \cap Ad_p\mathfrak{q}_{\mathbf{C}}$ can be identified with the tangent space to $G \cdot \bar{p}$ at $\bar{p}$.

(iv) The symmetric space $G \cdot \bar{p}$ is embedded in $G_{\mathbf{C}}/H_{\mathbf{C}}$ as a totally real submanifold of the same dimension as G/H and has the same rank and the same real rank as G/H.

(v) The orbit $G \cdot \bar{p}$ has minimal dimension among all G-orbits in $G_{\mathbf{C}}/H_{\mathbf{C}}$.

(vi) The slice representation at $\bar{p}$ is equivalent to the isotropy representation of the symmetric space $G \cdot \bar{p}$.

Proof. We only prove (v). For the rest we refer to Lemma 2.1 and to [6], Lemma 2.11 and Corollary 2.12. As we already remarked, the isotropy subgroup at $\bar{p}$ is given by $G_p = G \cap Ad_pH_{\mathbf{C}}$. It follows that $G_{\bar{p}}$ has the largest possible dimension $\dim G_{\bar{p}} = \dim H$ if and only if the decomposition $\mathfrak{g}_{\mathbf{C}} = Ad_p\mathfrak{h}_{\mathbf{C}} \oplus Ad_p\mathfrak{q}_{\mathbf{C}}$ of $\mathfrak{g}_{\mathbf{C}}$ under the involution τ_p is σ-stable. By Remark 3.5 in [6], this happens precisely when the point $\bar{p}$ satisfies conditions (5). In that case

$$\dim_{\mathbf{R}} G/G_{\bar{p}} = \dim_{\mathbf{R}} G/H = \dim_{\mathbf{C}} G_{\mathbf{C}}/H_{\mathbf{C}}.$$

Remark 2.3. Let G/L be an orbit of minimum dimension $\dim_{\mathbf{R}} G/L = \dim_{\mathbf{C}} G_{\mathbf{C}}/H_{\mathbf{C}}$.

(i) The orbit G/L intersects $\mathcal{A}$ (this fact follows from Remarks 3.5(i) and 3.16(i) in [6]). Let $p \in \mathcal{A} \cap G/L$. Then $G/G_{\bar{p}} \cong G/L$ is a symmetric space with symmetry τ_p.

(ii) If $H = K$ and G/K is a Riemannian symmetric space, then G/L is an ϵ-symmetric space (cf. [10]).

(iii) If G/L is a compactly causal symmetric space, then G is a group of Hermitian type (cf. [9], Thm.1.3.8).

Remark 2.4. Let $p = e^{A_0} \in \exp i\mathfrak{a}$ be a point satisfying conditions (3), but not conditions (5).

(i) The orbit $G \cdot \bar{p}$ is a homogeneous Cauchy-Riemann submanifold of $G_{\mathbf{C}}/H_{\mathbf{C}}$. The complex tangent space to $G \cdot \bar{p}$, that is the subspace of the tangent space which is invariant under the complex structure, is given by $T_{\mathbf{C}}(G \cdot \bar{p})_{\bar{p}} = \mathfrak{g}_{\mathbf{C}}^-$.

(ii) The G-orbits of points satisfying conditions (3) and not conditions (5) may have different dimensions.

(iii) Once more set $G^+ = Z_G(p^4)$. As we saw in Lemma 2.2 (vii), there exists a neighborhood U of $\bar{p}$ in $G_{\mathbf{C}}/H_{\mathbf{C}}$ consisting of points $\bar{x}$ whose G-orbits in $G_{\mathbf{C}}/H_{\mathbf{C}}$ have dimension greater or equal than $\dim G \cdot \bar{p}$. However, p is the *unique* point in U for which the symmetric space $G^+/G_{\bar{p}}$ satisfies

$$\dim G^+/G_{\bar{p}} = \text{codim } G \cdot \bar{p},$$

i.e. the isotropy representation of $G^+/G_{\bar{p}}$ at $\bar{p}$ is equivalent to the slice representation at $\bar{p}$. By Lemma 2.2(iv)-(vi), one also has the following inequalities

$$rank(G/H) < \dim G^+ \cdot p < \dim G/H < \dim G \cdot \bar{p}.$$

(iii) If $H = K$ and G/K is a Riemannian symmetric space, then $G^+ \cdot \bar{p}$ is an ϵ-symmetric space.

3. Cartan subsets.

Cartan subsets were introduced in [1] as cross-sections for the closed $G \times H_{\mathbf{C}}$-orbits in $G_{\mathbf{C}}$. Their images in $G_{\mathbf{C}}/H_{\mathbf{C}}$ under the canonical projection $G_{\mathbf{C}} \to G_{\mathbf{C}}/H_{\mathbf{C}}$ determine cross sections for the closed G-orbits in $G_{\mathbf{C}}/H_{\mathbf{C}}$. In this section, working out Matsuki's definition in our case, we give a description of Cartan subsets in terms of orthogonal systems of restricted root vectors in the real symmetric algebra $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \tau)$. By means of such description we prove the key result of the paper, namely that every Cartan subset admits a base point p satisfying conditions (3).

A description of Cartan subsets.

Definition 3.1. ([1], Sect.4) A *Cartan subspace* in $\mathfrak{g}$ is an abelian subspace consisting of semisimple elements. A *fundamental Cartan subset* in $(G_{\mathbf{C}}, \sigma, \tau)$ is a set $F = \exp i(\mathfrak{t} \oplus \mathfrak{a}) = \exp i\mathfrak{t} \exp i\mathfrak{a}$, where $\mathfrak{t} \oplus \mathfrak{a}$ is a maximally split θ-stable Cartan subspace in $\mathfrak{q}$. A *standard Cartan subset* in $(G_{\mathbf{C}}, \sigma, \tau)$ is a set $C = p \cdot \exp i\mathfrak{c}$, where

$p \in \mathcal{A} = \exp i\mathfrak{a}$ is a base point, and $\mathfrak{c} = \mathfrak{c}_{\mathfrak{k}} \oplus \mathfrak{c}_{\mathfrak{p}}$ is a θ-stable Cartan subspace in $\mathfrak{g} \cap Ad_p \mathfrak{q}_{\mathbf{C}}$ with $\dim \mathfrak{c} = \dim \mathfrak{t} \oplus \mathfrak{a}$, and inclusions $\mathfrak{c}_{\mathfrak{p}} \subset \mathfrak{a}$ and $\mathfrak{t} \subset \mathfrak{c}_{\mathfrak{k}}$.

Two standard Cartan subsets $C = \exp i\mathfrak{c}_{\mathfrak{k}} \exp i\mathfrak{c}_{\mathfrak{p}} \cdot p$ and $C' = \exp i\mathfrak{c}'_{\mathfrak{k}} \exp i\mathfrak{c}'_{\mathfrak{p}} \cdot p'$ are said to be conjugate if there exists an element $[(k,h)] \in N_{K \times H_{\mathbf{C}} \cap U}(F)/Z_{K \times H_{\mathbf{C}} \cap U}(F)$ such that

$$\exp i\mathfrak{c}'_{\mathfrak{k}} \cdot p' = k(\exp i\mathfrak{c}_{\mathfrak{k}} \cdot p)h^{-1},$$

where $N_{K \times H_{\mathbf{C}} \cap U}(F) := \{(k,h) \in K \times H_{\mathbf{C}} \cap U \mid k(\exp i\mathfrak{a})h^{-1} = \exp i\mathfrak{a}\}$ and $Z_{K \times H_{\mathbf{C}} \cap U}(F) := \{(k,h) \in K \times H_{\mathbf{C}} \cap U \mid kah^{-1} = a,\ \forall a \in \exp i\mathfrak{a}\}$.
By [1], Thm.3, every closed $G \times H_{\mathbf{C}}$-orbit in $G_{\mathbf{C}}$ intersects an element in a fixed set of representatives of the conjugacy classes of standard Cartan subsets. Such a Cartan subset is unique if the orbit has maximal dimension.

Let $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \tau)$ be a symmetric algebra, let $\mathfrak{a} \subset \mathfrak{q}_{\mathfrak{p}}$ be a maximal abelian subspace and let $\Delta_{\mathfrak{a}}$ be the corresponding restricted root system of $\mathfrak{g}$. Since the involution $\tau\theta$ is the identity on $\mathfrak{a}$, every restricted root space $\mathfrak{g}^{\alpha}$ is $\tau\theta$-stable. Consequently, it admits a basis of root vectors X_α satisfying either $\tau\theta X_\alpha = X_\alpha$ or $\tau\theta X_\alpha = -X_\alpha$.

Definition 3.2. ([11], p.344) An orthogonal system of restricted root vectors $Q = \{X_{\alpha_1}, \ldots, X_{\alpha_m}\}$ in $(\mathfrak{g}, \tau)$ is a set of restricted root vectors $X_{\alpha_j} \in \mathfrak{g}^{\alpha_j}$, for $\alpha_j \in \Delta_{\mathfrak{a}}$, satisfying the conditions

$$\alpha_1(X) = \ldots = \alpha_m(X) = 0,\ \forall\ X \in \mathfrak{a}, \qquad [X_{\alpha_j}, X_{\alpha_k}] = [X_{\alpha_j}, \theta X_{\alpha_k}] = 0, \qquad \tau\theta X_{\alpha_j} = \pm X_{\alpha_j},$$

for $j, k = 1, \ldots, m$.

Proposition 3.3. *Every standard Cartan subset C in $(G_{\mathbf{C}}, \sigma, \tau)$ can be described by an orthogonal system of restricted root vectors $Q = \{X_{\alpha_1}, \ldots, X_{\alpha_m}\}$ in $(\mathfrak{g}, \tau)$ as follows:*

$$C = \exp i\mathfrak{t}_Q \cdot A_Q,$$

where

$$\mathfrak{t}_Q = \mathfrak{t} \oplus \mathbf{R}(X_{\alpha_1} + \theta X_{\alpha_1}) \oplus \ldots \oplus \mathbf{R}(X_{\alpha_m} + \theta X_{\alpha_m}),$$

$$A_Q = \left\{ e^{iH} \in \exp i\mathfrak{a}_Q \mid \begin{cases} e^{2\alpha_j(H)} = -1, \text{ if } \tau\theta X_{\alpha_j} = X_{\alpha_j} \\ e^{2\alpha_j(H)} = 1, \text{ if } \tau\theta X_{\alpha_j} = -X_{\alpha_j} \end{cases} \right\} = \exp i\mathfrak{a}_Q \cdot a,$$

$$\mathfrak{a}_Q = \{H \in \mathfrak{a} \mid \alpha_j(H) = 0,\ j = 1, \ldots, m\}, \quad a = e^{iA_0}, \quad \begin{cases} \alpha_j(A_0) \equiv \pi/2 \mod \pi, \text{ if } \tau\theta X_{\alpha_j} = X_{\alpha_j} \\ \alpha_j(A_0) \equiv 0 \mod \pi, \text{ if } \tau\theta X_{\alpha_j} = -X_{\alpha_j}. \end{cases}$$

Proof. Let $\mathfrak{g}_{\mathbf{C}} \oplus i\mathfrak{g}_{\mathbf{C}} = \{\mathbf{Z} = \mathbf{X} + i\mathbf{Y} \mid \mathbf{X}, \mathbf{Y} \in \mathfrak{g}_{\mathbf{C}}\}$ be the complexification of $\mathfrak{g}_{\mathbf{C}}$ and let the fundamental Cartan subspace $i(\mathfrak{t} \oplus \mathfrak{a})$ act on $\mathfrak{g}_{\mathbf{C}} \oplus i\mathfrak{g}_{\mathbf{C}}$ by $[H, \mathbf{X} + i\mathbf{Y}] := [H, \mathbf{X}] + i[H, \mathbf{Y}]$, with $H \in i(\mathfrak{t} \oplus \mathfrak{a})$ and $\mathbf{X}, \mathbf{Y} \in \mathfrak{g}_{\mathbf{C}}$. For $\lambda \in i(\mathfrak{a} \oplus \mathfrak{t})^* \setminus \{0\}$, denote by $(\mathfrak{g}_{\mathbf{C}})^\lambda$ the λ-weight space

$$(\mathfrak{g}_{\mathbf{C}})^\lambda = \{\mathbf{Z}_\lambda \in \mathfrak{g}_{\mathbf{C}} \oplus i\mathfrak{g}_{\mathbf{C}} \mid [H, \mathbf{Z}_\lambda] = \lambda(H)\mathbf{Z}_\lambda, \quad H \in i(\mathfrak{t} \oplus \mathfrak{a})\}.$$

In [1], Sect.4.4, p.83, standard Cartan subsets in $(G_{\mathbf{C}}, \sigma, \tau)$ are described in terms of orthogonal systems $\mathbf{Q} = \{\mathbf{Z}_{\lambda_1}, \ldots, \mathbf{Z}_{\lambda_m}\}$ of weight vectors $\mathbf{Z}_{\lambda_i} \in (\mathfrak{g}_{\mathbf{C}})^{\lambda_i}$, with $\lambda_i|\mathfrak{t} \equiv 0$ for $i = 1, \ldots, m$, satisfying the conditions

$$\Theta\mathbf{Z}_{\lambda_i} = -\mathbf{Z}_{\lambda_i}, \quad \sigma\mathbf{Z}_{\lambda_i} = -\overline{\mathbf{Z}}_{\lambda_i}, \quad [\mathbf{Z}_{\lambda_i}, \mathbf{Z}_{\lambda_j}] = [\mathbf{Z}_{\lambda_i}, \sigma\mathbf{Z}_{\lambda_j}] = 0, \quad \sigma\tau\mathbf{Z}_{\lambda_i} = \mu_i\mathbf{Z}_{\lambda_i}, \quad \mu_i \in U(1),$$

for all $i, j = 1, \ldots, m$. A standard Cartan subset is obtained from $\mathbf{Q}$ as

$$C_{\mathbf{Q}} = \exp \mathfrak{t}_{\mathbf{Q}} A_{\mathbf{Q}},$$

where

$$\mathfrak{t}_{\mathbf{Q}} = i\mathfrak{t} \oplus \mathbf{R}(\mathbf{Z}_{\lambda_1} - \sigma\mathbf{Z}_{\lambda_1}) \oplus \ldots \oplus \mathbf{R}(\mathbf{Z}_{\lambda_m} - \sigma\mathbf{Z}_{\lambda_m}), \quad A_{\mathbf{Q}} = \{t \in A \mid t^{2\lambda_j} = \mu_j,\ j = 1, \ldots, m\}.$$

Observe that if $\lambda|i\mathfrak{t} \equiv 0$, then the weight space $(\mathfrak{g}_{\mathbf{C}})^\lambda$ is both Θ-stable and $\sigma\tau$-stable. We are going to show that $C_{\mathbf{Q}}$ can also be described by an orthogonal system of restricted root vectors Q in $(\mathfrak{g}, \tau)$ (see Definition 3.2). Write $\mathbf{Z}_\lambda = \mathbf{X}_\lambda + i\mathbf{Y}_\lambda \in \mathfrak{g}_{\mathbf{C}} \oplus i\mathfrak{g}_{\mathbf{C}}$, for $\mathbf{X}_\lambda, \mathbf{Y}_\lambda \in \mathfrak{g}_{\mathbf{C}}$. For $H \in i\mathfrak{a}$, one has that

$$\mathbf{Z}_\lambda \in (\mathfrak{g}_{\mathbf{C}})^\lambda \Leftrightarrow \begin{cases} [H, \mathbf{X}_\lambda] = -\mathrm{Im}\lambda \mathbf{Y}_\lambda \\ [H, \mathbf{Y}_\lambda] = \mathrm{Im}\lambda \mathbf{X}_\lambda; \end{cases}$$

$$\Theta\mathbf{Z}_\lambda = -\mathbf{Z}_\lambda \Leftrightarrow \quad \mathbf{X}_\lambda, \mathbf{Y}_\lambda \in (\mathfrak{g}_{\mathbf{C}})^{-\Theta} = i\mathfrak{u};$$

$$\sigma\mathbf{Z}_\lambda = -\overline{\mathbf{Z}_\lambda} \Leftrightarrow \begin{cases} \sigma\mathbf{X}_\lambda = -\mathbf{X}_\lambda \\ \sigma\mathbf{Y}_\lambda = \mathbf{Y}_\lambda \end{cases} \Leftrightarrow \begin{cases} \mathbf{X}_\lambda \in i\mathfrak{g} \\ \mathbf{Y}_\lambda \in \mathfrak{g}. \end{cases}$$

In particular, under the above conditions,

$$\mathbf{X}_\lambda \in i\mathfrak{k}, \quad \mathbf{Y}_\lambda \in \mathfrak{p}, \quad \text{and} \quad \mathbf{Z}_\lambda - \sigma\mathbf{Z}_\lambda = 2\mathbf{X}_\lambda \in i\mathfrak{k}.$$

Since the weight spaces $(\mathfrak{g}_{\mathbf{C}})^\lambda$ are $\sigma\tau$-stable and $(\sigma\tau)^2 = Id$, from $\sigma\tau\mathbf{Z}_\lambda = \mu\mathbf{Z}_\lambda$ one obtains $\mu = \pm 1$ (for example in the Riemannian case, where $\sigma\tau = \Theta$, one has $\mu = -1$, for all j). One has

$$\mathbf{X}_\lambda \in i(\mathfrak{q} \cap \mathfrak{k}), \quad \text{for } \mu = 1, \qquad\qquad \mathbf{X}_\lambda \in i(\mathfrak{h} \cap \mathfrak{k}), \quad \text{for } \mu = -1.$$

• *Claim.* One has $i2\mathbf{X}_\lambda = X_\alpha + \theta X_\alpha$, for some restricted root vector $X_\alpha \in \mathfrak{g}^\alpha$, with $\alpha := \mathrm{Im}(\lambda) \in \Delta_{\mathfrak{a}}$. Moreover

$$X_\alpha + \theta X_\alpha \in \mathfrak{k} \cap \mathfrak{q}, \quad \text{if } \mu = 1, \qquad X_\alpha + \theta X_\alpha \in \mathfrak{k} \cap \mathfrak{h}, \quad \text{if } \mu = -1.$$

Proof of the Claim: The vector $i\mathbf{X}_\lambda + \mathbf{Y}_\lambda \in \mathfrak{g}$ and for $iH \in \mathfrak{a}$, one has

$$[iH, i\mathbf{X}_\lambda + \mathbf{Y}_\lambda] = \mathrm{Im}\lambda(i\mathbf{X}_\lambda + \mathbf{Y}_\lambda).$$

This means that $X_\alpha := i\mathbf{X}_\lambda + \mathbf{Y}_\lambda$ is a root vector in $\mathfrak{g}^\alpha$, for the restricted root $\alpha := \mathrm{Im}\lambda \in \Delta_{\mathfrak{a}}$. Note that $\lambda|\mathfrak{t} \equiv 0$ implies $\alpha|\mathfrak{t} \equiv 0$. Since $\mathbf{Y}_\lambda \in \mathfrak{p}$, one has

$$i2\mathbf{X}_\lambda = X_\alpha + \theta X_\alpha,$$

as claimed. Moreover, one has

$$\sigma\tau\mathbf{X}_\lambda = \mathbf{X}_\lambda \Leftrightarrow \sigma\tau(X_\alpha + \theta X_\alpha) = -(X_\alpha + \theta X_\alpha) \Leftrightarrow \tau\theta X_\alpha = -X_\alpha \Leftrightarrow X_\alpha + \theta X_\alpha \in \mathfrak{q}_{\mathfrak{k}}, \text{ for } \mu = 1;$$

$$\sigma\tau\mathbf{X}_\lambda = -\mathbf{X}_\lambda \Leftrightarrow sigma\tau(X_\alpha + \theta X_\alpha) = (X_\alpha + \theta X_\alpha) \Leftrightarrow \tau\theta X_\alpha = X_\alpha \Leftrightarrow X_\alpha + \theta X_\alpha \in \mathfrak{h}_{\mathfrak{k}}, \text{ for } \mu = -1.$$

This concludes the proof of the claim.
It follows that $\mathfrak{t}_{\mathbf{Q}} = i\mathfrak{t} \oplus \mathbf{R}i(X_{\alpha_1} + \theta X_{\alpha_1}) + \ldots + \mathbf{R}i(X_{\alpha_m} + \theta X_{\alpha_m})$. One can check that $A_{\mathbf{Q}}$ can be written as $A_{\mathbf{Q}} = \exp i\mathfrak{a}_{\mathbf{Q}} \cdot a$, where $\mathfrak{a}_Q = \{H \in \mathfrak{a} \mid \alpha_j(H) = 0,\ j = 1, \ldots, m\}$, and $a = e^{iA_0} \in \exp i\mathfrak{a}$ is a point satisfying the conditions

$$\begin{cases} \alpha_j(A_0) \equiv 0 \mod \pi, \text{ if } \mu_j = 1 \\ \alpha_j(A_0) \equiv \pi/2 \mod \pi, \text{ if } \mu_j = -1 \\ j = 1, \ldots, m. \end{cases} \tag{6}$$

This concludes the proof of the Proposition.

Base points of Cartan subsets. In this subsection we prove that a Cartan subset C_Q associated to a given orthogonal system of restricted root vectors $Q = \{X_{\alpha_1}, \ldots, X_{\alpha_m}\}$ in $(\mathfrak{g}, \tau)$ admits a base point $p \in \exp i\mathfrak{a}_Q \cdot a$ satisfying conditions (3). Observe that, as long as $\dim \mathfrak{a}_Q \neq 0$, conditions (6) generally do not determine completely a base point of a Cartan subset C_Q, because the roots $\alpha_1, \ldots, \alpha_m$ take the same value at every point in $\exp i\mathfrak{a}_Q \cdot a$. On the other hand, if the cardinality of Q is equal to the real rank of $\mathfrak{g}$, then $\dim \mathfrak{a}_Q = 0$ and conditions (6) do uniquely determine the base point p. This shows that the statement of Lemma 3.4 is not a priori obvious but depends on the features of the restricted root system of $(\mathfrak{g}, \tau)$.

Lemma 3.4. *Let $C_Q = \exp i\mathfrak{c} \cdot p$ be a standard Cartan subset associated to an orthogonal system $Q = \{X_{\alpha_1}, \ldots, X_{\alpha_m}\}$. Then there exists a base point $p = e^{iH_0} \in \exp i\mathfrak{a}_Q \cdot a$ for C_Q satisfying conditions*

$$\begin{cases} \alpha_j(H_0) \equiv 0 \mod \pi, \text{ if } \mu_j = 1 \\ \alpha_j(H_0) \equiv \pi/2 \mod \pi, \text{ if } \mu_j = -1 \\ j = 1, \ldots, m. \end{cases} \quad \text{and} \quad 2\alpha(H_0) \equiv 0 \mod \pi/2, \quad \forall \alpha \in \Delta_{\mathfrak{a}}. \tag{7}$$

Proof. As we remarked, an arbitrary point in $\exp i\mathfrak{a}_Q \cdot a$ satisfies conditions (6). We need to show that there exists $p = e^{iH_0} \in \exp i\mathfrak{a}_Q \cdot a$, which satisfies the larger set of conditions (7). Without loss of generality, we may assume $\Delta_{\mathfrak{a}}$ irreducible. If $Q = \{X_{\alpha_1}, \ldots, X_{\alpha_m}\}$ is a system of orthogonal restricted root vectors in $(\mathfrak{g}, \tau)$, then $\{\alpha_1, \ldots, \alpha_m\}$ is a set of orthogonal restricted roots (cf. [11], p.344). In [6], Section 2.3, all sets of orthogonal roots in all irreducible restricted root systems were listed and the Lemma was proved in the Riemannian case, i.e. for orthogonal systems Q satisfying $\tau\theta X_{\alpha_j} = X_{\alpha_j}$, for all $j = 1, \ldots, m$. The same arguments prove the lemma for orthogonal systems $Q = \{X_{\alpha_1}, \ldots, X_{\alpha_m}\}$ satisfying $\tau\theta X_{\alpha_j} = -X_{\alpha_j}$, for all $j = 1, \ldots, m$. Here we are left to prove the lemma in the cases in which mixed conditions hold: $\tau\theta X_{\alpha_j} = -X_{\alpha_j}$, for some j, and $\tau\theta X_{\alpha_j} = X_{\alpha_j}$, for some other j.

Every root in $\Delta_{\mathfrak{a}}$ can be written as a linear combination of basic functionals $e_1, \ldots, e_r$, where r is the real rank of G/H (see [12]), and conditions (6) translate into a system of linear equations in the values $e_{i_1}(H_0), \ldots, e_{i_m}(H_0)$ for some $\{i_1, \ldots, i_m\} \subset \{1, \ldots, r\}$. The point is to show that there exists $H_0 \in \mathfrak{a}$ for which the values $e_1(H_0), \ldots, e_r(H_0)$ are compatible with conditions (7). The equations in $e_{i_1}(H_0), \ldots, e_{i_m}(H_0)$ arising from (6) are of the form

$$e_k(H_0) \pm e_h(H_0) = s\frac{\pi}{2}, \quad \text{or} \quad e_i(H_0) = t\frac{\pi}{2}, \qquad t, s \in \mathbf{Z},$$

where different equations involve e_i's with different indices, except for pairs

$$\begin{cases} e_k(H_0) + e_h(H_0) = a\frac{\pi}{2} \\ e_k(H_0) - e_h(H_0) = b\frac{\pi}{2} \end{cases}, \qquad a, b \in \mathbf{Z}.$$

If a, b are both odd or both even, then the values $e_k(H_0),\ e_h(H_0)$ are integer multiples of $\frac{\pi}{2}$ and of π, respectively. If a and b have different parity (this actually happens in the pseudo-Riemannian case), then the values $e_k(H_0),\ e_h(H_0)$ are odd multiples of $\frac{\pi}{4}$.

A tedious but straightforward check of the various cases shows that every Cartan subset indeed admits a base point $p = e^{iH_0}$ satisfying conditions (7). As an example, consider the Cartan subset associated to an orthogonal system $Q = \{X_{\alpha_1}, \ldots, X_{\alpha_8}\}$, where $\{\alpha_1, \ldots, \alpha_8\}$ is a maximal set of strongly orthogonal roots in E_8 (see [6], page 631). In this case, conditions (6) become

$$\begin{cases} \lambda_1(H_0) = n_1\frac{\pi}{2} \\ \ldots \\ \lambda_8(H_0) = n_8\frac{\pi}{2} \end{cases}, \qquad n_1, \ldots, n_8 \in \mathbf{Z}.$$

They uniquely determine the values $\{e_i(H_0)\}_{i=1,\ldots,8}$ and those of the simple roots $\{\gamma_i(H_0)\}_{i=1,\ldots,8}$ at H_0, which turn out to be equal to

$$(-n_3 + n_2 - n_4 + n_1)\frac{\pi}{4}, \quad (n_2 + n_4 - n_6 + n_5)\frac{\pi}{4}, \quad (-n_1 + n_4 - n_7 - n_5)\frac{\pi}{4}, \quad (n_1 - n_2 + n_3 - n_4)\frac{\pi}{4},$$

$$(-n_1 + n_2 + n_6 - n_7)\frac{\pi}{4}, \quad (n_1 - n_2 + n_4 - n_3)\frac{\pi}{4}, \quad (-n_1 + n_5 + n_7 - n_4)\frac{\pi}{4}, \quad (-n_6 - n_7 + n_8 - n_5)\frac{\pi}{4},$$

respectively. Now it is clear that conditions (7) hold.

Remark 3.5. Let $p = e^{iH_0} \in \exp i\mathfrak{a}$ be a point satisfying conditions (3). Set $\mathfrak{g}^+ = Z_{\mathfrak{g}}(p^4)$ and let $(\mathfrak{g}^+ = \mathfrak{g}_p \oplus \mathfrak{m}, \tau_p)$ be the associated symmetric algebra ($\mathfrak{g}^+ = \mathfrak{g}$ under conditions (5)). By Lemma 2.1, the following equality holds

$$\mathfrak{m} = \mathfrak{g}^+ \cap Ad_p\mathfrak{q}_{\mathbf{C}} = \mathfrak{g} \cap Ad_p\mathfrak{q}_{\mathbf{C}}$$

and the fundamental Cartan subspace $\mathfrak{t} \oplus \mathfrak{a}$ is contained in $\mathfrak{m}$. By the arguments of Proposition 3.3, more "compact" θ-stable standard Cartan subspaces $\mathfrak{c} \subset \mathfrak{m}$ arise from orthogonal systems of restricted root vectors $Q = \{X_{\alpha_1}, \ldots, X_{\alpha_m}\}$ satisfying $\alpha_i(H_0) \equiv 0 \bmod \pi$, for those i's for which $\tau X_{\alpha_i} = -\theta X_{\alpha_i}$, and $\alpha_i(H_0) \equiv \pi/2 \bmod \pi$, for those i's for which $\tau X_{\alpha_i} = \theta X_{\alpha_i}$. Under such conditions in fact $X_{\alpha_i} + \theta X_{\alpha_i}$ lies in $\mathfrak{g}^+$ and $\tau_p(X_{\alpha_i}) = -\theta X_{\alpha_i}$, for $i = 1, \ldots, m$.

Now recall from Lemma 2.1 that $i\mathfrak{m}$ can be identified with an $Ad_{G_{\bar{p}}}$-invariant complementary subspace of $T(G \cdot \bar{p})_{\bar{p}}$ in $T(G_{\mathbf{C}}/H_{\mathbf{C}})_{\bar{p}}$. In this way Cartan subsets $C = \exp i\mathfrak{c} \cdot p$ based at a point p are in 1-1 correspondence with cross sections for the closed $G_{\bar{p}}$-orbits in the slice representation $(Ad_{G_{\bar{p}}}, i\mathfrak{m})$.

4. Orbit structure of $G_{\mathbf{C}}/H_{\mathbf{C}}$

Let $S = \{p_1, \ldots, p_n\}$ be the finite set of points in $\mathcal{A} = \exp i\mathfrak{a}$ defined by conditions (3). In this section we show that suitable G-invariant neighborhoods of the orbits

$$G \cdot \bar{p}_1, \ldots, G \cdot \bar{p}_m$$

in $G_{\mathbf{C}}/H_{\mathbf{C}}$ determine the G-orbit structure of $G_{\mathbf{C}}/H_{\mathbf{C}}$. Let p be an arbitrary point in S. By Lemma 2.1, the orbit of $\bar{p} \in G_{\mathbf{C}}/H_{\mathbf{C}}$ under the group $G^+ = Z_G(p^4)$ is a symmetric space with involution τ_p. Let $(\mathfrak{g}^+ = \mathfrak{g}_{\bar{p}} \oplus \mathfrak{m}, \tau_p)$ be the corresponding symmetric algebra, where $\mathfrak{m} = \mathfrak{g}^+ \cap Ad_p\mathfrak{q}_{\mathbf{C}}$. By Lemma 2.1(vi), the adjoint action of $G_{\bar{p}}$ on $\mathfrak{m}$ is equivalent to the isotropy representation of the symmetric space $G^+/G_{\bar{p}}$ and to the slice representation at $\bar{p}$ in $G_{\mathbf{C}}/H_{\mathbf{C}}$. Define the twisted bundle $G \times_{G_{\bar{p}}} \mathfrak{m}$ as the orbit space of $G \times \mathfrak{m}$ under the $G_{\bar{p}}$-action given by $\gamma \cdot (g, X) := (g\gamma^{-1}, Ad_\gamma X)$, for $\gamma \in G_{\bar{p}}$. The group G acts on $G \times_{G_{\bar{p}}} \mathfrak{m}$ by $g' \cdot [g, X] := [g'g, X]$, for $g' \in G$. Observe that every G-orbit in $G \times_{G_{\bar{p}}} \mathfrak{m}$ intersects $\mathfrak{m}$. Moreover if $X \in \mathfrak{m}$, then $G \cdot X$ is closed in $G \times_{G_{\bar{p}}} \mathfrak{m}$ if and only if $G_{\bar{p}} \cdot X$ is closed in $\mathfrak{m}$. Hence there is an identification

$$(G \times_{G_{\bar{p}}} \mathfrak{m}) \| G \cong \mathfrak{m} \| G_{\bar{p}},$$

where $\|$ denotes the quotient with respect to closed orbits. Define

$$\omega = \{X \in \mathfrak{m} \mid |\mathrm{Re}\lambda| < \pi/8, \ \forall \lambda \in spec(\mathrm{ad}_X)\},$$

where $ad_X \colon \mathfrak{g}_{\mathbf{C}} \to \mathfrak{g}_{\mathbf{C}}$ is the adjoint map $ad_X(Y) = [X, Y]$ and $spec(\mathrm{ad}_X)$ is its spectrum. Define $\phi_p \colon G \times_{G_{\bar{p}}} \mathfrak{m} \to G_{\mathbf{C}}/H_{\mathbf{C}}$ by $[g, X] \mapsto g \exp iX \cdot pH_{\mathbf{C}}/H_{\mathbf{C}}$, where exp is the exponential map $\mathfrak{g}_{\mathbf{C}} \to G_{\mathbf{C}}$. Denote by

$$D_p := \phi_p(G \times_{G_{\bar{p}}} \mathfrak{m}) \tag{8}$$

the image of D_p in $G_{\mathbf{C}}/H_{\mathbf{C}}$ under the map ϕ_p.

Lemma 4.1. *The map ϕ_p is G-equivariant, and its restriction to $G \times_{G_{\bar{p}}} \omega$ is injective. The set D_p is a G-invariant saturated set in $G_{\mathbf{C}}/H_{\mathbf{C}}$ with non-empty interior. The G-orbit structure of D_p is modelled on the isotropy representation of the symmetric space $G^+/G_{\bar{p}}$.*

Proof. The map ϕ_p is clearly G-equivariant. Given two elements $[g_1, X], \ [g_2, Y] \in G \times_{G_{\bar{p}}} \omega$, one has that $\phi([g_1, X]) = \phi([g_2, Y])$ if and only if

$$g \exp(iX) = \exp(iY) h_p, \qquad g = g_2^{-1} g_1 \in G, \ h_p \in Ad_p H_{\mathbf{C}}. \tag{9}$$

By applying the map $\eta_{\tau_p}(z) = z\tau_p(z)^{-1}$ to both terms of the above equality, one obtains

$$g \exp(i2X) = \exp(i2Y) \tau_p(g).$$

Write $g = \exp(i2Y)\tau_p(g)\exp(i2X)^{-1}$. From $\sigma(g) = g$ one gets $\sigma\tau_p(g)\exp(i4X) = \exp(i2Y)\tau_p(g)$ and

$$Ad_{p^4}\tau_p(g)\exp(i4X) = \exp(i4Y)\tau_p(g),$$

since $\sigma\tau_p = Ad_{p^4}\tau_p\sigma$ (cf. [6], Lemma 2.14). By applying $\sigma\tau_p$ to both terms of the above equality and the fact that $\sigma\tau_p Ad_{p^4} = Ad_{p^4}\sigma\tau_p$ (by Lemma 2.1(ii)), one gets

$$Ad_{p^4}(g)\exp(-i4X) = \exp(-i4Y)g \;\Leftrightarrow\; g = \exp(i4Y)^{-1}Ad_{p^4}(g)\exp(i4X).$$

From $\sigma(g) = g$ one gets

$$\exp(i8Y) = \exp(iAd_{\psi(g)}8X), \qquad \psi(g) = Ad_{p^4}(g).$$

Since $X, Y \in \omega$, both $i8Y$ and $Ad_{\psi(g)}8X$ are contained in the injectivity set of the exponential map exp: $\mathfrak{g}_{\mathbf{C}} \to G_{\mathbf{C}}$. This implies $Y = Ad_{\psi(g)}X = (p^4gp^{-1})Y(p^4g^{-1}p^{-1})$ and

$$Y = Ad_gX,$$

since $Ad_{p^4} = Id$ on $\mathfrak{m}$. The above relation plugged in (9) yields $g = h_p \in G \cap Ad_pH_{\mathbf{C}} = G_{\bar{p}}$ and $g_2 = g_1h_p^{-1}$. In particular $[g_1, X] = [g_2, Y]$. This concludes the proof of the first statement.

It follows that D_p is a G-invariant set in $G_{\mathbf{C}}/H_{\mathbf{C}}$, with non-empty interior and orbit structure modelled on that of $G \times_{G_{\bar{p}}} \mathfrak{m}$. In particular D_p is saturated.

Proposition 4.2. *Let $S = \{p_1, \ldots, p_n\}$ be the finite set of points in $\mathcal{A} = \exp i\mathfrak{a}$ defined by conditions (3). Then there exist G-invariant saturated neighbourhoods $D_1, \ldots, D_m$ of the orbits $G \cdot \bar{p}_1, \ldots, G \cdot \bar{p}_m$ which cover $G_{\mathbf{C}}/H_{\mathbf{C}}$. The orbit structure of each D_i is modelled on the isotropy representation of the real reductive symmetric space $Z_G(p_i^4)/G_{\bar{p}_i}$.*

Proof. For $p_i \in S$, set $G_i^+ := Z_G(p_i^4)$ and let $(\mathfrak{g}_i^+ = \mathfrak{g}_{\bar{p}_i} \oplus \mathfrak{m}_i, \tau_{p_i})$ denote the associated symmetric algebra (cf. Lemma 2.1). As G-invariant saturated neighborhoods of the orbits $G \cdot \bar{p}_1, \ldots, G \cdot \bar{p}_m$ in $G_{\mathbf{C}}/H_{\mathbf{C}}$ take the sets $D_1, \ldots, D_m$ defined in (8) (see Lemma 4.1). It remains to show that they cover $G_{\mathbf{C}}/H_{\mathbf{C}}$. Since they are saturated, it is sufficient to prove that they intersect all closed G-orbits in $G_{\mathbf{C}}/H_{\mathbf{C}}$. In view of Matsuki's Thm.3 in [1], it is sufficient to show that every standard Cartan subset $C = \exp i\mathfrak{c} \cdot p$ can be embedded in one of the sets $D_1, \ldots, D_m$. By Lemma 3.4, the base point of C can be assumed to satisfy conditions (3), and to coincide with a point in S, say p_i. By Remark 3.5, one has that $\mathfrak{c}$ is a Cartan subspace in $\mathfrak{m}_i$, and therefore C embeds in D_i. This concludes the proof of the proposition.

Remark 4.3. For $i = 1, \ldots, m$, define $S_i := \exp i\mathfrak{m}_i \cdot p_i$. Then the set $\overline{S}_i \subset G_{\mathbf{C}}/H_{\mathbf{C}}$ is transversal to the orbit $G \cdot \bar{p}_i$ at $\bar{p}_i$. More precisely, $\overline{S}_i$ is contained in $(G_i^+)^c \cdot \bar{p}_i$, the orbit of $\bar{p}_i$ under the c-dual group $(G_i^+)^c$. Such orbit is in fact the c-dual symmetric space of $G_i^+/G_{\bar{p}_i}$ (see Sect.2). One should compare the sets $\overline{S}_i$ with the so-called *transversals* defined in Section 3 of [3] and the above Proposition 4.2 with Theorem 5.6 therein.

Remark 4.4. Given $p_1 = e^{iH_1}$, $p_2 = e^{iH_2}$ in $\mathcal{A} = \exp i\mathfrak{a}$, by [1], Thm.3, the points $\bar{p}_1$, $\bar{p}_2$ in $G_{\mathbf{C}}/H_{\mathbf{C}}$ lie on the same G-orbit if and only if p_1, p_2 lie on the same orbit under the Weyl group $W(\mathcal{A}) \cong N_{K\times H_{\mathbf{C}}\cap U}(A)/Z_{K\times H_{\mathbf{C}}\cap U}(\mathcal{A})$, where $N_{K\times H_{\mathbf{C}}\cap U}(\mathcal{A}) = \{(k,h) \in K\times H_{\mathbf{C}}\cap U \mid k\mathcal{A}l^{-1} = A\}$ and $Z_{K\times H_{\mathbf{C}}\cap U}(\mathcal{A}) = \{(k,h) \in K \times H_{\mathbf{C}} \cap U \mid kal^{-1} = a,\ \forall a \in \mathcal{A}\}$.

Generalizing the arguments of [13], Prop.6, one can show that two points $x = e^{iH_1}$, $y = e^{iH_2} \in \mathcal{A}$, lie on the same $W(\mathcal{A})$-orbit if and only if there exists $n \in N_K(\mathfrak{a})$ and $\gamma \in \mathcal{A}$, with $\gamma^4 = e$, such that

$$y = Ad_nx \cdot \gamma. \tag{10}$$

At Lie algebra level, the action (10) is given by

$$H_2 = Ad_nH_1 + H, \quad n \in N_K(\mathfrak{a}), \quad \gamma = \exp iH.$$

In other words, $W(\mathcal{A})$ is isomorphic to the semidirect product $W_K(\mathfrak{a}) \cdot \Gamma$, where $W_K(\mathfrak{a}) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ and Γ is the lattice in $\mathfrak{a}$ defined by

$$\Gamma = \bigoplus_{\alpha\in\Delta_{\mathfrak{a}}} \mathbf{Z}\frac{\pi}{2}\frac{2H_\alpha}{\langle H_\alpha, H_\alpha\rangle}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the Killing form and for every root α, the vector $H_\alpha \in \mathfrak{a}$ is defined by $\alpha(X) = \langle X, H_\alpha \rangle$, for all $X \in \mathfrak{a}$.
By the above discussion, in Proposition 4.2 one may take the set $S = \{p_1, \dots, p_n\}$ in a fundamental domain of $W(\mathcal{A})$ in $\mathcal{A}$.

5. Applications.

(1) *Parametrization of all G-orbits in $G_{\mathbf{C}}/H_{\mathbf{C}}$.* The result of Proposition 4.1 reduces the problem of parametrizing all G-orbits in $G_{\mathbf{C}}/H_{\mathbf{C}}$ to the study of the slice representation at the finite set of points $\bar{p}_1, \dots, \bar{p}_m$, for $p_i \in S$, satisfying conditions (3) (see also Remark 4.4). Any G-orbit is contained in a G-invariant set D_p, for some point $p \in S$. Assume for simplicity that p satisfies conditions (5) and $G \cdot \bar{p}$ is a symmetric space with symmetric algebra $\mathfrak{g} = \mathfrak{g}_{\bar{p}} \oplus \mathfrak{m}$. Every G-orbit in D_p admits a reference point $\bar{x} = \overline{\exp iX \cdot p}$, for some $X \in \mathfrak{m}$, and it is closed if and only if the $G_{\bar{p}}$-orbit of X in $\mathfrak{m}$ is closed. This happens if and only if X is semisimple in $\mathfrak{m}$. Let $X = X_s + X_n$ be the Jordan decomposition of X in $\mathfrak{m}$. If $X_n \neq 0$, one has that $G \cdot \overline{\exp i(X_s + X_n) \cdot p}$ is a non-closed orbit containing the closed orbit $G \cdot \overline{\exp i(X_s) \cdot p}$ in its closure. For example, the non-closed orbits containing $G \cdot \bar{p}$ in their closure are in one-to-one correspondence with the nilpotent $G_{\bar{p}}$-orbits in $\mathfrak{m}$. It is well known that there are finitely many of them [14]. Finally observe that the decomposition of $x = \exp iX \cdot p = x_s x_n \in G_{\mathbf{C}}$, where $x_s = \exp iX_s \cdot p$ and $x_n = \exp iX_n$, coincides with the Jordan decomposition of x defined by Matsuki in [1], Prop.2(ii), p.66. That was obtained by lifting to $G_{\mathbf{C}}$ the Jordan decomposition of the image of x in the real algebraic group $Aut_{\mathbf{R}}(\mathfrak{g}_{\mathbf{C}})$ via the map

$$\eta \colon G_{\mathbf{C}} \to Aut_{\mathbf{R}}(\mathfrak{g}_{\mathbf{C}}), \quad x \mapsto \sigma \circ Ad_x \tau Ad_{x^{-1}}.$$

The result of Proposition 4.2 was used for example in [8] to completely determine the orbit structure of $G_{\mathbf{C}}/K_{\mathbf{C}}$, when G/K is an irreducible Riemannian symmetric space of rank-one.

(2) *Cauchy-Riemann structure of principal G-orbits in $G_{\mathbf{C}}/H_{\mathbf{C}}$.* In [6] the Cauchy-Riemann structure of principal G-orbits in the complexification of a Riemannian symmetric space $G_{\mathbf{C}}/K_{\mathbf{C}}$ was investigated. A crucial ingredient in the calculations was the fact that every Cartan subset admits a base point in S. Because of Lemma 3.4, the results of [6] hold in the more general setting of the complexification of an arbitrary semisimple symmetric space G/H.

(3) *Properness of G-action.* There exists a G-invariant region in $G_{\mathbf{C}}/H_{\mathbf{C}}$ where G acts properly if and only if the slice representation space at some point $\bar{p}$, with $p \in S = \{p_1, \dots, p_n\}$, contains a region with proper $G_{\bar{p}}$-action. As we remarked in Lemma 2.1, the slice representation at $\bar{p}$ is equivalent to the isotropy representation of the symmetric space $Z_G(p^4)/G_{\bar{p}}$. Hence, a sufficient condition for proper G-action in some region in $G_{\mathbf{C}}/H_{\mathbf{C}}$ is the existence of a point $p \in S$ such that the orbit $G/G_{\bar{p}}$ is either a Riemannian or a compactly causal symmetric space, or such that $Z_G(p^4)/G_{\bar{p}}$ is a compactly causal symmetric space.

To see an example of the latter case, consider a non-compact irreducible Hermitian symmetric space G/K, of non-tube type. In this case the restricted roots system $\Delta_{\mathfrak{a}}$ is of type BC_r. Denote by $\Pi_{\mathfrak{a}} = \{\gamma_1, \dots, \gamma_r\}$ a set of simple roots in $\Delta_{\mathfrak{a}}$, by $\lambda = \sum_{i=1}^r k_i \gamma_i$ the highest root and by $\omega_1, \dots, \omega_r \in \mathfrak{a}$ the set of dual roots, defined by $\gamma_i(\omega_j) = \delta_{ij}$. Set $p = e^{i\frac{\pi}{2}\frac{\omega_r}{2}}$. One can easily check that the point $\bar{p}$ lies on the boundary of the crown domain $\Xi \subset G_{\mathbf{C}}/H_{\mathbf{C}}$ (cf. [13]). It satisfies conditions (3), but not conditions (5), implying that $G \cdot \bar{p}$ is a non-symmetric orbit (see for example [15], p.1338). Moreover, the Lie algebra $Z_{\mathfrak{g}}(p^4)$ is given by

$$Z_{\mathfrak{g}}(p^4) = Z_{\mathfrak{g}}(\mathfrak{a}) \oplus \bigoplus_{\substack{\alpha(\pi\omega_r) \equiv 0 \\ \mathrm{mod}\ 2\pi}} \mathfrak{g}^\alpha, \tag{11}$$

and is a proper subalgebra of $\mathfrak{g}$. Write $BC_r = \{\pm e_i, \pm 2e_i, \pm(e_i \pm e_j),\ 1 \leq i < j \leq r\}$. Then $\omega_r = (e_1 + \ldots + e_r)$ and it is easy to check that the non-zero roots appearing in the decomposition (11) are precisely $\{\pm 2e_i, \pm(e_i \pm e_j),\ 1 \leq i < j \leq r\}$. There are three cases:

$$\mathfrak{g} = \mathfrak{su}(r, m),\ (r < m), \quad Z_{\mathfrak{g}}(p^4) = \mathfrak{u}(m - r) \oplus \mathfrak{su}(r, r), \quad \mathfrak{g}_p = \mathfrak{u}(m - r) \oplus \mathfrak{sl}(r, \mathbf{C}) \oplus \mathbf{R};$$

$$\mathfrak{so}^*(2r),\ (r \text{ odd}),\quad Z_{\mathfrak{g}}(p^4) = \mathbf{R} \oplus \mathfrak{so}^*(2(r-1)),\quad \mathfrak{g}_p = \mathbf{R} \oplus \mathfrak{sl}(r-1,\mathbf{H}) \oplus \mathbf{R};$$

$$\mathfrak{e}_{6(-14)},\ (r=2),\quad Z_{\mathfrak{g}}(p^4) = \mathbf{R} \oplus \mathfrak{so}(2,8),\quad \mathfrak{g}_p = \mathbf{R} \oplus \mathfrak{so}(1,1) \oplus \mathfrak{so}(1,7).$$

The slice representation at $\bar{p}$ is equivalent to the isotropy representation of the symmetric space of Cayley type given by

$$\mathfrak{su}(r,r)/\mathfrak{sl}(r,\mathbf{C}) \oplus \mathbf{R},\qquad \mathfrak{so}^*(2(r-1))/\mathbf{R} \oplus \mathfrak{sl}(r-1,\mathbf{H}) \oplus \mathbf{R},\qquad \mathfrak{so}(2,8)/\mathfrak{so}(1,1) \oplus \mathfrak{so}(1,7), \tag{12}$$

respectively (cf. [9], p.89) . Write $(\mathfrak{g}^+ = \mathfrak{g}_p \oplus \mathfrak{m}, \tau_p)$ for the symmetric algebra associated to any of the spaces in (12). Denote by $V^\pm$ the maximal proper elliptic $Ad_{G_{\bar{p}}}$-invariant cones in $\mathfrak{m}$. Then the point $\bar{p}$ also lies in the boundary of the G-invariant domains $W^\pm = \overline{G \exp iV^\pm \cdot p}$ in $G_{\mathbf{C}}/K_{\mathbf{C}}$, on which the group G acts properly. The domains $W^\pm$ are generally not Stein (see [8], for the rank-1 case) and contain no G-invariant Stein subdomains, but they are contained in a larger G-invariant Stein domain on which G-acts properly (see [16]).

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