Complex extensions of semisimple symmetric spaces.

Laura Geatti
Dipartimento di Matematica, Università di Roma 2 Tor Vergata, 00133 Roma
e-mail: geatti@mat.uniroma2.it

Abstract: Let G/H be a pseudo-Riemannian semisimple symmetric space. The tangent bundle $T(G/H)$ contains a maximal G-invariant neighbourhood Ω of the zero section where the adapted-complex structure exists. Such Ω is endowed with a canonical G-invariant pseudo-Kähler metric of the same signature as the metric on G/H. We use the polar map $\phi\colon \Omega \to G^{\mathbb{C}}/H^{\mathbb{C}}$ to define a G-invariant pseudo-Kähler metric on distinguished G-invariant domains in $G^{\mathbb{C}}/H^{\mathbb{C}}$ or on coverings of principal orbit strata in $G^{\mathbb{C}}/H^{\mathbb{C}}$. In the rank-one case, we show that the polar map is globally injective and the domain $\phi(\Omega) \subset G^{\mathbb{C}}/H^{\mathbb{C}}$ is an increasing union of q-complete domains.

Introduction.

Let G/H be a non-compact semisimple symmetric space embedded in its complexification $G^{\mathbb{C}}/H^{\mathbb{C}}$. It is natural to ask whether there exists a G-invariant open set

$$G/H \ \subset \ D \ \subset \ G^{\mathbb{C}}/H^{\mathbb{C}},$$

whose complex analytic properties reflect the geometry and the harmonic analysis of G/H. Since $G^{\mathbb{C}}/H^{\mathbb{C}}$ contains G-invariant open sets with very different complex analytic properties (cf.[Ge]), one should expect D to be a proper subdomain of $G^{\mathbb{C}}/H^{\mathbb{C}}$. The situation is best understood in the case of an irreducible Riemannian symmetric space G/K. In this case there is a distinguished G-invariant subdomain

$$G/K \ \subset \ D \ \subset \ G^{\mathbb{C}}/K^{\mathbb{C}},$$

which in many respects may be considered the canonical complexification of G/K. The domain D, introduced by Akhiezer, Gindikin in [AG] and intensively studied in recent years, has several remarkable properties: every eigenfunction of the algebra of G-invariant differential operators on G/K admits a holomorphic extension to D and every unitary spherical representation of G/K can be realized on a Hilbert space of holomorphic functions on D. Moreover D is Stein, carries plenty of G-invariant plurisuharmonic functions, and is a maximal connected set where the G-action is proper (see [B],[BHH],[H],[KS1],[KS2]).

The Akhiezer–Gindikin domain D can be described as follows. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. The tangent bundle $T(G/K)$ of G/K may be identified with the homogeneous vector bundle $G \times_K \mathfrak{p} \to G/K$, where K acts on $\mathfrak{p}$ by the Adjoint representation. Then the polar map

$$\phi\colon G \times_K \mathfrak{p} \longrightarrow G^{\mathbb{C}}/K^{\mathbb{C}}, \quad [g, X] \mapsto g \exp iXK^{\mathbb{C}}$$

may be viewed as a G-equivariant map from $T(G/K)$ with values in $G^{\mathbb{C}}/K^{\mathbb{C}}$. The domain D is the image in $G^{\mathbb{C}}/K^{\mathbb{C}}$ of the largest connected neighbourhood Ω of G/K in $T(G/K)$, where the differential of the polar map ϕ has maximum rank. It turns out that the domain Ω also coincides with the maximal connected neighbourhood of G/K in $T(G/K)$ where the so-called adapted complex structure exists (see [LS],[Sz1] [GS1], [GS2]). As a result, Ω carries a canonical G-invariant Kähler structure extending the Riemannian structure of G/K. Since the polar map is globally injective on Ω, such a G-invariant Kähler structure can be pushed-forward onto D. This makes the domain D interesting also from the geometric point of view.

In this paper we consider a semisimple pseudo-Riemannian symmetric space G/H, embedded in its complexification $G^{\mathbb{C}}/H^{\mathbb{C}}$, with the aim of determining how the above facts generalize to this situation. If

G/H is a compactly causal symmetric space, then $G^{\mathbb{C}}/H^{\mathbb{C}}$ contains a maximal G-invariant Stein domain D having G/H in its Shilov boundary. The domain D, which may be considered as a one-sided complexification of G/H, has been investigated in [Ne1].

However, in the general case the Stein manifold $G^{\mathbb{C}}/H^{\mathbb{C}}$ might contain no Stein G-invariant subdomains, and that the G-action might fail to be proper on every invariant open subset of $G^{\mathbb{C}}/H^{\mathbb{C}}$. This happens for example when G/H is a real hyperboloid $SO(p,q)_0/SO(p-1,q)$, with $p,q>2$. Let $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}$ be the decomposition of $\mathfrak{g}$ induced by the symmetry of G/H at the base point. Identify the tangent bundle $T(G/H)$ with the homogeneous vector bundle $G\times_H\mathfrak{q}\to G/H$, where H acts on $\mathfrak{q}$ by the Adjoint representation. Consider then the corresponding polar map

$$\phi\colon G\times_H\mathfrak{q}\longrightarrow G^{\mathbb{C}}/H^{\mathbb{C}},\quad [g,X]\mapsto g\exp iXH^{\mathbb{C}}.$$

Like in the Riemannian case, the maximal connected set $G/H\subset\Omega$ where the polar map has maximum rank is a proper G-invariant subdomain in $T(G/H)$. The domain Ω also coincides with the maximal connected neighbourhood of G/H in $T(G/H)$ where the adapted complex structure exists (cf. [Sz2], [HI]). As a result, Ω carries a canonical G-invariant pseudo-Kähler structure extending the pseudo-Riemannian structure of G/H.

If G/H is a pseudo-Riemannian symmetric space of rank one, we show that the restriction of the polar map $\phi\colon\Omega\longrightarrow G^{\mathbb{C}}/H^{\mathbb{C}}$ is globally injective. In particular, the canonical G-invariant pseudo-Kähler structure of Ω can be pushed forward onto $D=\phi(\Omega)$. Moreover, if the metric on G/H has signature (p,q), the domain D an increasing union of q-complete domains (by our convention, 0-complete is Stein). In general, D is not a Stein domain.

In the higher rank case, the polar map $\phi\colon\Omega\longrightarrow G^{\mathbb{C}}/H^{\mathbb{C}}$ in generally not injective. However, ϕ is injective on close orbits of maximal dimension in Ω. Moreover, the restriction of ϕ to distinguished G-invariant subsets of Ω defines G-equivariant coverings of principal orbit strata in $D=\phi(\Omega)$.

As a result, a canonical G-invariant pseudo-Kähler structure is defined on coverings of principal orbit strata in D or on suitable neighbourhoods of closed orbits of maximal dimension in D. These results extend the ones obtained by Fels in the group case, using different methods (cf. [Fe]).

The paper is organized as follows. In section 1, we recall some definitions and set up the notation. In section 2, we give several characterizations of the singular set of the differential of the polar map and we define the distinguished G-invariant neighbourhood Ω of G/H in its tangent bundle $T(G/H)$. In section 3, we briefly recall the definition and the main properties of the adapted complex structure on $T(G/H)$. In section 4, we prove two preliminary lemmas about the G-action on $G^{\mathbb{C}}/H^{\mathbb{C}}$, which may be of independent interest. These lemmas are used to prove the main results in sections 5 and 6. In section 5, we deal with semisimple symmetric spaces of rank one. We also work out in detail a family of examples. In section 6, we deal with symmetric spaces of rank higher than one.

1. Preliminaries.

A semisimple symmetric space is a coset space G/H, where G is a real semisimple Lie group and $H \subset G$ is an open subgroup of the fixed point group of an involution $\tau: G \longrightarrow G$.

In what follows, we consider *semisimple symmetric spaces G/H which admit a G-equivariant embedding into a simply connected complexification $G^{\mathbb{C}}/H^{\mathbb{C}}$.* They arise in the following way. Start with a simply connected complex semisimple Lie group $G^{\mathbb{C}}$ endowed with a Cartan involution Θ, a conjugation σ (different from Θ) and a holomorphic involution τ satisfying the commutativity relations

$$\sigma\tau = \tau\sigma, \quad \Theta\sigma = \sigma\Theta, \quad \Theta\tau = \tau\Theta. \tag{1.1}$$

Denote by $U = Fix(\Theta, G^{\mathbb{C}})$ the corresponding compact real form, by $G = Fix(\sigma, G^{\mathbb{C}})$ the corresponding non-compact real form and by $H^{\mathbb{C}} = Fix(\tau, G^{\mathbb{C}})$ the complex fixed point subgroup of τ. By (1.1), the restriction of Θ to G defines a Cartan involution θ of G so that the maximal compact subgroup of G is given by $K = G \cap U$. Similarly, the restriction of τ to G defines an involution of G commuting with θ, whose fixed point subgroup is given by $H = G \cap H^{\mathbb{C}}$. In this way, the space G/H admits an equivariant embedding in the complex symmetric space $G^{\mathbb{C}}/H^{\mathbb{C}}$ as the G-orbit of the base point $eH^{\mathbb{C}}$.

The product involution $\sigma^c := \sigma\tau$ defines a conjugation of $G^{\mathbb{C}}$ with real form denoted by G^c; since $\sigma^c\tau = \tau\sigma^c$, the restriction of τ to G^c defines an involution of G^c, with fixed point subgroup $G^c \cap H^{\mathbb{C}} = H$. The restriction of Θ to G^c defines a Cartan involution θ^c of G^c, commuting with τ. The G-orbit and G^c-orbit of the base point $eH^{\mathbb{C}} \in G^{\mathbb{C}}/H^{\mathbb{C}}$ define transversal totally real embeddings

$$G/H \hookrightarrow G^{\mathbb{C}}/H^{\mathbb{C}} \hookleftarrow G^c/H$$

of so-called *c-dual symmetric spaces* [HO]. One has that

$$\dim_{\mathbb{R}} G/H = \dim_{\mathbb{R}} G^c/H = \dim_{\mathbb{C}} G^{\mathbb{C}}/H^{\mathbb{C}}.$$

Troughout the paper, the Lie algebra of a group is denoted by the corresponding gothic letter. For example, $\mathfrak{g}$ and $\mathfrak{g}^{\mathbb{C}}$ denote the Lie algebras of G and $G^{\mathbb{C}}$, respectively. An involution of a group and the derived involution of its Lie algebra are denoted by the same symbol. The commutativity relations (1.1) ensure that the decompositions induced by Θ, σ and τ on $\mathfrak{g}^{\mathbb{C}}$ and by their restrictions on $\mathfrak{g}$ are all compatible with each other. For example, if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is the decomposition of $\mathfrak{g}$ induced by τ, then both $\mathfrak{h}$ and $\mathfrak{q}$ are θ-stable and $\mathfrak{g}$ has a combined decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \cap \mathfrak{k} \ \oplus \ \mathfrak{h} \cap \mathfrak{p} \ \oplus \ \mathfrak{q} \cap \mathfrak{k} \ \oplus \ \mathfrak{q} \cap \mathfrak{p}.$$

In this setting, the c-dual symmetric spaces G/H and G^c/H are the analogues of the non-compact Riemannian symmetric space G/K and its compact dual U/K in $G^{\mathbb{C}}/K^{\mathbb{C}}$. The decompositions of $\mathfrak{g}$ and $\mathfrak{g}^c$ by τ are given by

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \quad \text{and} \quad \mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q},$$

respectively. If $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \tau)$ is a symmetric algebra, a Cartan subspace of $\mathfrak{q}$ is by definition a maximal abelian subspace $\mathfrak{c} \subset \mathfrak{q}$ consisting of semisimple elements. The rank of a symmetric space G/H is the dimension of an arbitrary Cartan subspace in $\mathfrak{q}$.

In this paper, we do not deal with the group case, i.e. with symmetric spaces of the form $G \times G/Diag(G)$. Such spaces have already been investigated in [Br2],[Fe].

2. A distinguished G-invariant neighbourhood of G/H in $T(G/H)$.

Let G/H be a semisimple symmetric space. Identify the tangent bundle $T(G/H)$ with the homogeneous vector bundle $G\times_H \mathfrak{q}$, defined as the quotient of $G\times\mathfrak{q}$ by the H-action $h\cdot(g,X):=(gh^{-1},Ad_hX)$. Identify G/H with the zero section in $G\times_H\mathfrak{q}$. In this way, the *polar map*

$$\phi\colon G\times_H\mathfrak{q}\longrightarrow G^{\mathbb{C}}/H^{\mathbb{C}},\quad [g,X]\mapsto g\exp iXH^{\mathbb{C}}, \tag{2.1}$$

defines a G-equivariant map from the tangent bundle of G/H with values in $G^{\mathbb{C}}/H^{\mathbb{C}}$.

The next proposition gives several characterizations of the set where the map ϕ has non-singular differential. By the G-equivariance of ϕ it is sufficient to consider the differential $d\phi$ at the points $[e,X]$, with $X\in\mathfrak{q}$.

Proposition 2.1. *Let ϕ be the map defined in (2.1).*
(i) Let $X\in\mathfrak{q}$. Then the differential $d\phi_{[e,X]}$ is non-singular if and only if

$$Ad_{\exp iX}\mathfrak{h}\cap i\mathfrak{q}=\{0\}. \tag{2.2}$$

(ii) Let $X\in\mathfrak{q}$. Then $Ad_{\exp iX}\mathfrak{h}\cap i\mathfrak{q}=\{0\}$ if and only if $ad_X\colon\mathfrak{g}\to\mathfrak{g}$ has no real eigenvalue

$$\lambda\in\mathbb{R},\quad \lambda\equiv\pi/2 \bmod \pi.$$

(iii) Let $X\in\mathfrak{q}$ and let $X=X_s+X_n$ be its Jordan decomposition, with X_s semisimple, X_n nilpotent, X_s, $X_n\in\mathfrak{q}$. Then $Ad_{\exp iX}\mathfrak{h}\cap i\mathfrak{q}=\{0\}$ if and only if

$$Ad_{\exp iX_s}\mathfrak{h}\cap i\mathfrak{q}=\{0\}. \tag{2.3}$$

If the semisimple element X_s sits in a Cartan subspace $\mathfrak{c}\subset\mathfrak{q}$ and $\Delta_{\mathfrak{c}}$ denotes the restricted root system of $\mathfrak{g}^{\mathbb{C}}$ under $\mathfrak{c}^{\mathbb{C}}$, condition (2.3) is satisfied if and only if

$$\alpha(X_s)\not\equiv\pi/2 \bmod \pi,\quad \textit{for all } \alpha\in\Delta_{\mathfrak{c}}.$$

Proof.
(i) The proof is similar to that of Prop.3 in [AG], where the compact real form U of $G^{\mathbb{C}}$ is replaced by the c-dual real form G^c. Let G^c/H the c-dual symmetric space of G/H. Observe that for $X\in\mathfrak{q}$, one has that

$$u=\exp iX\in G^c\quad\text{and}\quad uH\in G^c/H.$$

Consider the diagram

$$\begin{tikzcd}
G\times G^c \arrow[r, "\pi"] \arrow[d, "1\times\rho_u^{-1}"] & G\times_H(G^c/H) \arrow[r, "\phi"] & G^{\mathbb{C}}/H^{\mathbb{C}} \\
G\times G^c \arrow[r, "\psi_u"] & G^{\mathbb{C}}/H^{\mathbb{C}} \arrow[ur, "\tau_u"] &
\end{tikzcd}$$

where the maps are defined as follows
$\pi\colon G\times G^c\longrightarrow G\times_H G^c/H,\quad (g,v)\mapsto[g,vH]=[gh^{-1},hvH],\quad h\in H;$
$\phi\colon G\times_H(G^c/H)\longrightarrow G^{\mathbb{C}}/H^{\mathbb{C}},\quad [g,vH]\mapsto gvH^{\mathbb{C}};$
$1\times\rho_{u^{-1}}\colon G\times G^c\longrightarrow G\times G^c,\quad (g,v)\mapsto(g,vu^{-1}),\quad u\in G^c;$
$\psi_u\colon G\times G^c\longrightarrow G^{\mathbb{C}}/H^{\mathbb{C}},\quad (g,v)\mapsto u^{-1}gvuH^{\mathbb{C}},\quad u\in G^c;$
$\tau_u\colon G^{\mathbb{C}}/H^{\mathbb{C}}\longrightarrow G^{\mathbb{C}}/H^{\mathbb{C}},\quad x\mapsto uxH^{\mathbb{C}},\quad u\in G^c.$
One easily checks that the diagram is commutative:

$$\phi\pi(g,v)=\tau_u\psi_u 1\times\rho_{u^{-1}}(g,v),\qquad \text{for all } (g,v)\in G\times G^c,\ \text{and } u\in G^c.$$

In order to determine the points $[e,X]\in G^{\mathbb{C}}\times_H\mathfrak{q}$ where the differential $d\phi$ has maximum rank, we examine the rank of $d\phi$ at the points $[e,uH]$, $u=\exp iX\in G^c$.

Since $1 \times \rho_{u^{-1}}$ and τ_u are diffeomorphisms and $d\pi$ is onto, such rank is maximum if and only if the map ψ_u has differential of maximum rank at (e,e). The differential $d\psi_{u,(e,e)}: \mathfrak{g} \oplus \mathfrak{g}^c \longrightarrow \mathfrak{q}^{\mathbb{C}}$ is given by

$$d\psi_{u,(e,e)}(A,B) = Ad_{u^{-1}}(A+B) \mod \mathfrak{h}^{\mathbb{C}}, \quad (A,B) \in \mathfrak{g} \oplus \mathfrak{g}^c, \quad \mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q},$$

and the kernel is given by

$$\ker d\psi_{u,(e,e)} = \{(C,-C) \mid C \in \mathfrak{h}\} \oplus \{(0, Ad_u D) \mid D \in \mathfrak{h}\} \oplus \{(E,0) \mid E \in Ad_u i\mathfrak{h} \cap \mathfrak{q}\}.$$

The differential $d\psi_{u,(e,e)}$ has maximum rank if and only if $\dim(\ker d\psi_{u,(e,e)}) = 2\dim \mathfrak{h}$. This happens precisely when

$$Ad_u \mathfrak{h} \cap i\mathfrak{q} = \{0\}.$$

(ii) Let $X \in \mathfrak{q}$. The operator $Ad_{\exp iX}: \mathfrak{g}^c \longrightarrow \mathfrak{g}^c$ can be written as

$$Ad_{\exp iX} = \exp ad_{iX} = \cos ad_X + i \sin ad_X. \tag{2.4}$$

Since $\cos ad_X(\mathfrak{h}) \subset \mathfrak{h}$ and $i \sin ad_X(\mathfrak{h}) \subset i\mathfrak{q}$, the condition $Ad_{\exp iX}\mathfrak{h} \cap i\mathfrak{q} = \{0\}$ is equivalent to the injectivity of the map $\cos ad_X|\mathfrak{h}: \mathfrak{h} \to \mathfrak{h}$, namely

$$\cos ad_X(H) \neq 0, \qquad \text{for all } H \in \mathfrak{h},\ H \neq 0. \tag{2.5}$$

Indeed if $\cos ad_X: \mathfrak{g} \to \mathfrak{g}$ is injective, condition (2.5) is clearly satisfied. Conversely, assume that $\cos ad_X: \mathfrak{g} \to \mathfrak{g}$ is not injective. Then $\cos ad_X$ has an eigenvalue $\mu = 0$ and ad_X has a *real* real eigenvalue $\lambda \equiv \pi/2 \mod \pi$. In particular, there exists a λ-eigenvector $Z \in \mathfrak{g}$, and $\tau Z \neq \pm Z$. Since $\ker \cos ad_X$ is τ-stable, the vectors $H := Z + \tau Z$ and $Q := Z - \tau Z$ define non-zero elements in $\ker \cos ad_X \cap \mathfrak{h}$ and $\ker \cos ad_X \cap \mathfrak{q}$, respectively. It follows that $\cos ad_X|\mathfrak{h}: \mathfrak{h} \to \mathfrak{h}$ is not injective either. In conclusion, $\cos ad_X|\mathfrak{h}$ is injective if and only if $\cos ad_X: \mathfrak{g} \to \mathfrak{g}$ is injective, and this happens if and only if ad_X has no eigenvalues

$$\begin{cases} \lambda \in \mathbb{R} \\ \lambda \equiv \pi/2 \mod \pi. \end{cases}$$

(iii) We already saw in (ii) that condition (2.2) is equivalent to the injectivity of the operator $\cos ad_X: \mathfrak{g} \longrightarrow \mathfrak{g}$. So we need to prove that $\cos ad_X$ is injective if and only if $\cos ad_{X_s}$ is injective. From the decomposition $X = X_s + X_n$ and the fact that $[X_s, X_n] = 0$, it follows that

$$\cos ad_X = \cos ad_{X_s} + (\cos ad_{X_s}(\cos ad_{X_n} - I) - \sin ad_{X_s} \sin ad_{X_n}) \qquad (2.6).$$

Since $\cos ad_{X_s}$ is semisimple, $\cos ad_{X_s}(\cos ad_{X_n} - I) - \sin ad_{X_s} \sin ad_{X_n}$ is nilpotent and these operators commute, equation (2.6) is the Jordan decomposition of $\cos ad_X$. It follows that $\cos ad_X$ is injective if and only if $\cos ad_{X_s}$ is injective, as requested.

Remark 2.2. By Proposition 2.1, the regular set of $d\phi_{[e,X]}$ is a *proper* Ad_H*-invariant subdomain* of $\mathfrak{q}$. A result by Halversheid (cf. [Ha], p.17), implies that the singular subset of $d\phi_{[e,X]}$ in $\mathfrak{q}$ disconnects $\mathfrak{q}$. So the connected component of the regular set of $d\phi$ containing G/H is a proper G-invariant subdomain of $G \times_H \mathfrak{q}$, namely

$$\Omega = G \times_H \omega, \qquad \omega = \{X \in \mathfrak{q} \mid |\lambda| < \pi/2, \text{ for all } \lambda \in spec(ad_X) \cap \mathbb{R}\}. \tag{2.7}$$

(Here $spec(L)$ denotes the spectrum of an operator L). Since ω is starlike, Ω is smoothly retractible to G/H.

By Proposition 2.1(ii), one has that $d\phi_{[e,0]}$ is non-singular and $d\phi_{[e,N]}$ is non-singular for every nilpotent element $N \in \mathfrak{q}$. In this framework, one can consider the map $p: \mathfrak{q} \longrightarrow \mathfrak{q}||Ad_H$, which associates to $X \in \mathfrak{q}$ the unique closed Ad_H-orbit in the closure of $Ad_H(X)$ (see [Br1]). Each fiber of this map contains a unique closed orbit, which is also the unique orbit of minimum dimension. Recall that an Ad_H-orbit in $\mathfrak{q}$ is closed if and only if X is semisimple (cf. [vD]). By Proposition 2.1(iii), both ω and its boundary $\partial\omega$ are Ad_H-saturated sets, i.e. satisfy $p^{-1}p(\omega) = \omega$ (resp. $p^{-1}p(\partial\omega) = \partial\omega$).

Proposition 2.3. *Let $\mathfrak{q} = \mathfrak{q}\cap\mathfrak{k}\oplus\mathfrak{q}\cap\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{q}$, let $\mathfrak{a}\subset\mathfrak{q}\cap\mathfrak{p}$ be a maximal abelian subspace (not necessarily maximal abelian in $\mathfrak{q}$) and let $\Delta_{\mathfrak{a}} = \Delta_{\mathfrak{a}}(\mathfrak{g},\mathfrak{a})$ be the corresponding restricted root system. Define $\omega_0 = \{A\in\mathfrak{a} \mid |\alpha(A)| < \pi/2,\ \forall\alpha\in\Delta_{\mathfrak{a}}\}$. Then the set ω defined in (2.7) is given by*

$$\omega = p^{-1}(p(Ad_H(\mathfrak{q}\cap\mathfrak{k}\oplus\omega_0))).$$

Proof. Both sets $Ad_H(\mathfrak{q}\cap\mathfrak{k}\oplus\omega_0)$ and ω are Ad_H-stable. So we need to show that they have the same semisimple elements. Let $X\in Ad_H(\mathfrak{q}\cap\mathfrak{k}\oplus\omega_0)$. If $\lambda\in spec(ad_X)\cap\mathbb{R}$, then $|\lambda|<\pi/2$. This shows that $Ad_H(\mathfrak{q}\cap\mathfrak{k}\oplus\omega_0)\subset\omega$.
To prove the converse statement, let $X\in\omega$ be a semisimple element. Then X is H-conjugate to an element $S = S_{\mathfrak{k}} + S_{\mathfrak{p}} = Ad_h X$ in a standard Cartan subspace $\mathfrak{c} = \mathfrak{c}_k\oplus\mathfrak{c}_{\mathfrak{p}}$, with $c_{\mathfrak{k}}\subset\mathfrak{q}\cap\mathfrak{k}$ and $\mathfrak{c}_{\mathfrak{p}}\subset\mathfrak{a}$ (see [Ma], p.79). In particular $S_{\mathfrak{p}}\in\omega_0\subset\mathfrak{a}$ and $X\in Ad_H(\mathfrak{q}\cap\mathfrak{k}\oplus\omega_0)$.

3. The adapted complex structure on Ω.

The notion of an adapted complex structure on the tangent bundle of a Riemannian manifold was introduced and developed in [LS], [Sz1], [GS1], [GS2]. Its definition and many of its features have a straightforward generalization to the pseudo-Riemannian case [Sz2].

Definition 3.1. Let (M,g) be a pseudo-Riemannian manifold and let U be an open neighbourhood of M in its tangent bundle TM. A complex structure J on U is called *adapted* if for every geodesic $\gamma\colon\mathbb{R}\longrightarrow M$, the differential

$$d\gamma\colon T\mathbb{R}\cong\mathbb{C}\longrightarrow TM,\quad (x,y)\mapsto(\gamma(x), y\gamma'(x))$$

is holomorphic on $d\gamma^{-1}(U)$.

A semisimple symmetric space G/H is in a natural way a pseudo-Riemannian manifold. The tangent space $T(G/H)_{eH}$ to G/H at the base point eH, can be identified with $\mathfrak{q} = \mathfrak{q}\cap\mathfrak{k}\oplus\mathfrak{q}\cap\mathfrak{p}$; the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{q}\times\mathfrak{q}$

$$B_{\mathfrak{g}}|\mathfrak{q}(X,Y) = Tr(ad_X\circ ad_Y),\quad X,Y\in\mathfrak{q}$$

induces on G/H a G-invariant metric g of signature (σ^+,σ^-), where $\sigma^+ = \dim\mathfrak{q}\cap\mathfrak{p}$ and $\sigma^- = \dim\mathfrak{q}\cap\mathfrak{k}$.

Let Ω be the domain in $G\times_H\mathfrak{q}\cong T(G/H)$ defined in (2.7). In analogy with the Riemannian case, one has that (cf. [HI], [Sz2])

(a) *The complex structure on Ω given by the pull-back of the complex structure of $G^{\mathbb{C}}/H^{\mathbb{C}}$ by the polar map (2.1) is adapted.*
(b) *Ω is the "largest" G-invariant connected subset of $T(G/H)$ containing G/H and carrying an adapted complex structure.*
(c) *The energy function*

$$E\colon T(G/H)\longrightarrow\mathbb{R},\quad E(x,v) := \frac{1}{2}g_x(v,v)^2,\quad x\in G/H,\ v\in T(G/H)_x \tag{3.1}$$

is a smooth G-invariant function on Ω. Its complex Hessian of E has σ^+ positive and σ^- negative eigenvalues.
(d) *The formula*

$$h(Z,W) := -\frac{i}{2}\partial\bar\partial E(Z,\overline{W}),\quad Z,W\in T^{\mathbb{C}}\Omega. \tag{3.2}$$

defines a G-invariant pseudo-Kähler metric on Ω with the same signature as g.
(e) *The function $\sqrt{|E|}$ satisfies the homogeneous complex Monge-Ampere equation*

$$(\partial\bar\partial\sqrt{|E|})^n\equiv 0,$$

outside the null set in Ω.

4. The G-action on $G^{\mathbb{C}}/H^{\mathbb{C}}$.

In this section we prove two preliminary lemmas regarding the G-action on $G^{\mathbb{C}}/H^{\mathbb{C}}$. These are used in the next sections to prove the main results.

Resume the notation introduced in section 1. Denote by $Aut_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ the group of the real automorphisms of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$. Define a map $\eta: G^{\mathbb{C}} \to Aut_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ by $\eta(x) = \sigma Ad_x \tau Ad_{x^{-1}}$ (see [Ma], p.51). An element $x \in G^{\mathbb{C}}$ is called *regular semisimple with respect to* σ, τ if its image $\eta(x)$ is a regular semisimple element in the real algebraic group $Aut_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ or equivalently when it sits on a closed $G \times H^{\mathbb{C}}$-orbit of maximal dimension in $G^{\mathbb{C}}$. Let $\bar{x}$ denote the image of x under the canonical projection $\pi: G^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}$. Then x is regular semisimple with respect to σ, τ if and only if its image $\bar{x} \in G^{\mathbb{C}}/H^{\mathbb{C}}$ sits on a closed G-orbit of maximal dimension in $G^{\mathbb{C}}/H^{\mathbb{C}}$ (see [Ma]). Let $X \in \mathfrak{q}$, such that $x = \exp iX \in G^{\mathbb{C}}$ is a regular semisimple with respect to σ, τ. Then X sits in some Cartan subspace $\mathfrak{c}$ in $\mathfrak{q}$ and can be characterized in terms of the restricted roots $\Delta_{\mathfrak{c}} = \Delta_{\mathfrak{c}}(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$ as follows (see [Ge], Prop. 3.14): X sits in the complement in $\mathfrak{c}$ of the set

$$\bigcup_{\alpha \in \Delta_{\mathfrak{c}}^r} \{\alpha(X) \equiv 0 \bmod \pi/2\} \bigcup_{\alpha \in \Delta_{\mathfrak{c}}^i} \{\alpha(X) = 0\} \bigcup_{\alpha \in \Delta_{\mathfrak{c}}^c} \left\{ \begin{cases} \mathrm{Re}\alpha(X) \equiv 0 \bmod \pi/2 \\ \mathrm{Im}\alpha(X) = 0 \end{cases} \right\} \tag{4.1}$$

(here $\Delta_{\mathfrak{c}}^r$, $\Delta_{\mathfrak{c}}^i$, $\Delta_{\mathfrak{c}}^c$ denote the sets of roots which restricted to $\mathfrak{c}$ take real, imaginary or complex values, respectively). Observe that one such X is in particular regular semisimple in $\mathfrak{q}$, which by definition means that the centralizer $Z_{\mathfrak{q}}(X)$ is abelian and equal to $\mathfrak{c}$. This last condition is characterized by $\alpha(X) \neq 0$, for all $\alpha \in \Delta_{\mathfrak{c}}$.

Remark 4.1. Let $X = X_s + X_n$ be the Jordan decomposition of X in $\mathfrak{q}$. Let $x = \exp iX = x_s x_n$ be the corresponding Jordan decomposition in $G^{\mathbb{C}}$, with $x_s = \exp iX_s$ and $x_n = \exp iX_n$. One has that

$$\eta(x) = Ad_{x_n^{-2}} \sigma\tau Ad_{x_s^{-2}} = Ad_{x_n^{-2}} \eta(x_s).$$

Set $u = Ad_{x_n^{-2}}$ and $s = \sigma\tau Ad_{x_s^{-2}} = Ad_{x_s^{-2}} \sigma\tau$. It is easy to check that u is unipotent, s is semisimple, and $su = us$. So $\eta(x) = us$ is the Jordan decomposition of $\eta(x)$ in $Aut_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$. Moreover, one has that

$$\sigma(iX_n) = Ad_{x_s} \tau Ad_{x_s^{-1}}(iX_n) = -iX_n.$$

Then by [Ma], Prop.2 (ii), p.66, the decomposition $x = x_s x_n$ also coincides with the lifting to $G^{\mathbb{C}}$ of the Jordan decomposition of $\eta(x)$ in $Aut_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$.

Lemma 4.2. *Let $X \in \mathfrak{q}$. Let $x = \exp iX$ be the corresponding element in $G^{\mathbb{C}}$ and $\bar{x}$ its image in $G^{\mathbb{C}}/H^{\mathbb{C}}$. Then*

(i) The isotropy subgroup of $\bar{x}$ in G is given by $G_{\bar{x}} = \{g \in G \mid gx^2 = x^2\tau(g)\}$;

(ii) $G_{\bar{x}} \subset Z_G(x^4)$;

(iii) $Z_G(x^4) = Z_G(x^2)$ implies $G_{\bar{x}} = Z_H(x^2)$;

(iv) Assume that $[e, X] \in G \times_H \mathfrak{q}$ lies in the regular set of $d\phi$. Then $G_{\bar{x}} = Z_H(x^2)$;

(v) Let $\mathfrak{c}$ be a Cartan subspace in $\mathfrak{q}$ and let $X \in \mathfrak{c}$. Assume that $x = \exp iX$ is a regular semisimple element with respect to σ, τ. Then $G_{\bar{x}} = Z_H(x) = Z_H(X) = Z_H(\mathfrak{c})$.

Proof.

(i) By definition, $g \in G_{\bar{x}}$ if there exists $h_c \in H^{\mathbb{C}}$ such that $gx = xh_c$. Write $h_c = x^{-1}gx$. Since $h_c \in H^{\mathbb{C}}$, one has that $\tau(h_c) = h_c$. This is equivalent to

$$x\tau(g)x^{-1} = x^{-1}gx \quad \text{and} \quad x^2\tau(g) = gx^2.$$

Conversely, assume that $g \in \{g \in G \mid gx^2 = x^2\tau(g)\}$. Then

$$gx = gx^2 \cdot x^{-1} = x^2\tau(g)x^{-1} = x \cdot x\tau(g)x^{-1}.$$

Define $h_c := x\tau(g)x^{-1}$. One has that

$$\tau(h_c) = \tau(x\tau(g)x^{-1}) = x^{-1}gx = x^{-1}gx^2x^{-1} = x^{-1}x^2\tau(g)x^{-1} = x\tau(g)x^{-1} = h_c.$$

So $h_c \in H^{\mathbb{C}}$ and $g \in G_{\bar{x}}$, as desired.
(ii) Let $g \in G_{\bar{x}}$. Then by (i) one has that $gx^2 = x^2\tau(g)$ and $g = x^2\tau(g)x^{-2}$. Since $g \in G$, one has that $\sigma(g) = g$, which is equivalent to

$$\sigma(x^2)\tau(g)\sigma(x^{-2}) = x^2\tau(g)x^{-2} \quad \Leftrightarrow \quad x^4\tau(g) = \tau(g)x^4 \quad \Leftrightarrow \quad x^4 g = gx^4.$$

In other words, $g \in G_{\bar{x}}$ implies $g \in Z_G(x^4)$.
(iii) Assume that $Z_G(x^2) = Z_G(x^4)$. By (ii) one has that $G_{\bar{x}} \subset Z_G(x^2)$. This together with (i) implies that $gx^2 = x^2\tau(g) = x^2 g$. Then $g \in H$ and $G_{\bar{x}} = Z_H(x^2)$.
(iv) By (iii) it is sufficient to show that $Z_{G^{\mathbb{C}}}(x^4) = Z_{G^{\mathbb{C}}}(x^2)$, and actually that $Z_{G^{\mathbb{C}}}(x^4) \subset Z_{G^{\mathbb{C}}}(x^2)$, the opposite inclusion being obvious. Assume first that X is a *semisimple* element in some Cartan subspace $\mathfrak{c} \subset \mathfrak{q}$. Since $G^{\mathbb{C}}$ is simply connected, the centralizers $Z_{G^{\mathbb{C}}}(x^2)$ and $Z_{G^{\mathbb{C}}}(x^4)$ are connected (cf. [Hu]) and hence determined by their Lie algebras. Denote by $\Delta_{\mathfrak{c}} = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$ the restricted root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{c}^{\mathbb{C}}$. Let $\mathfrak{g}^{\mathbb{C}} = Z_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{c}^{\mathbb{C}}) \oplus \bigoplus_{\alpha \in \Delta_{\mathfrak{c}}} \mathfrak{g}^{\alpha}$ be the corresponding root decomposition. The Lie algebra of $Z_{G^{\mathbb{C}}}(x^4)$ is given by

$$Z_{\mathfrak{g}^{\mathbb{C}}}(x^4) = \{Z \in \mathfrak{g}_{\mathbb{C}} \mid Ad_{x^4}Z = Z\} = Z_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{c}) \oplus \bigoplus_{\alpha(X)=0} \mathfrak{g}^{\alpha} \oplus \bigoplus_{\substack{\alpha(X)\neq 0 \\ \alpha(4X)\equiv 0 \mod 2\pi}} \mathfrak{g}^{\alpha}.$$

By Prop. 2.1(iii), the element $[e, X]$ lies in the regular set of $d\phi$ if and only if $\alpha(X) \neq (2k+1)\pi/2,\ k \in \mathbb{Z}$, for all $\alpha \in \Delta_{\mathfrak{c}}$. As a consequence, $\beta(4X) \equiv 0 \mod 2\pi$, for some root β, if and only if

$$\beta(X) = m\pi/2, \quad \text{for some } m \in 2\,\mathbb{Z},\ m \neq 0.$$

It follows that $\beta(2X) \equiv 0 \mod 2\pi$, which means that $Z_{\mathfrak{g}^{\mathbb{C}}}(x^4) \subset Z_{\mathfrak{g}^{\mathbb{C}}}(x^2)$ and $Z_{G^{\mathbb{C}}}(x^4) \subset Z_{G^{\mathbb{C}}}(x^2)$, as requested.

Assume now that X is *non-semisimple*. Let $X = X_s + X_n$ be its Jordan decomposition in $\mathfrak{q}$, with $X_s, X_n \in \mathfrak{q}$ and $[X_s, X_n] = 0$. Write $x = x_s x_n = \exp iX_s \exp iX_n$. By Remark 4.1 and [Ma], Prop.2, p.66, the equation

$$g x_s x_n = x_s x_n h_c, \quad g \in G,\ h_c \in H^{\mathbb{C}}$$

is equivalent to the system

$$\begin{cases} g \exp iX_s = \exp iX_s h_c \\ Ad_g X_n = X_n. \end{cases}$$

In particular, $g \in G_{\bar{x}_s} \cap Z_G(X_n)$. Since $[e, X_s]$ lies in the regular set of $d\phi$ (by Proposition 2.1(iii)), one has that $g \in H$. It follows that

$$G_{\bar{x}} = G_{\bar{x}_s} \cap Z_G(X_n) = Z_H(x_s^2) \cap Z_G(X_n) \subset H, \quad \text{and} \quad G_{\bar{x}} = Z_H(x^2),$$

as requested.
(v) If $x = \exp iX$ is a regular semisimple element with respect to σ, τ, then X is a semisimple element in some Cartan subspace $\mathfrak{c} \subset \mathfrak{q}$ and $[e, X]$ lies in the regular set of $d\phi$. The same argument used in (iv) shows that

$$Z_{\mathfrak{g}^{\mathbb{C}}}(x^4) = Z_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{c}^{\mathbb{C}}) = Z_{\mathfrak{g}^{\mathbb{C}}}(x) = Z_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{c}) = Z_{\mathfrak{g}^{\mathbb{C}}}(X).$$

As a consequence, the σ-stable connected groups $Z_{G^{\mathbb{C}}}(x)$, $Z_{G^{\mathbb{C}}}(x^4)$, $Z_{G^{\mathbb{C}}}(\mathfrak{c})$ and $Z_{G^{\mathbb{C}}}(X)$ all coincide (for the connectedness of such groups, see [Hu][St]). In particular (see also [Ge], Sect.3.3), one has that

$$G_{\bar{x}} = H \cap Z_{G^{\mathbb{C}}}(x) = Z_H(x) = H \cap Z_{G^{\mathbb{C}}}(X) = Z_H(X) = Z_H(\mathfrak{c}). \tag{4.2}$$

Lemma 4.3. *Let $X, Y \in \mathfrak{q}$. Let $x = \exp iX$, $y = \exp iY$ be the corresponding elements in $G^{\mathbb{C}}$ and $\bar{x}$, $\bar{y}$ their images in $G^{\mathbb{C}}/H^{\mathbb{C}}$. One has that:*
(i) $\bar{x}$, $\bar{y}$ sit on the same G-orbit if and only if $gx^2 = y^2\tau(g)$, for some $g \in G$;
(ii) $\bar{x}$, $\bar{y}$ sit on the same H-orbit in $G^{\mathbb{C}}/H^{\mathbb{C}}$ if in addition $gx^2 = y^2 g$ holds;
(iii) if $\bar{x}$, $\bar{y}$ sit on the same G-orbit, then $y^4 = gx^4g^{-1}$, for some $g \in G$.
Let X, Y be semisimple elements in the same Cartan subspace $\mathfrak{c} \subset \mathfrak{q}$. Assume that x, y are regular semisimple elements with respect to σ, τ and that $\bar{x}$, $\bar{y}$ sit on the same G-orbit, i.e. (4.3) holds. Then:
(iv) $g\tau(g)^{-1} \in Z_H(\mathfrak{c})$ and $\tau(g) = zg$, for some $z \in Z_H(\mathfrak{c})$, with $z^2 = 1$;
(v) $Ad_gX \in \mathfrak{c}$ and $g \in N_G(\mathfrak{c})$;
(vi) $gxg^{-1} = yq$, with $g \in N_G(\mathfrak{c})$ and $q^4 = 1$.

Proof.
(i) By definition, $\bar{x}$, $\bar{y}$ sit on the same G-orbit in $G^{\mathbb{C}}/H^{\mathbb{C}}$ if

$$gx = yh_c, \quad \text{for some } g \in G,\ h_c \in H^{\mathbb{C}}. \tag{4.3}$$

Write $h_c = y^{-1}gx$. Then $\tau(h_c) = h_c$ implies $gx^2 = y^2\tau(g)$. Conversely, assume that $gx^2 = y^2\tau(g)$, for some $g \in G$. Write $gx = gx^2x^{-1} = y^2\tau(g)x^{-1} = yy\tau(g)x^{-1}$ and set $h_c = y\tau(g)x^{-1}$. One can check that $\tau(h_c) = h_c$ and $h_c \in H^{\mathbb{C}}$. This shows that (4.3) is satisfied and $\bar{x}$, $\bar{y}$ sit on the same G-orbit.
(ii) If $g \in H$, then $\tau(g) = g$ and one has $gx^2 = y^2g$ by statement (i). Conversely, assume that (4.3) holds and that moreover $gx^2 = y^2g$, for $g \in G$. Then by (i) $\tau(g) = g$ and $g \in H$.
(iii) By (i), our assumption is equivalent to $gx^2 = y^2\tau(g)$, for some $g \in G$. Write $g = y^2\tau(g)x^{-2}$. Then $\sigma(g) = g$ implies $\tau(g)x^4 = y^4\tau(g)$. By applying the involution $\sigma\tau$ to both terms of the equality, we get $y^4 = Ad_gx^4$, as desired.
(iv) By applying the map $\eta_{\sigma\tau}(x) := x\sigma\tau(x)^{-1}$ to both terms of equation (4.3), we get

$$g\tau(g)^{-1}y = yh_c\sigma(h_c)^{-1}.$$

This means that $g\tau(g)^{-1}$ is an element in $G_{\bar{y}}$, the isotropy subgroup in G of $\bar{y}$. Since y is regular semisimple with respect to σ, τ, by Lemma 4.2(v), one has that $g\tau(g)^{-1} \in Z_H(\mathfrak{c})$. Equivalently, $\tau(g) = zg$, for some $z \in Z_H(\mathfrak{c})$. Since $\tau(z) = z = z^{-1}$, it follows that $z^2 = 1$.
(v) We first prove that $Ad_gX \in \mathfrak{q}$. We need to show that $\tau(Ad_gX) = -Ad_gX$, which is equivalent to $g^{-1}\tau(g) \in Z_G(X)$. From $\sigma\tau(y) = y$, $\sigma\tau(y^4) = y^4$ and $y^4 = Ad_gx^4$ (by (iii)), we obtain

$$y^4 = Ad_gx^4 = Ad_{\tau(g)}x^4 \quad \text{and} \quad g^{-1}\tau(g) \in Z_G(x^4).$$

Since x is regular semisimple with respect to σ, τ, by Lemma 4.2(v), one has that $Z_G(x^4) = Z_G(X)$. Then $g^{-1}\tau(g) \in Z_G(X)$ and $Ad_gX \in \mathfrak{q}$, as requested.
By (4.1), the operator ad_{4Y} has no real eigenvalue $\lambda = k2\pi$, $k \in \mathbb{Z} \setminus \{0\}$. As a consequence, the element $i4Y$ lies in the regular set of the differential of the exponential map exp: $\mathfrak{g}^{\mathbb{C}} \to G^{\mathbb{C}}$ (cf.[Va]). From $\exp i4Y = Ad_g \exp i4X = \exp i4Ad_gX$, by [Ne2], one obtains

$$[Y, Ad_gX] = 0, \quad \text{or equivalently} \quad Ad_gX \in Z_{\mathfrak{q}}(Y) = \mathfrak{c}.$$

Since X, Ad_gX are regular semisimple in $\mathfrak{q}$, from

$$\mathfrak{c} = Z_{\mathfrak{q}}(X) = Z_{\mathfrak{q}}(Ad_gX) = Ad_gZ_{\mathfrak{q}}(X) = Ad_g\mathfrak{c},$$

it follows that $g \in N_G(\mathfrak{c})$, as requested.
(vi) By (i) and (iv) we can write $gx^2g^{-1} = y^2\tau(g)g^{-1} = y^2z$, for some $z \in Z_H(\mathfrak{c})$, with $z^2 = 1$. By (v), we can also write $\exp i2(Ad_gX - Y) = \exp\gamma$, for some γ in the square lattice $\frac{1}{2}\Gamma$ (here Γ denotes the unit lattice in $i\mathfrak{c} \cap \mathfrak{u} \subset \mathfrak{g}^{\mathbb{C}}$). It follows that $Ad_gX = Y + \eta$, for some $\eta \in \frac{1}{4}\Gamma$, and $gxg^{-1} = yq$, for some $q \in \exp i\mathfrak{c}$, with $q^4 = 1$.

5.1. The rank-1 case.

Let G/H be a semisimple symmetric space of rank one. Let $\Omega \subset G \times_H \mathfrak{q}$ be the domain defined in (2.7). The main goal of this section is to prove that the polar map $\phi\colon \Omega \to G^{\mathbb{C}}/H^{\mathbb{C}}$ is globally injective (cf. Proposition 5.4). This shows that the image domain $D = \phi(\Omega)$ in $G^{\mathbb{C}}/H^{\mathbb{C}}$ is the direct generalizazion of the Akhiezer-Gindikin domain. However, in the pseudo-Riemannian case the domain D is generally not Stein.

Recall that in tthe rank-one case every Cartan subspace in $\mathfrak{q}$ is one-dimensional. Up to Ad_H-conjugacy, there are precisely two θ-stable Cartan subspaces in $\mathfrak{q}$: a compact one $\mathfrak{t} \subset \mathfrak{q} \cap \mathfrak{k}$ and a non-compact one $\mathfrak{a} \subset \mathfrak{q} \cap \mathfrak{p}$. As a consequence, given a semisimple element in $X \in \mathfrak{q}$, the eigenvalues of ad_X are either all real or all imaginary. Let $\mathfrak{a}$ be the non-compact Cartan subspace in $\mathfrak{q}$ and let $\Delta_{\mathfrak{a}} = \Delta_{\mathfrak{a}}(\mathfrak{g}, \mathfrak{a})$ denote the corresponding restricted root system. Then either $\Delta_{\mathfrak{a}} = \{\pm\alpha\}$ is of type A_1 (reduced case) or $\Delta_{\mathfrak{a}} = \{\pm\alpha, \pm 2\alpha\}$ is of typre BC_1 (non-reduced case). A list of all rank-1 semisimple symmetric algebras and their restricted root systems can be found in [OS]. Here is the list:

- $\mathfrak{so}(p+1, q+1)_0/\mathfrak{so}(p+1, q)$, for $p \geq 0,\ q \geq 0$
$q = 0$ Riemannian
$\Delta_{\mathfrak{a}} = A_1$.

- $\mathfrak{su}(p+1, q+1)/\mathfrak{s}(\mathfrak{u}(p+1, q) \times \mathfrak{u}(1))$, for $p \geq 0,\ q \geq 0$
$q = 0$ Riemannian
$\Delta_{\mathfrak{a}} = BC_1$.

- $\mathfrak{sp}(p+1, q+1)/\mathfrak{sp}(p+1, q) \times \mathfrak{sp}(1))$, for $p \geq 0,\ q \geq 0$
$q = 0$ Riemannian
$\Delta_{\mathfrak{a}} = BC_1$.

- $\mathfrak{sl}(n+2, \mathbb{R})/\mathfrak{gl}(n+1, \mathbb{R})$, for $n \geq 0$
$\Delta_{\mathfrak{a}} = BC_1$.

- $\mathfrak{sp}(n+2, \mathbb{R})/\mathfrak{sp}(n+1, \mathbb{R}) \times \mathfrak{sp}(1, \mathbb{R})$, for $n \geq 0$
$\Delta_{\mathfrak{a}} = BC_1$.

- $\mathfrak{f}_{4(-20)}/\mathfrak{so}(8, 1)$
$\Delta_{\mathfrak{a}} = BC_1$.

- $\mathfrak{f}_{4(4)}/\mathfrak{so}(5, 4)$
$\Delta_{\mathfrak{a}} = BC_1$.

Let $\mathfrak{l} = \mathfrak{b} \oplus \mathfrak{a} = \mathfrak{b}_{\mathfrak{k}} \oplus \mathfrak{b}_{\mathfrak{p}} \oplus \mathfrak{a}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ extending $\mathfrak{a}$ and let $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{l}^{\mathbb{C}})$ denote the corresponding root system. By [OS], Sect. 3.2, there exists a positive system Δ^+ in Δ which is stable both under $-\tau$ and $-\theta$. Let $\Pi = \{\lambda_i\}_{i=1,\dots,n}$ be the corresponding set of simple roots. Let Γ be the inverse lattice in $i\mathfrak{l}_{\mathbb{R}}$

$$\Gamma = \bigoplus_{\lambda_i \in \Pi} \mathbb{Z} 2\pi i h_{\lambda_i} \subset i\mathfrak{l}_{\mathbb{R}}, \quad \lambda_i(h_{\lambda_i}) = 2.$$

Recall that in a simply connected compact Lie group U, the lattice Γ coincides with the unit lattice $\exp_U^{-1}(e)$. Denote by $\langle \cdot, \cdot \rangle$ the euclidean inner product induced on $\mathfrak{l}_{\mathbb{R}}$ by the restriction of the Killing form of $\mathfrak{g}$. Then the following relations hold

$$\|h_{\lambda_i}\| = \frac{4}{\|\lambda_i\|}, \quad \langle h_{\lambda_i}, h_{\lambda_j} \rangle = \frac{2}{\|\lambda_i\|} c_{ij} = \frac{2}{\|\lambda_j\|} c_{ji}, \qquad c_{ij} \in \{0, -1, -2, -3\}.$$

Lemma 5.1. *Let Δ be a root system, all of whose roots have the same lenght (namely $A_n,\ D_n,\ E_6\ ,E_7,\ E_8$).*
(i) The generators $\{i\pi h_{\lambda_i}\}$ of $\frac{1}{2}\Gamma$ are vectors of shortest length in $\frac{1}{2}\Gamma$.
(ii) Let τ be an involutive automorphism of $\frac{1}{2}\Gamma$, which is an isometry and satisfies the conditions

$$\langle h_{\lambda_i}, \tau h_{\lambda_i}\rangle \geq 0, \quad i = 1, \ldots, n.$$

Then $X := i\pi(h_{\lambda_i} - \tau h_{\lambda_i})$ is the shortest vector in $\mathbb{R}X \cap \frac{1}{2}\Gamma$.

Proof. Without loss of generality we can prove the corresponding statements for a lattice L with basis vectors $\{v_1, \ldots, v_n\}$, satisfying

$$\|v_i\|^2 = 2, \qquad \langle v_i, v_j\rangle = 0, -1.$$

(i) Let $v = \sum_i m_i v_i,\ m_i \in\ \mathbb{Z}$. The formula

$$\|v\|^2 = 2(\sum_i m_i^2 - \sum_{i<j} \epsilon(i,j) m_i m_j) \geq 2, \quad \epsilon(i,j) = 0, 1$$

show that the square of the norm of every vector in the lattice satisfies is a positive even number. Hnece the basis vectors are vectors of shortest length as requested.

(ii) Write $v := v_i - \tau v_i$. The formula

$$\|v\|^2 = \|v_i\|^2 + \|\tau v_i\|^2 - 2\langle v_i, \tau v_i\rangle = 2(\|v_i\|^2 - \langle v_i, \tau v_i\rangle)0$$

implies that the inner product $\langle v_i, \tau v_i\rangle$ can only take the values 1 or 0. In the first case, $\|v\|^2 = 2$ and v is a vector of shortest length. In the second case, $\|v\|^2 = 4$. Assume that there is a vector $v' \in \mathbb{R}v \cap L$ which is shorter than v. By (i), $\|v'\|^2 = 2$. Since $v' = qv$, for some rational number $q \in \mathbb{Q}$, the equation

$$\|v'\|^2 = q^2 4 = 2$$

implies that $q = \frac{1}{\sqrt{2}}$. This is absurd.

Lemma 5.2. *Let Δ be the root system B_n. A generator $\{i\pi h_{\lambda_{i_0}}\}$ of $\frac{1}{2}\Gamma$ is a vector of shortest length if and only if λ_{i_0} is a long root in Δ.*

Proof. Without loss of generality we can prove the corresponding statement for a lattice L with basis $\{v_1, \ldots, v_{n-1}, v_n\}$ satisfying

$$\|v_1\|^2 = \ldots = \|v_{n-1}\|^2 = 2,\ \|v_n\|^2 = 4,$$

$$\langle v_i, v_j\rangle = -1,\ 1 \leq i < j \leq n-1,\ \ \langle v_{n-1}, v_n\rangle = -2.$$

The first $n-1$ vectors correspond to the long roots in Δ, while v_n corresponds to the short root. Observe that if $v = \sum_i m_i v_i$ is an element in L, then $\|v\|^2$ is a positive even integer. Consider then $v = v_i - \tau v_i$, for some $i = 1, \ldots, n-1$. The inequality

$$\|v\|^2 = 2(\|v_i\|^2 - \langle v_i, \tau v_i\rangle) > 0,$$

implies that $\langle v_i, \tau v_i\rangle$ can only take the values 1 or 0. In the first case $\|v\|^2 = 2$ and v is a vector of shortest length. In the second case, $\|v\|^2 = 4$. Since no rational number satisfies $q^2 = \frac{1}{2}$, one has that $v = v_i - \tau v_i$ is the shortest vector in $\mathbb{R}v \cap L$. On the other hand, one easily checks that if $v = v_n - \tau v_n$, then $\frac{1}{2}v$ is the shortest vector in $\mathbb{R}v \cap L$.

Lemma 5.3. *Let γ be a non-zero vector in $\mathfrak{a}\cap i\frac{1}{2}\Gamma$. Then for every root α in the restricted root system $\Delta_{\mathfrak{a}}$, one has that*
(i) $\alpha(\gamma) \in \mathbb{Z}\pi$;
(ii) $|\alpha(\gamma)| \geq \pi$;
(iii) $|\alpha(\gamma)| \geq 2\pi$, *if $\Delta_{\mathfrak{a}}$ is reduced.*

Proof. It is sufficient to prove the lemma for the simple root in $\alpha \in \Delta_{\mathfrak{a}}$. Recall that α can be written as $\alpha = \frac{1}{2}(\lambda_{i_0} - \tau\lambda_{i_0})$, for some simple root $\lambda_{i_0} \in \Pi$. Let $\gamma = \pi \sum m_i h_{\lambda_i} \in i\frac{1}{2}\Gamma$, with $m_i \in \mathbb{Z}$. Then

$$\alpha(\gamma) = \frac{1}{2}(\lambda_{i_0} - \tau\lambda_{i_0})(\gamma) = \frac{1}{2}(\lambda_{i_0}(\gamma) - \tau\lambda_{i_0}(\gamma)) = \frac{1}{2}(\lambda_{i_0}(\gamma) - \lambda_{i_0}(\tau\gamma)) =$$

$$= \lambda_{i_0}(\gamma) = \pi \sum m_i \lambda_{i_0}(h_{\lambda_i}) \in \pi\ \mathbb{Z},$$

proving (i). To prove (ii), we need to show that

$$\alpha(\gamma) \neq 0, \quad \text{for all } \gamma \in i\frac{1}{2}\Gamma \cap \mathfrak{a},\ \gamma \neq 0.$$

Observe that the vector $\gamma_0 = \pi(h_{\lambda_{i_0}} - \tau h_{\lambda_{i_0}})$ is a non-zero vector $\mathfrak{a}\cap i\frac{1}{2}\Gamma$, and that every $\gamma \in \mathfrak{a}\cap i\frac{1}{2}\Gamma$ is of the form $\gamma = q\gamma_0$, for some $q \in \mathbb{Q}$. So we need to show that

$$\alpha(\gamma_0) = \lambda_{i_0}(\gamma_0) = \pi(2 - \lambda_{i_0}(\tau h_{\lambda_{i_0}})) = \pi(2 - \lambda_{i_0}(h_{\tau\lambda_{i_0}})) \neq 0.$$

Suppose by contradiction that $\lambda_{i_0}(h_{\tau\lambda_{i_0}}) = 2$. Then both $\lambda_{i_0} - \tau\lambda_{i_0}$ and $\lambda_{i_0} - 2\tau\lambda_{i_0}$ are roots in Δ. Since by assumption λ_{i_0} and $\tau\lambda_{i_0}$ have non-zero restrictions to $\mathfrak{a}$, it means that

$$\frac{1}{2}(\lambda_{i_0} - \tau\lambda_{i_0})|\mathfrak{a} = \alpha, \quad (\lambda_{i_0} - \tau\lambda_{i_0})|\mathfrak{a} = 2\alpha, \quad (\lambda_{i_0} - 2\tau\lambda_{i_0})|\mathfrak{a} \neq 0, \alpha, 2\alpha, \tag{5.1}$$

which is absurd. This concludes the proof of (ii).
(iii) Assume now that $\Delta_{\mathfrak{a}}$ is reduced. Formulas (5.1) imply that $\lambda_{i_0}(h_{\tau\lambda_{i_0}}) \leq 0$. If $\lambda_{i_0}(\tau h_{\lambda_{i_0}}) < 0$, one has that $\tau\lambda = -\lambda$ (see [OS], Lemma 3.10). Then $\lambda_{i_0}(h_{\tau\lambda_{i_0}}) = -2$ and $\lambda_{i_0}(\gamma_0) = 4\pi$.
If $\lambda_{i_0}(\tau h_{\lambda_{i_0}}) = 0$, one has that $\lambda_{i_0}(\gamma_0) = 2\pi$. To conclude the proof, we need to show that in both cases, γ_0 is the shortest vector in $\mathfrak{a}\cap i\frac{1}{2}\Gamma$. By the classification results in [OS], Sect. (5.8), the restricted root system $\Delta_{\mathfrak{a}}$ is reduced if and only if $(\mathfrak{g},\mathfrak{h}) = (\mathfrak{so}(p+1,q+1), \mathfrak{so}(p+1,q)$. In this case, the root system Δ is either of type B_n or of type D_n, depending on whether $p+q$ is odd or even. In the first case, any root λ_{i_0} whose restriction to $\mathfrak{a}$ is equal to α is a long root ([OS], p.466, [Wa], p.30). In the second case, all roots in Δ have the same length. By Lemma 5.1 and Lemma 5.2, in both cases γ_0 is the shortest vector in $\mathfrak{a}\cap i\frac{1}{2}\Gamma$ and the proof of (iii) is complete.

Proposition 5.4. *Let G/H be a rank-one symmetric space. Then the polar map $\phi\colon \Omega \longrightarrow G^{\mathbb{C}}/H^{\mathbb{C}}$ is globally injective.*

Proof. Let $[g_1, X], [g_2, Y] \in \Omega$ be two points with the same image $\phi([g_1, X]) = \phi([g_2, Y])$.This means that

$$g_1 \exp iX = g_2 \exp iY h_c, \quad \text{for some } h_c \in H^{\mathbb{C}},\ \text{and } g_1, g_2 \in G,$$

or equivalently

$$\exp iX = g \exp iY h_c \quad \text{for} \quad h_c \in H^{\mathbb{C}},\ \text{and } g = g_1^{-1}g_2 \in G. \tag{5.2}$$

We want to show that $[g_1, X] = [g_2, Y]$, i.e. that there exists $h \in H$ such that

$$\begin{cases} g_1 = g_2 h^{-1} \\ X = Ad_h Y. \end{cases}$$

Let $X = X_s + X_n$ and $Y = Y_s + Y_n$ be the Jordan decomposition of X and Y in $\mathfrak{q}$. Write $x = \exp iX = x_s x_n$, with $x_s = \exp iX_s$ and $x_n = \exp iX_n$, and similarly $y = \exp iY = y_s y_n$, with $y_s = \exp iY_s$ and $y_n = \exp iY_n$. By Remark 4.1 and [Ma], Prop.2 (ii)a, p.66, equation (5.2) is equivalent to the system

$$\begin{cases} \exp iX_n = g \exp iY_n g^{-1} \\ \exp iX_s = g \exp iY_s h_c. \end{cases} \tag{5.3}$$

Let $\omega \subset \mathfrak{q}$ be the subset defined in (2.7). Observe that in the rank-one case, for every non-zero semisimple element $Z \in \omega$, the element $z = \exp iZ \in G^{\mathbb{C}}$ is necessarily regular semisimple with respect to σ, τ. In particular Z is regular semisimple in $\mathfrak{q}$. As a consequence, the elements X, Y in (5.2) are either both nilpotent or both semisimple. In the first case, system (5.3) reduces to

$$\begin{cases} \exp iX_n = g \exp iY_n g^{-1} \\ g h_c = e. \end{cases}$$

Since the exponential map is injective on the set of nilpotent elements, from the equation $\exp iX_n = g \exp iY_n g^{-1} = \exp iAd_g Y_n$, we obtain

$$X_n = Ad_g Y_n, \quad \text{with } g \in G \cap H^{\mathbb{C}} = H,$$

as requested. If X, Y are both semisimple, system (5.3) reduces to the equation

$$\exp iX = g \exp iY h_c \quad g \in G,\ h_c \in H^{\mathbb{C}}. \tag{5.4}$$

Moreover, by [Ma], Thm.3, the elements X, Y may be assumed to sit in the same Cartan subspace in $\mathfrak{q}$. By Lemma 4.3(iii), equation (5.4) implies then

$$\exp i4X = g \exp i4Y g^{-1} = \exp i4Ad_g Y. \tag{5.5}$$

Recall that the compact Cartan subspace $\mathfrak{t}$ of $\mathfrak{q}$ is all contained in ω and that the restriction of the exponential map $\exp\colon i\mathfrak{t} \to G^{\mathbb{C}}$ is injective. Hence, if $X, Y \in \mathfrak{t}$, equation (5.5) implies

$$X = Ad_g Y, \quad \text{and} \quad x = \exp iX = g \exp iY g^{-1} = gyg^{-1}.$$

From the last relation one obtains that $g^{-1} = h_c \in G \cap H^{\mathbb{C}} = H$, as requested.

Assume now that $X, Y \in \omega \cap \mathfrak{a}$, where $\mathfrak{a}$ is the non-compact Cartan subspace in $\mathfrak{q}$. By Lemma 4.3(v), one has that $Ad_{g^{-1}} X \in \mathfrak{a}$ and $g \in N_G(\mathfrak{a})$. Moreover, by Lemma 4.3(iv) one has that $g^{-1}\tau(g) \in Z_H(\mathfrak{a})$. Therefore from (5.2) and Lemma 4.3(ii) it follows that

$$y^2 = g^{-1} x^2 \tau(g) = g^{-1} x^2 g c, \quad \text{for } c = g^{-1}\tau(g) \in Z_H(\mathfrak{a}),\ c^2 = 1.$$

Write the element $c = g^{-1} x^{-2} g y^2 \in \exp i\mathfrak{a} \cap \{x^2 = 1\}$ as

$$c = \exp i\gamma, \quad \text{with } \gamma = 2(Y - Ad_{g^{-1}} X) \in \mathfrak{a} \cap \frac{1}{2} i\Gamma.$$

Let α be the simple root $\alpha \in \Delta_{\mathfrak{a}}$. Since $X, Y \in \mathfrak{a} \cap \omega$, it follows from (2.7) that

$$\begin{cases} |\alpha(\gamma)| = |\alpha(2(Y - Ad_g X))| < 2\pi, \text{in the reduced case} \\ |\alpha(\gamma)| = |\alpha(2(Y - Ad_g X))| < \pi, \text{in the non-reduced case.} \end{cases}$$

By Lemma 5.3, the element c is necessarily the identity element in G and

$$y^2 = g x^2 g^{-1}. \tag{5.6}$$

By Lemma 4.3(ii), condition (5.6) implies that $g \in H$. Finally, recall that 2ω is contained in the injectivity set of $\exp\colon \mathfrak{g}^{\mathbb{C}} \to G^{\mathbb{C}}$. Then from (5.6) or equivalently from $\exp i2Y = \exp i2Ad_gX$, it follows that $Y = Ad_gX,\ g \in H$, as requested.

The preceeding proposition shows that in the rank-one case, the domain $D = \phi(\Omega)$ is an analogue of the Akhiezer-Gindikin domain. The push-forward of the canonical pseudo-Kähler metric of Ω, deriving from the adapted complex structure, defines a G-invariant pseudo-Kähler metric on D, of the same signature (σ^+, σ^-) as the metric on G/H. Next we show that D is an increasing union of q-complete smoothly bounded domains, where $q = \sigma^-$. As we shall see in the examples at the end of this section, in general D is not a Stein domain. (Recall that an n-dimensional complex manifold X is called q-complete if it admits an exhaustion function of class C^2, whose complex Hessian has at least $n - q$ non-negative eigenvalues. By this definition, a Stein manifold is 0-complete).

Let $E\colon \Omega \longrightarrow \mathbb{R}$ be the Energy function defined in (3.1). Since E is G-invariant, one has that $E([g, X]) = \frac{1}{2}B_{\mathfrak{g}}(X, X)$, where $B_{\mathfrak{g}}$ denotes the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{q}$. It follows that $E([g, N]) = 0$, for every nilpotent element $N \in \mathfrak{q}$. Let $\mathfrak{t}$ be the compact Cartan subspace in $\mathfrak{q}$ and let $\Delta_{\mathfrak{t}} = \Delta_{\mathfrak{t}}(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ denote the corresponding restricted root system. Then $\mathfrak{t} \subset \omega$ and for $X \in \mathfrak{t}$, one has

$$E([g, X]) = -(\mathrm{Im}\alpha(X))^2(\dim \mathfrak{g}^{\alpha} + 4 \dim \mathfrak{g}^{2\alpha}) \leq 0, \quad \alpha \in \Delta_{\mathfrak{t}},\ (\mathfrak{g}^{2\alpha} \text{ possibly trivial}).$$

In particular, E is non-positive and $E([g, X]) \to -\infty$, for $|\mathrm{Im}\alpha(X)| \to \infty$. Let $\mathfrak{a}$ be the non-compact Cartan subspace in $\mathfrak{q}$ and let $\Delta_{\mathfrak{a}} = \Delta_{\mathfrak{a}}(\mathfrak{g}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}})$ denote the corresponding restricted root system. For $X \in \omega \cap \mathfrak{a}$, one has

$$E([g, X]) = \alpha(X)^2(\dim \mathfrak{g}^{\alpha} + 4 \dim \mathfrak{g}^{2\alpha}), \quad \alpha \in \Delta_{\mathfrak{a}},\ (\mathfrak{g}^{2\alpha} \text{ possibly trivial}).$$

Let S be the supremum of E on Ω. For $0 < s < S$, define

$$\Omega_s = \{[g, X] \in \Omega \mid E([g, X]) < s\}, \quad \text{and} \quad D_s = \phi(\Omega_s).$$

Proposition 5.5. *For every $s \in\ ]0, S[$, the G-invariant domain D_s is q-complete, for $q = \sigma^-$. The domain D is an increasing union of q-complete domains*

$$D = \bigcup_{0<s<S} D_s.$$

Proof. It is clear that $D = \bigcup_{0<s<S} D_s$. It remains to show that each domain D_s is q-complete, for $q = \sigma^-$. For every $s \in\ ]0, S[$, the boundary ∂D_s is a regular orientable hypersurface. It consists of one or two closed hypersurface G-orbits intersecting the slice $A = \exp i\mathfrak{a}$. We can compute the Levi form of the boundary of D_s by computing the Levi form of these orbits. Let $x_0 = \exp iX_0 \in \partial D_s$ be a base point, with $X_0 \in \mathfrak{a} \cap \omega$. By [Ge], Prop. 5.14(i), the Levi form of ∂D_s at x_0 is a Hermitian matrix whose coefficients, up to a positive scalar multiple, are given by

$$L(\partial D_s)_{x_0} \sim \begin{pmatrix} I_{m^+(\alpha)} & 0 & 0 & 0 \\ 0 & -I_{m^-(\alpha)} & 0 & 0 \\ 0 & 0 & I_{m^+(2\alpha)} & 0 \\ 0 & 0 & 0 & -I_{m^-(2\alpha)} \end{pmatrix}.$$

Here the numbers $m^+(\alpha),\ m^-(\alpha),\ m^+(2\alpha),\ m^-(2\alpha)$ are the dimensions of the ±1-eigenspaces of the involution $\tau\theta$ on the root spaces $\mathfrak{g}^{\alpha}$ and $\mathfrak{g}^{2\alpha}$. They are called the "signatures" of the restricted root spaces (cf. [OS]). In our case, the numbers $m^+(\alpha),\ m^-(\alpha),\ m^+(2\alpha),\ m^-(2\alpha)$ are given by

$$\mathfrak{m}_+(\alpha) + m_+(2\alpha) = \dim \mathfrak{q} \cap \mathfrak{p} - 1 = \sigma^+ - 1, \qquad m_-(\alpha) + m_-(2\alpha) = \dim \mathfrak{q} \cap \mathfrak{k} = \sigma^-.$$

Observe that the Levi form is positive definite when $\tau = \theta$ and $\dim \mathfrak{q} \cap \mathfrak{k} = 0$. By [EVS], Thm. 3.8, p.421, a smoothly bounded open set in a Stein manifold, satisfying the above conditions, is q-complete, for $q = \sigma^-$.

Remark 5.6. The boundary ∂D of D is not smooth. In the examples below, the Levi form of ∂D at the smooth points is indefinite and degenerate. This shows that in general D is not a Stein domain. In all such examples the manifold $G^{\mathbb{C}}/H^{\mathbb{C}}$ contains no G-invariant Stein subdomains.

Example 5.7. *The real hyperboloids.*
Let p, q be positive integers, $p, q > 2$. In $\mathbb{C}^{p+q}$ consider the manifold

$$\mathbf{X} = \{Z \in \mathbb{C}^{p+q} \mid z_1^2 + \ldots + z_p^2 - z_{p+1}^2 - \ldots - z_{p+q}^2 = -1\}, \qquad \dim_{\mathbb{C}} \mathbf{X} = p + q - 1.$$

The group $G^{\mathbb{C}} = SO(p, q, \mathbb{C})$ acts transitively on $\mathbf{X}$. Taking as a base point $\mathbf{x} = (0, \ldots, 0, 1)$, there is an identification

$$\mathbf{X} = G^{\mathbb{C}}/H^{\mathbb{C}} = SO(p, q, \mathbb{C})/SO(p, q - 1, \mathbb{C}).$$

Consider on $\mathbf{X}$ the action of the connected real form $G = SO(p, q)_0$. It turns out that there are two pseudo-Riemannian G-symmetric spaces, embedded in $\mathbf{X}$ as totally real submanifolds of maximal dimension. To each of them there is associated a domain, image of the corresponding polar map. We determine such domains and examine their complex analytic properties.

We begin by describing the G-orbit structure of $\mathbf{X}$. The G-orbits in $\mathbf{X}$ are in one-to-one correspondence with the following set

$$\begin{array}{ccccccccc} & & & \mathbf{n} & & \mathbf{m} & & & \\ & & Q(s) & * & P(t) & * & R(\sigma) & & \\ ---- & & \overline{\qquad\qquad} & \bullet & \overline{\qquad\qquad} & \bullet & \overline{\qquad\qquad} & & ---- \\ & & & G/H & & G/L & & & \end{array} \tag{5.7}$$

The left black dot corresponds to the G-orbit of the point $\mathbf{x} = (0, \ldots, 0, 1)$, diffeomorphic to the pseudo-Riemannian rank-one symmetric space $G/H = SO(p, q)_0/SO(p, q - 1)$, of signature $(p, q - 1)$; the right black dot corresponds to the G-orbit of the point $\mathbf{y} = (i, 0, \ldots, 0)$, diffeomorphic to the pseudo-Riemannian rank-one symmetric space $G/L = SO(p, q)_0/SO(p - 1, q)$, of signature $(p - 1, q)$. Both G/H and G/L are totally real submanifolds of $\mathbf{X}$, of real dimension

$$\dim_{\mathbb{R}} G/H = \dim_{\mathbb{R}} G/L = \dim_{\mathbb{C}} \mathbf{X}.$$

The central segment and the two halflines parametrize the slices meeting the three types of closed orbits of maximal dimension. Since G/H and G/L has rank one, closed orbits of maximal dimension are real hypersurfaces in $\mathbf{X}$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the symmetric algebra associated to G/H and $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$ the symmetric algebra associated to G/L. The points

$$Q(s) = (0, \ldots, 0, i \sinh s, \cosh s), \quad \text{for } s \in]-\infty, 0[,$$

parametrize the orbits intersecting the slice $C = \exp i\mathfrak{t} \cdot \mathbf{x}$, where $\mathfrak{t}$ is the compact Cartan subspace in $\mathfrak{q}$; the points

$$P(t) = (i \sin t, 0, \ldots, 0, \cos t), \quad \text{for } t \in]0, \pi/2[,$$

parametrize the orbits intersecting the slice $A = \exp i\mathfrak{a}$, where $\mathfrak{a}$ is the non-compact Cartan subspace both in $\mathfrak{q}$ and in $\mathfrak{m}$; the points

$$R(\sigma) = (i \cosh \sigma, \sinh \sigma, 0, \ldots, 0), \quad \text{for } \sigma \in]0, \infty[,$$

parametrize the orbits intersecting the slice $C' = \exp i\mathfrak{t}' \cdot \mathbf{y}$, where $\mathfrak{t}'$ is the compact Cartan subspace in $\mathfrak{m}$. Observe that

$$Q(0) = P(0) = \mathbf{x}, \qquad P(\frac{\pi}{2}) = Q(0) = \mathbf{y}.$$

In addition to the closed orbits, there are two nilpotent hypersurface orbits: one with base point

$$\mathbf{n} = (i, 0, \ldots, 0_p, i, 0, \ldots, 1)$$

having G/H in its closure, and the other with base point

$$\mathbf{m} = (i, 0, \ldots, 1_p, 1, 0, \ldots, 0),$$

having G/L in its closure (here the subscript p marks the p^{th} coordinate of a point).

The domain D in $G^{\mathbb{C}}/H^{\mathbb{C}}$ determined by G/H is given by

$$D = \bigcup_{s<0} G \cdot Q(s) \ \bigcup \ G/H \cup G \cdot \mathbf{n} \ \bigcup_{t \in]0, \frac{\pi}{2}[} G \cdot P(t), \qquad \partial D = G/L \cup G \cdot \mathbf{m},$$

and corresponds to the following subset of diagram (5.7)

$$\begin{array}{ccccc} & & \mathbf{n} & & \\ & Q(s) & * & P(t) & \\ ----- & \underline{\qquad\qquad} & \bullet & \underline{\qquad\qquad} & \\ & & G/H & & \end{array}$$

Consider now the real-valued G-invariant function

$$F \colon \mathbf{X} \longrightarrow \mathbb{R}, \quad Z \mapsto F(Z) := |z_1|^2 + \ldots + |z_p|^2 - |z_{p+1}|^2 - \ldots - |z_{p+q}|^2.$$

Evaluating F on the base points of all G-orbits in $\mathbf{X}$, it is easy to see that F separates the closed G-orbits

$$\begin{aligned} -\infty \le F(Q_s) = -(\sinh^2 s + \cosh^2 s) < F(\mathbf{x}) = -1 < F(P_t) = 1 - 2\cos^2 t < \\ < F(\mathbf{y}) = 1 < F(R_\sigma) = \sinh^2 \sigma + \cosh^2 \sigma \le +\infty. \end{aligned} \tag{5.8}$$

The domain D and its boundary ∂D can be easily described by means of F:

$$D = \{Z \in \mathbf{X} \mid F(Z) < 1\}, \qquad \partial D = \{Z \in \mathbf{X} \mid F(Z) = 1\}.$$

The boundary ∂D is not smooth (the set of smooth points coincides with the orbit $G \cdot \mathbf{m}$). The Levi form of ∂D at the smooth points is degenerate with $p-2$ positive eigenvalues, $q-1$ negative eigenvalues and 1 zero eigenvalue. Hence, for p, q sufficiently large, D is not a Stein domain. Let $R \in]-1, 1[$. For every such R, the subdomain $D_R = \{Z \in D \mid F(Z) < R\}$ has smooth boundary, with non-degenerate Levi form of signature $(p-1, q-1)$. By [EVS], Thm. 3.8, the domain D_R is $(q-1)$-complete. As a result, the domain D is an increasing union of $(q-1)$-complete domains

$$D = \bigcup_{R<1} D_R.$$

The domain D' in $G^{\mathbb{C}}/H^{\mathbb{C}}$ determined by G/L is given by

$$D' = \bigcup_{t \in]0, \frac{\pi}{2}[} G \cdot P(t) \ \bigcup \ G/L \cup G \cdot \mathbf{m} \ \bigcup_{\sigma>0} G \cdot R(\sigma), \qquad \partial D = G/H \cup G \cdot \mathbf{n},$$

and corresponds to the following subset of diagram (5.7)

$$\begin{array}{ccccc} & & \mathbf{m} & & \\ & P(t)) & * & R(\sigma) & \\ & \underline{\qquad\qquad} & \bullet & \underline{\qquad\qquad} & ----- \\ & & G/L & & \end{array}$$

In terms of F, the domain D' and its boundary are given by

$$D' = \{Z \in \mathbf{X} \mid -F(Z) < 1\}, \qquad \partial D' = \{Z \in \mathbf{X} \mid F(Z) = -1\}.$$

The set of smooth points in $\partial D'$ coincides with the orbit $G \cdot \mathbf{n}$ and has degenerate Levi form with $p-1$ positive eigenvalues, $q-2$ negative eigenvalues and 1 zero eigenvalue. Hence, for p,q sufficiently large, the domain D' is not Stein either. Let $R \in]-1,1[$. For every such R, the subdomain $D'_R = \{Z \in D \mid -F(Z) < -R\}$ has smooth boundary with non-degenerate Levi form of signature $(q-1, p-1)$. By [EVS], the domain D' an the increasing union of $(p-1)$-complete domains: $D' = \cup_R D'_R$.

A few more remarks: for every $s \in]-\infty,-1[$, the level hypersurface $\mathbf{X}_R = \{F(Z) = R\}$ consists of one orbit with base point $Q_s = (0,\dots,0, i\sinh s, \cosh s)$, for $s \in]-\infty, 0[$; its Levi form is non-degenerate with signature $(p, q-2)$. For every $R \in]-1,+\infty[$, the level hypersurface $\mathbf{X}_R = \{F(Z) = R\}$ consists of one orbit with base point $R_\sigma = (i\cosh\sigma, \sinh\sigma, 0,\dots,0)$, for $\sigma \in]0,+\infty[$; its Levi form is non-degenerate with signature $(p-2, q)$. From these computations we conclude that for p,q sufficiently large, the complex Hessian of every G-invariant function on $\mathbf{X}$ has both positive and negative eigenvalues at all points. As a consequence, no G-invariant open set in $\mathbf{X}$ admits G-invariant plurisubharmonic functions. One can easily check that the G-action fails to be proper on every G-invariant open subset of $\mathbf{X}$. So no G-invariant open subset of $\mathbf{X}$ carries a G-invariant Kähler structure.

6. The higher rank case.

Let G/H be a pseudo-Riemannian symmetric space of rank higher than one. Let $\Omega \subset G \times_H \omega$ be the domain defined in (2.7). In this case, the polar map

$$\phi\colon \Omega \longrightarrow G^{\mathbb{C}}/H^{\mathbb{C}}$$

is generally non-injective. As a result the domain Ω with the adapted complex structure is a non-injective Riemann domain over $G^{\mathbb{C}}/H^{\mathbb{C}}$.

One may see this as follows: let $\mathfrak{c} = \mathfrak{c}_k \oplus \mathfrak{c}_\mathfrak{p}$ be a maximally split θ-stable Cartan subspace in $\mathfrak{q}$, with both $\mathfrak{c}_\mathfrak{k}$ and $\mathfrak{c}_\mathfrak{p}$ different from $\{0\}$, and let $\Delta_\mathfrak{c} = \Delta_\mathfrak{c}(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})$ denote the corresponding restricted root system. Fix an element $X \in \mathfrak{c}_\mathfrak{k}$ with the property that $\mathrm{Im}\alpha(X) \neq 0$, for all imaginary and complex roots in $\Delta_\mathfrak{c}$. This is possible by taking X in the complement of the finite set of hyperplanes $\{H \in \mathfrak{c}_\mathfrak{k} \mid \mathrm{Im}\alpha(H) = 0\}_{\alpha\in\Delta_\mathfrak{c}}$ in $\mathfrak{c}_\mathfrak{k}$. Let γ be an element in the intersection $\mathfrak{c}_\mathfrak{p} \cap i\Gamma$, where Γ denotes the unit lattice in $\mathfrak{u} \subset \mathfrak{g}^{\mathbb{C}}$. One such element γ can be constructed as follows: let $\mathfrak{b} \oplus \mathfrak{c}$ be a τ,θ-stable Cartan subalgebra in $\mathfrak{g}$ extending $\mathfrak{c}$. Let λ be a root in $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}} \oplus \mathfrak{c}^{\mathbb{C}})$ with non-zero restriction to $\mathfrak{c}$ and let $h_\lambda \in \mathfrak{b}_\mathbb{R} \oplus \mathfrak{c}_\mathbb{R}$ be its inverse root (with $\lambda(h_\lambda) = 2$). Then $\gamma = 2\pi(h_\lambda + \theta h_\lambda + \tau h_\lambda + \theta\tau h_\lambda)$ lies in $\mathfrak{c}_\mathfrak{p} \cap i\Gamma$. The elements X and $X+\gamma$ satisfy the conditions

$$X,\ X+\gamma \in \omega \quad \text{and} \quad \exp iX = \exp i(X+\gamma).$$

As a consequence, the corresponding elements $[e, X]$ and $[e, X+\gamma]$ in Ω have the same image under the polar map. Moreover, since the inclusion $Z_H(X) \subset Z_H(x^2)$ may be a proper one, by Lemma 4.2(iv), the polar map ϕ may also fail to be injective on some G-orbits in Ω.

In the next proposition, we show that the polar map is injective on every closed orbit of maximal dimension and is a covering map, when restricted to certain distinguished G-invariant subsets of Ω. Such sets are coverings of principal orbit strata in D.

Recall that when G acts on $G^{\mathbb{C}}/H^{\mathbb{C}}$, closed orbits of maximal dimension come in a finite number of orbit types. Such orbits are called principal orbits, since their union is an open dense subset of $G^{\mathbb{C}}/H^{\mathbb{C}}$. The set of principal orbits of a given type is an open subset in $G^{\mathbb{C}}/H^{\mathbb{C}}$, generally disconnected. It is referred to as a principal orbit stratum. The domain $D = \phi(\Omega) \subset G^{\mathbb{C}}/H^{\mathbb{C}}$ contains a number of connected components of principal orbit strata. To be more precise, let $\mathfrak{c}$ be a Cartan subspace of $\mathfrak{q}$. Denote by $\mathfrak{c}_{rs}$ the set of all elements $X \in \mathfrak{c}$ with the property that $x = \exp iX \in G^{\mathbb{C}}$ is a regular semisimple element with respect to σ, τ (cf.(4.1)). The set $G \exp i\mathfrak{c}_{rs} H^{\mathbb{C}}$ is an open subset of $G^{\mathbb{C}}/H^{\mathbb{C}}$, consisting of closed orbits of maximal dimension, all of the same type. One of its connected components is contained in $D = \phi(\Omega)$.

Consider now the subset

$$\Omega_{\mathfrak{c}} := G \times_{N_H(\mathfrak{c})} \mathfrak{c}_{rs} \quad \subset \quad G \times_H \mathfrak{q}.$$

Observe that $\mathfrak{c}_{rs}$ is stable under the group $N_H(\mathfrak{c})$, so the set $\Omega_{\mathfrak{c}}$ is well defined and the restriction of the polar map ϕ to $\Omega_{\mathfrak{c}}$ has non-singular differential.

Proposition 6.1. *The restriction of the polar map to $\Omega_{\mathfrak{c}}$*

$$\phi \colon \Omega_{\mathfrak{c}} \longrightarrow G^{\mathbb{C}}/H^{\mathbb{C}}$$

is a G-equivariant covering map.

Proof. First we prove that the restriction of ϕ to every G-orbit in $\Omega_{\mathfrak{c}}$ is injective. Let $[e,X] \in \Omega_{\mathfrak{c}}$ and let $\bar{x} = \phi([e,X])$ be its image in $G^{\mathbb{C}}/H^{\mathbb{C}}$. Since the map ϕ is equivariant, it induces an inclusion of the corresponding isotropy subgroups $G_{[e,X]} \hookrightarrow G_{\bar{x}}$. The injectivity of ϕ on the orbit $G \cdot [e,X]$ is equivalent to showing that $G_{[e,X]} = G_{\bar{x}} = Z_H(X)$. This follows from Lemma 4.2(v). Since the map

$$\mathfrak{c}_{rs} \longrightarrow \exp i\mathfrak{c}_{rs} \longrightarrow \exp i\mathfrak{c}_{rs}/\exp i\mathfrak{c}_{rs} \cap H^{\mathbb{C}}$$

is a covering map, the proof of the proposition is complete.

In the next proposition, we show that the polar map is always injective on a smaller G-invariant open subdomain $\Omega' \subset \Omega$, defined by

$$\Omega' = G \times_H \omega', \qquad \omega' = \{X \in \mathfrak{q} \mid |\mathrm{Re}\lambda| < \pi/4, \ \ \text{for all } \lambda \in \mathit{spec}(ad_X)\}. \tag{6.1}$$

Proposition 6.2. *The polar map ϕ is injective on the domain $\Omega' = G \times_H \omega'$.*

Proof. Let $[g_1,X],[g_2,Y] \in G \times_H \omega'$ be points with the same image in $G^{\mathbb{C}}/H^{\mathbb{C}}$, i.e.

$$g \exp iX = \exp iYh, \quad \text{for } h \in H^{\mathbb{C}}, \ g = g_2^{-1}g_1 \in G. \tag{6.2}$$

Lemma 4.3(iii) implies that

$$\exp i4Y = g \exp i4Xg^{-1} = \exp iAd_g 4X.$$

Moreover, since $4\omega'$ is contained in the injectivity set of $\exp \colon \mathfrak{g}^{\mathbb{C}} \to G^{\mathbb{C}}$, one has that $Y = Ad_g X$. This relation together with equation (6.2) implies that $g = h \in G \cap H^{\mathbb{C}} = H$. In other words

$$\begin{cases} g_2 = g_1 h^{-1} \\ Y = Ad_h X \end{cases}, \quad h \in H,$$

as requested.

Corollary 6.3. By Proposition 6.1, a canonical G-invariant pseudo-Kähler metric is defined on coverings of principal orbit strata in D. Proposition 6.1 and 6.2 extend similar results obtained in [Fe] and [Br2] for the group case, by different methods.

Remark 6.4. *If G/H is a non-Riemannian semisimple symmetric space, the domains $D = \phi(\Omega)$ and $D' = \phi(\Omega')$ cannot be hyperbolic.* Both D and D' contain the complex homogeneous subvariety

$$K^{\mathbb{C}}/(K \cap H)^{\mathbb{C}} \quad \hookrightarrow \quad D' \quad \subset \quad D,$$

embedded as the $K^{\mathbb{C}}$-orbit of the base point $eH^{\mathbb{C}}$. The homogeneous subvariety $K^{\mathbb{C}}/(K \cap H)^{\mathbb{C}}$ is also the image of the set $K \times_{K \cap H} \mathfrak{k} \cap \mathfrak{q} \subset \Omega' \subset \Omega$ by the map ϕ. Indeed, for every $X \in \mathfrak{k} \cap \mathfrak{q}$, the eigenvalues of ad_X are all purely imaginary and the restriction of ϕ to $K \times_{K \cap H} \mathfrak{k} \cap \mathfrak{q}$ has both non-singular differential and it is injective. It can be viewed as the complexification of the compact symmetric space $K/K\cap H \hookrightarrow G/H$, embedded in G/H as the K-orbit of the base point eH. One has that

$$\dim_{\mathbb{C}} K^{\mathbb{C}}/(K\cap H)^{\mathbb{C}} = \dim_{\mathbb{R}} K/K\cap H = \dim_{\mathbb{R}} \mathfrak{q}\cap\mathfrak{k}.$$

References

[AG] Akhiezer D., Gindikin S., *On Stein extensions of real symmetric spaces*, Math. Ann., **286** (1990), 1–12.

[Ba] Barchini L., *Stein extensions of real symmetric spaces and the geometry of the flag manifold*, Math. Ann., **326** (2003) 331–346.

[BHH] Burns D., Halverscheid S., Hind R., *The Geometry of Grauert Tubes and Complexification of Symmetric Spaces*, Duke Math. J., **118** (2003) 465–491.

[Br1] Bremigan R., *Quotients for algebraic group actions over non-algebraically closed fields*, J. reine angew. Math. **453** (1994) 21–47.

[Br2] Bremigan R., *Pseudo-Kähler forms on complex Lie groups*, Documenta Math. **5** (2000), 595–611.

[EVS] Eastwood M., Vigna-Suria G., *Cohomologically complete and pseudoconvex domains*, Comm. Math. Helv., **55** (1980) 413–426.

[Fe] Fels G., *Pseudo-Kählerian structure on domains over a complex semisimple Lie group*, Math. Ann. **232** (2002), 1–29.

[H] Huckleberry A., *On certain domains in cycle spaces of flag manifolds*, Math. Ann., **323** (2002) 797–810.

[Ha] Halverscheid S., *On Maximal Domains of Definition of Adapted Complex Structures for Symmetric Spaces of Non-compact Type,* Thesis Ruhr-Universität Bochum, 2001.

[HI] Halverscheid, S., Iannuzzi, A., *Maximal complexifications of certain Riemannian homogeneous spaces.* Trans. of the AMS, **355**, 11 (2003) 4581–4594.

[HO] Hilgert J., Òlafsson G., Causal symmetric spaces. Geometry and Harmonic Analysis, Perspectives in Math. **18**, Academic Press, San Diego, CA, 1997.

[Hu] Humphreys J.E., Conjugacy Classes in Semisimple Algebraic groups, AMS Math. Surveys and Monographs, Vol. 43, Providence RI, 1995.

[KS1] Krötz B., Stanton R., *Holomorphic extensions of representations I: Automorphic functions*, Ann. of Math. **159**, 2 (2004), 641–724.

[KS2] Krötz B., Stanton R., *Holomorphic extensions of representations II: geometry and harmonic analysis*, GAFA, Geom. Funct. Anal., **15** (2005), 190-245.

[Ge] Geatti L., *Invariant domains in the complexification of a non-compact Riemannian symmetric space*, J. of Algebra, **251** (2002) 619–685.

[GS1] Guillemin V., Stenzel M.,*Grauert tubes and the homogeneous Monge-Ampère equation*, J. Diff. Geom. **34** (1991) 561–570.

[GS2] Guillemin V., Stenzel M.,*Grauert tubes and the homogeneous Monge-Ampère equation*, J. Diff. Geom. **35** (1992) 627–641.

[LS] Lempert L., Szőke R.,*Global solutions of the homogeneous complex Monge-Ampère equation and complex structures on the tangent bundles of Riemannian manifolds*, Math. Ann. **290** (1991) 689–712.

[Ma] Matsuki T., *Double coset decomposition of reductive Lie groups arising from two involutions*, J. of Algebra, **197** (1997) 49–91.

[Ne1] Neeb K.H., *On the complex geometry of invariant domains in complexified symmetric spaces*, Ann. Inst. Fourier, Grenoble, **49**, 1 (1999), 177–225.

[Ne2] Neeb K.H., Holomorphy and convexity in Lie theory, de Gruyter Expositions in Mathematics 28, Walter de Gruyter Co., Berlin, 2000.

[OS] Oshima T., Sekiguchi J., *The restricted root system of a semisimple symmetric pair*, Advanced Studies in Pure Math. 4, Group Representations and Systems of Differential Equations, 1984, pp. 433–497.

[St] Steinberg R., *Torsion in reductive groups*, Adv. in Math., **15** (1975), 13–92.

[Sz1] Szőke R.,*Complex structures on the tangent bundles of Riemannian manifolds*, Math. Ann. **291** (1991) 409–428.

[Sz2] Szőke R.,*Canonical complex structures associated to connections and complexifications of Lie groups*, Math. Ann. **329** (2004) 553–591.

[vD] van Dijk G., *Orbits of real affine symmetric spaces I: the infinitesimal case*, Indag. Math. **45** (1983) 51–66.

[Va] Varadarajan V.S., Lie groups, Lie algebras and their representations, Springer Verlag, New York 1984.

[Wa] Warner G., Harmonic analysis on Semi-Simple Lie groups I, Springer Verlag, Berlin-Heidelberg-New York 1972.