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Polar orthogonal representations of real reductive algebraic groups

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Abstract

We prove that a polar orthogonal representation of a real reductive algebraic group has the same closed orbits as the isotropy representation of a pseudo-Riemannian symmetric space. We also develop a partial structural theory of polar orthogonal representations of real reductive algebraic groups which slightly generalizes some results of the structural theory of real reductive Lie algebras.

Keywords: Polar representations; Real reductive algebraic groups; Pseudo-Riemannian symmetric spaces

1. Introduction

A representation of a complex reductive algebraic group G on a finite-dimensional complex vector space V is called *polar* if there exists a subspace $c \subset V$ consisting of semisimple elements such that $\dim c = \dim V//G$ (the categorical quotient), and for a dense subset of c, the tangent spaces to the orbits are parallel [DK85]; then it turns out that every closed orbit of G meets c [DK85, Prop. 2.2]. The class of polar representations was introduced and studied by Dadok and Kac in [DK85], and it is very important in invariant theory because it includes the adjoint actions, the representations associated to symmetric spaces studied by Kostant and Ral-

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lis [KR71] as well as, more generally, the representations associated to automorphisms of finite order (θ-groups) introduced by Vinberg [Vin76] (see also [Kac80]). At present, there is no complete classification of polar representations although the paper [DK85] contains very important partial results.

A complex (resp. real) representation admitting a complex-valued (resp. real-valued) invariant non-degenerate symmetric bilinear form is called *orthogonal*. It is well known that a complex orthogonal representation admits a real form invariant under a maximal compact subgroup. Consider in particular the complex polar orthogonal representations and the class of compact real forms they originate. Since the complex reductive algebraic groups are exactly the complexifications of the compact Lie groups, one can equivalently define directly the concept of a real polar representation of a compact Lie group in the differential-geometric setting (as in e.g. [PT87]) and obtain the same class. Note that orbits of polar representations of compact Lie groups are very important in submanifold geometry and Morse theory [BS58,Con71,Sze84,PT87,DO01,GT03]. Now, such representations were classified by Dadok in [Dad85], and the following very nice characterization was deduced: *A polar representation of a compact Lie group has the same orbits as the isotropy representation of a Riemannian symmetric space.*

The purpose of this paper is to study *noncompact* real forms of complex polar orthogonal representations. Equivalently, we define a representation of a real reductive algebraic group (in the sense of [BH62, §1]) to be *polar* if and only if its algebraic complexification is polar. In Section 3 we prove the following theorem.

Theorem 1. *A polar orthogonal representation of a connected real reductive algebraic group has the same closed orbits as the isotropy representation of a pseudo-Riemannian symmetric space.*

In Section 4, we discuss some aspects of the submanifold geometry of the closed orbits of the polar orthogonal representations of the real reductive algebraic groups in that we relate them to a notion of pseudo-Riemannian isoparametric submanifold of a pseudo-Euclidean space (compare [Hah84,Mag85]).

Finally in Section 5, independently of classification results, we develop a partial structural theory of polar orthogonal representations of real reductive algebraic groups that generalizes some results of the structural theory of real reductive Lie algebras. In this regard, we propose to replace adjoint actions by polar orthogonal ones. The results we prove are slight generalizations of well known results for the adjoint actions, but we believe our proofs are more geometric. In particular, we show that a polar orthogonal representation of a real reductive algebraic group admits finitely many pairwise inequivalent so-called Cartan subspaces in standard position with respect to a compact real form such that the union of those subspaces meets all the closed orbits and always orthogonally (Theorem 15 and Corollary 18). We also construct the so-called Cayley transformations that relate different equivalence classes of Cartan subspaces (Section 5.3), and use those to show that the equivalence classes of Cartan subspaces in the two extremal positions with respect to the compact real form are unique (Corollary 25).

Unless explicit mention to the Zariski topology is made, we use throughout the classical topology. We always use lowercase gothic letters to denote Lie algebras. For a given homomorphism of groups, we denote the induced homomorphism on the Lie algebra level by the same letter whenever the context is clear. Sometimes it is useful to call a representation *orthogonalizable* if it admits an invariant non-degenerate symmetric bilinear form but we do not want to fix such a form.

2. Preliminaries

Let G be a connected complex reductive algebraic group. Let $\tau : G \to GL(V)$ be a complex representation. A vector $v \in V$ is called *semisimple* if the orbit Gv is closed. Not every orbit of G in V is closed, but the closure of any orbit contains a unique closed orbit. An element is called *regular* if it is semisimple and $\dim Gv \geqslant \dim Gx$ for all semisimple $x \in V$. The representation τ is called *stable* or is said to admit *generically closed orbits* if there exists an open and dense subset of V consisting of closed orbits. An orthogonalizable representation is necessarily stable (see [Sch80, Cor. 5.9] or [Lun72,Lun73]).

Let $\mathbf{C}[V]$ be the polynomial algebra of V, and let $\mathbf{C}[V]^G$ be the algebra of G-invariant polynomials. It does not contain nilpotents, and is finitely generated by a theorem of Hilbert, so it is the coordinate ring of an affine algebraic variety denoted by $V/\!/G$ and called the *categorical quotient* of V by G. The embedding $\mathbf{C}[V]^G \to \mathbf{C}[V]$ induces a surjective morphism of affine algebraic varieties $\pi : V \to V/\!/G$. Every fiber of π contains a unique closed orbit. It follows that $V/\!/G$ can be seen as the parameter set of closed G-orbits in V, and then $\pi(v)$ represents the unique closed orbit in the closure of Gv [PV94, §4].

For semisimple $v \in V$, set

$$c_v = \{x \in V \mid \mathfrak{g} \cdot x \subset \mathfrak{g} \cdot v\}.$$

Then c_v consists entirely of semisimple elements, and the isotropy subalgebras satisfy $\mathfrak{g}_x \supset \mathfrak{g}_v$ for $x \in c_v$ [DK85, Lem. 2.1]. The representation τ is called *polar* if a semisimple v can be chosen so that $\dim c_v = \dim V/\!/G$. In this case, c_v is called a *Cartan subspace*. The Cartan subspaces of a polar representation are all G-conjugate [DK85, Thm. 2.3].

The group G can be seen simply as the complexification of a compact connected Lie group U; compare [Sch80, §5] or [BH62, Rmk. 3.4]. Then U is a maximal compact (necessarily connected) subgroup of G, and every maximal compact subgroup of G is G-conjugate to U. It is easy to see that a representation τ is orthogonalizable if and only if it admits a real form $\tau_u : U \to GL(W)$ [Sch80, Prop. 5.7]. The group U must be the fixed point group G^θ of a unique anti-holomorphic involutive automorphism θ of G, which is called a *Cartan involution* of G. Also, the subspace W is the fixed point set $V^{\tilde\theta}$ of a conjugate-linear involutive automorphism $\tilde\theta$ of V, the equation $\tilde\theta(g \cdot v) = \theta(g) \cdot \tilde\theta(v)$ holds for $g \in G$ and $v \in V$, and an invariant form $\langle \cdot,\cdot \rangle$ can be chosen on V so that it is real-valued on $V^{\tilde\theta}$.

More generally, we consider real forms of $\tau : G \to GL(V)$ given by a pair $(\sigma, \tilde\sigma)$ where σ is an anti-holomorphic involution of G and $\tilde\sigma$ is a real structure on V satisfying $\tilde\sigma(g \cdot v) = \sigma(g) \cdot \tilde\sigma(v)$ for $g \in G$, $v \in V$. The fixed point subgroup G^σ is a (not necessarily connected) real reductive algebraic group, and τ of course restricts to a representation of $G^\sigma \to GL(V^{\tilde\sigma})$, where $V^{\tilde\sigma}$ is the fixed point set of $\tilde\sigma$ in V. We say that two real forms $(\sigma, \tilde\sigma)$ and $(\sigma', \tilde\sigma')$ *commute* if they commute componentwise. If τ is orthogonal with respect to $\langle \cdot,\cdot \rangle$ and a real form $(\sigma, \tilde\sigma)$ is given, then $\langle \cdot,\cdot \rangle$ is said to be *defined over* $\mathbf{R}$ *with respect to* $\tilde\sigma$ and $(\sigma, \tilde\sigma)$ is called an *orthogonal real form* if $\langle \cdot,\cdot \rangle$ is real-valued on $V^{\tilde\sigma}$. Note that the latter condition is equivalent to having

$$\langle \tilde\sigma x, \tilde\sigma y \rangle = \overline{\langle x, y \rangle}$$

for $x, y \in V$. A *Cartan pair* of τ is an orthogonal real form $(\theta, \tilde\theta)$ such that θ is a Cartan involution of G and $\langle \cdot,\cdot \rangle$ is real-valued and negative-definite on $V^{\tilde\theta}$. Note that $(\theta, \tilde\theta)$ is a Cartan pair of τ with respect to $\langle \cdot,\cdot \rangle$ if and only if $(\theta, -\tilde\theta)$ is a Cartan pair of τ with respect to $-\langle \cdot,\cdot \rangle$. The following result is essentially proved in [Bre93, 7.4], but we find it convenient to include a proof here because we will need to refer to some of its techniques.

Proposition 2 *(Bremigan). Let $\tau : G \to O(V, \langle\cdot,\cdot\rangle)$ be an orthogonal representation, and suppose that $(\sigma, \tilde\sigma)$ is an orthogonal real form. Then there exists a Cartan pair $(\theta, \tilde\theta)$ which commutes with $(\sigma, \tilde\sigma)$.*

Proof. It is well known that there exists a Cartan involution θ of G such that $\theta\sigma = \sigma\theta$. Let $U = G^\theta$ be the associated maximal compact subgroup of G. Consider the realification V^r of V, and denote the invariant complex structure on V^r by J so that $V = (V^r, J)$. Note that

$$\tilde\sigma\tau(g)\tilde\sigma^{-1} = \tau\big(\sigma(g)\big), \qquad J\tau(g)J^{-1} = \tau(g) \quad \text{and} \quad J\tilde\sigma J^{-1} = -\tilde\sigma$$

for $g \in G$. Let G^* be the subgroup of $GL(V^r)$ generated by $\tau(G)$, $\tilde\sigma$ and J. Then G^* contains $\tau(G)$ as a normal subgroup of finite index. Due to $\theta\sigma = \sigma\theta$, we have also that $\tilde\sigma$ normalizes $\tau(U)$. Let U^* be the subgroup of G^* generated by $\tau(U)$, $\tilde\sigma$ and J. Then U^* is a compact subgroup of G^*, so we can find a U^*-invariant positive-definite real inner product on V^r which we denote by "$\cdot$." Set

$$(x, y) = x \cdot y + i(x \cdot Jy)$$

for $x, y \in V^r$. Then it is easily checked that $(\cdot,\cdot)$ is a U-invariant positive-definite Hermitian form on $(V^r, J) = V$ which is real-valued on $V^{\tilde\sigma}$. In particular, $i\mathfrak{u}$ acts on V by Hermitian endomorphisms. Next, define a conjugate-linear automorphism $\tilde\theta$ of V by setting

$$(x, \tilde\theta y) = -\langle x, y\rangle \tag{1}$$

for $x, y \in V$. Then

$$\big(x, \tilde\theta(gy)\big) = -\langle x, gy\rangle = -\big\langle g^{-1}x, y\big\rangle = \big(g^{-1}x, \tilde\theta y\big) = \big(x, \theta(g)\tilde\theta(y)\big), \tag{2}$$

so $\tilde\theta\tau(g) = \tau(\theta(g))\tilde\theta$ for $g \in G$. Moreover

$$(x, \tilde\theta\tilde\sigma y) = -\langle x, \tilde\sigma y\rangle = -\overline{\langle\tilde\sigma x, y\rangle} = \overline{(\tilde\sigma x, \tilde\theta y)} = (x, \tilde\sigma\tilde\theta y) \tag{3}$$

implying that $\tilde\theta\tilde\sigma = \tilde\sigma\tilde\theta$. We also have that

$$\begin{aligned}
\big(\tilde\theta^2 x, y\big) &= \overline{\big(y, \tilde\theta^2 x\big)} = -\overline{\langle y, \tilde\theta x\rangle} = -\overline{\langle\tilde\theta x, y\rangle} \\
&= \overline{(\tilde\theta x, \tilde\theta y)} = (\tilde\theta y, \tilde\theta x) = \cdots \\
&= \big(x, \tilde\theta^2 y\big)
\end{aligned}$$

for $x, y \in V$. It follows that $\tilde\theta^2 : V \to V$ is a G^*-equivariant $\mathbf{C}$-linear Hermitian automorphism. Hence there exists a $\langle\cdot,\cdot\rangle$- and $(\cdot,\cdot)$-orthogonal G^*-invariant decomposition $V = \bigoplus_j V_j$ such that $\tilde\theta^2|_{V_j} = \lambda_j \,\mathrm{id}_{V_j}$ where $\lambda_j \in \mathbf{R} \setminus \{0\}$ and the λ_j's are pairwise distinct.

Note that $\lambda_j(x,x) = (\tilde\theta x, \tilde\theta x) > 0$ if $x \in V_j \setminus \{0\}$, so we also have $\lambda_j > 0$. If we change $(\cdot,\cdot)$ by a factor of $\lambda_j^{1/2}$ on $V_j \times V_j$, as we do, $\tilde\theta$ is changed by a factor of $\lambda_j^{-1/2}$ on V_j, and then the resulting $\tilde\theta$ satisfies $\tilde\theta^2 = \mathrm{id}_V$. Note that Eqs. (2) and (3) are unchanged. Now $(\theta, \tilde\theta)$ is a real form of (G, V) commuting with $(\sigma, \tilde\sigma)$. Further,

$$\langle\tilde\theta x, \tilde\theta y\rangle = -\big(\tilde\theta x, \tilde\theta^2 y\big) = -(\tilde\theta x, y) = -\overline{(y, \tilde\theta x)} = \overline{\langle y, x\rangle} = \overline{\langle x, y\rangle}$$

for $x, y \in V$ and

$$\langle x, x\rangle = -(x, \tilde\theta x) = -(x, x) < 0$$

for $x \in V^{\tilde\theta} \setminus \{0\}$. This completes the proof. □

Proposition 3. *Let $\tau : G \to O(V, \langle\cdot,\cdot\rangle)$ be an orthogonal representation, and suppose that θ is a Cartan involution of G. Then there can be at most one real structure $\tilde\theta$ on V such that $(\theta, \tilde\theta)$ is a Cartan pair of τ.*

Proof. Suppose that $(\theta, \tilde\theta)$ and $(\theta, \tilde\theta')$ are two Cartan pairs of τ. Define

$$h(x, y) = -\langle x, \tilde\theta y\rangle \quad \text{and} \quad h'(x, y) = -\langle x, \tilde\theta' y\rangle$$

for $x, y \in V$. It is easy to see that h and h' are two U-invariant positive-definite Hermitian forms. Diagonalizing h' with respect to h, we get a U-invariant, h-orthogonal splitting $V = \bigoplus_j V_j$ such that $\tilde\theta' = \lambda_j \tilde\theta$ on V_j, where $\lambda_j > 0$ and the λ_j's are pairwise distinct. Using $(\tilde\theta')^2 = \tilde\theta^2 = 1$, we finally see that $\tilde\theta' = \tilde\theta$. □

Corollary 4. *Let $\tau : G \to O(V, \langle\cdot,\cdot\rangle)$ be an orthogonal representation. Then any two Cartan pairs of τ are G-conjugate*; *moreover, if the underlying Cartan involutions commute with a real form σ of G, then the Cartan pairs are $(G^\sigma)^\circ$-conjugate.*

Proof. Let $(\theta_1, \tilde\theta_1)$ and $(\theta_2, \tilde\theta_2)$ be two Cartan pairs of τ. It is known that there exists $g \in G$ such that $\theta_2 = \mathrm{Inn}_g\, \theta_1\, \mathrm{Inn}_g^{-1}$. Of course, $(\mathrm{Inn}_g\, \theta_1\, \mathrm{Inn}_g^{-1}, g\tilde\theta_1 g^{-1})$ is also a Cartan pair. Proposition 3 implies that $\tilde\theta_2 = g\tilde\theta_1 g^{-1}$. Further, if both θ_1 and θ_2 commute with σ, it is known that g can be taken in the identity component of G^σ. □

3. The classification

Let $\hat G_{\mathbf R}/G_{\mathbf R}$ be a semisimple pseudo-Riemannian symmetric space. Here $\hat G_{\mathbf R}$ is a connected real semisimple Lie group, $\hat\tau$ is a non-trivial involutive automorphism of $\hat G_{\mathbf R}$ and $G_{\mathbf R}$ is an open subgroup of the fixed point group of $\hat\tau$. The automorphism $\hat\tau$ induces an automorphism of the Lie algebra $\hat{\mathfrak g}_{\mathbf R}$ of $\hat G_{\mathbf R}$ which we denote by the same letter. Let $\hat{\mathfrak g}_{\mathbf R} = \mathfrak g_{\mathbf R} + V_{\mathbf R}$ be the decomposition into ± 1-eigenspaces of $\hat\tau$. Of course, $\mathfrak g_{\mathbf R}$ is the Lie algebra of $G_{\mathbf R}$. The restriction of the Killing form of $\hat{\mathfrak g}_{\mathbf R}$ to $V_{\mathbf R} \times V_{\mathbf R}$ is $\mathrm{Ad}_{G_{\mathbf R}}$-invariant and non-degenerate, so it induces a $\hat G_{\mathbf R}$-invariant pseudo-Riemannian metric on $\hat G_{\mathbf R}/G_{\mathbf R}$. The adjoint action of $G_{\mathbf R}$ on $V_{\mathbf R}$ is equivalent to the isotropy representation of $\hat G_{\mathbf R}/G_{\mathbf R}$ at the base-point.

Next, extend $\hat{\tau}$ complex-linearly to an automorphism of the complexification $\hat{\mathfrak{g}} = (\hat{\mathfrak{g}}_{\mathbf{R}})^c$ denoted by the same letter and consider the corresponding decomposition $\hat{\mathfrak{g}} = \mathfrak{g} + V$ into ± 1-eigenspaces. Let $\hat{G}$ be the simply-connected complex Lie group with Lie algebra $\hat{\mathfrak{g}}$, view $\hat{\tau}$ as an involution of $\hat{G}$, and let G be the fixed point group of $\hat{\tau}$ in $\hat{G}$. Note that G is connected. The adjoint action of G on V is a complex polar action whose Cartan subspaces coincide with the maximal Abelian subspaces of V consisting of semisimple elements (indeed, this is a θ-group (see [DK85, Introd.] or [PV94, 8.5, 8.6]); no relation here to the aforementioned Cartan involution θ). Further, it is an orthogonal action with respect to the restriction of the Killing form of $\hat{\mathfrak{g}}$ to V. By passing from $\hat{G}_{\mathbf{R}}$ to a finite covering if necessary, we may assume that $\hat{G}_{\mathbf{R}}$ embeds into $\hat{G}$ and $G_{\mathbf{R}}$ embeds into G, so we can view the adjoint action of $G_{\mathbf{R}}$ on $V_{\mathbf{R}}$ as an orthogonal real form of the adjoint action of G on V. We deduce that the isotropy representation of a pseudo-Riemannian symmetric space is a polar representation. In this section, we prove Theorem 1 which is essentially a converse to this result.

Before giving the proof of Theorem 1, we prove four lemmas. In the remaining of this section, let G be a complex reductive algebraic group defined over $\mathbf{R}$ and denote by $G_{\mathbf{R}}$ the identity component of its real points.

Lemma 5. *Let $\tau : G \to GL(V)$ be a polar representation, where $V = V_1 \oplus V_2$ is a G-invariant decomposition. Assume that the induced representations $\tau_i : G \to GL(V_i)$ are stable. Then*:

(a) *τ_i is polar, $i = 1, 2$.*
(b) *Every Cartan subspace of τ is of the form $c = c_1 \oplus c_2$, where c_i is a Cartan subspace of τ_i, $i = 1, 2$.*
(c) *The closed orbits of G on V_2 coincide with those of G_{v_1}, where v_1 is any semisimple vector of V_1. In particular, V_1 and V_2 are inequivalent representations.*
(d) *Fix a Cartan subspace $c = c_1 \oplus c_2$, let $\mathfrak{h}_1$ (resp. $\mathfrak{h}_2$) denote the centralizer of c_2 (resp. c_1) in $\mathfrak{g}$, and denote by H_i the connected subgroup of G corresponding to $\mathfrak{h}_i$. Then the closed orbits of τ coincide with those of $\hat{\tau} : H_1 \times H_2 \to GL(V_1 \oplus V_2)$, where $\hat{\tau}(g_1, g_2)(v_1 + v_2) = \tau_1(g_1)v_1 + \tau_2(g_2)v_2$.*

Proof. Parts (a) and (b) are Prop. 2.14 in [DK85], and (c) is Cor. 2.15 of that paper. Let us prove (d). Select a regular element $v_1 + v_2 \in c_1 \oplus c_2$ for τ. Then $\mathfrak{h}_1 = \mathfrak{g}_{v_2}$, $\mathfrak{h}_2 = \mathfrak{g}_{v_1}$ and (c) implies that $\mathfrak{h}_1 \cdot v_1 = \mathfrak{g} \cdot v_1$ and $\mathfrak{h}_2 \cdot v_2 = \mathfrak{g} \cdot v_2$ (note that v_i is semisimple for τ_i since c_i is a Cartan subspace). This implies that $\mathfrak{g} = \mathfrak{h}_1 + \mathfrak{h}_2$, so $G = H_1 \cdot H_2 = H_2 \cdot H_1$ by connectedness of G. For any $u_1 + u_2 \in c_1 \oplus c_2$, we now have that $G(u_1 + u_2) \subset (H_1 \times H_2)(u_1 + u_2)$. Since $\mathfrak{g} = \mathfrak{h}_1 + \mathfrak{h}_2$, and $G(u_1 + u_2)$, $H_1 u_1 \times H_2 u_2$ are closed and connected, it follows that the two orbits coincide. □

Lemma 6. *Let $\rho : G_{\mathbf{R}} \to GL(V_{\mathbf{R}})$ be a polar representation, where $V_{\mathbf{R}} = (V_{\mathbf{R}})_1 \oplus (V_{\mathbf{R}})_2$ is a $G_{\mathbf{R}}$-invariant decomposition. Assume that the induced representations $\rho_i : G_{\mathbf{R}} \to GL((V_{\mathbf{R}})_i)$ are orthogonalizable. Then there exist closed connected subgroups H_i' of $G_{\mathbf{R}}$, $i = 1, 2$, such that the restricted representations $\rho_i|_{H_i'} : H_i' \to GL((V_{\mathbf{R}})_i)$ are polar and the closed orbits of ρ coincide with those of $\hat{\rho} : H_1' \times H_2' \to GL((V_{\mathbf{R}})_1 \oplus (V_{\mathbf{R}})_2)$, where $\hat{\rho}(g_1, g_2)(v_1 + v_2) = \rho_1(g_1)v_1 + \rho_2(g_2)v_2$.*

Proof. The complexification $\tau = \rho^c : G \to GL(V)$ is polar and each $\tau_i = \rho_i^c$ is orthogonalizable, hence stable. By Lemma 5, the closed orbits of τ coincide with those of $\hat{\tau} : H_1 \times H_2 \to$

$GL(V_1 \oplus V_2)$, where $V_i = (V_{\mathbf{R}})_i^c$, the group H_1 (resp. H_2) is the connected centralizer of c_2 (resp. c_1) in G, and $c = c_1 \oplus c_2 \subset V_1 \oplus V_2$ is a Cartan subspace of τ. As usual, suppose that ρ is defined by $(\sigma, \tilde{\sigma})$. Now c can be taken to be $\tilde{\sigma}$-stable due to Lemma 13 below. In this case, H_i is σ-stable; set H_i' to be subgroup of $G_{\mathbf{R}}$ given by the identity component of $(H_i)^\sigma$. It is clear that the groups H_i' have the desired properties. □

Given a representation $\rho : G_{\mathbf{R}} \to GL(V_{\mathbf{R}})$, denote by $\rho^* : G_{\mathbf{R}} \to GL(V_{\mathbf{R}}^*)$ the dual representation. Note that $\rho \oplus \rho^* : G_{\mathbf{R}} \to GL(V_{\mathbf{R}} \oplus V_{\mathbf{R}}^*)$ is always orthogonal with respect to

$$\big\langle \big(v_1, v_1^*\big), \big(v_2, v_2^*\big)\big\rangle = v_1^*(v_2) + v_2^*(v_1). \tag{4}$$

The proof of the following lemma is simple and we omit it.

Lemma 7. *Let $\rho : G_{\mathbf{R}} \to GL(V_{\mathbf{R}})$ be orthogonalizable. Then there exists an irreducible decomposition*

$$V_{\mathbf{R}} = (V_{\mathbf{R}})_1 \oplus \cdots \oplus (V_{\mathbf{R}})_r \oplus (V_{\mathbf{R}})_{r+1} \oplus (V_{\mathbf{R}})_{r+1}^* \oplus \cdots \oplus (V_{\mathbf{R}})_s \oplus (V_{\mathbf{R}})_s^*,$$

where $(V_{\mathbf{R}})_1, \ldots, (V_{\mathbf{R}})_r$ are orthogonalizable and $(V_{\mathbf{R}})_{r+1}, \ldots, (V_{\mathbf{R}})_s$ are not orthogonalizable.

The following lemma will be used to show that certain polar representations have the same closed orbits as a the isotropy representation of a symmetric space.

Lemma 8. *Suppose $\tau : G \to GL(V)$ is a polar orthogonalizable representation, U is a maximal compact subgroup of G and $\tau_u : U \to GL(W)$ is a real form. Suppose also that U' is a connected closed subgroup of U and $G' \subset G$ is the complexification of U'. If $\tau_u|_{U'}$ has the same orbits in W as τ_u, then $\tau|_{G'}$ has the same closed orbits in V as τ. If, in addition, $\rho : G_{\mathbf{R}} \to GL(V_{\mathbf{R}})$ is a real form of τ and $G'_{\mathbf{R}} \subset G_{\mathbf{R}}$ is a connected real form of G', then $\rho|_{G'_{\mathbf{R}}}$ has the same closed orbits in $V_{\mathbf{R}}$ as ρ.*

Proof. The assertion about ρ immediately follows from that about τ and the facts that $G_{\mathbf{R}}v$ is closed if and only if Gv is closed [Bir71] and $\dim_{\mathbf{R}} G_{\mathbf{R}}v = \dim Gv$ for $v \in V_{\mathbf{R}}$. Let us prove the assertion about τ. We first claim that if $v \in V$ and Gv is closed, then $G'v = Gv$. Of course, we already have that $G'v \subset Gv$. In the case in which $v \in W$, we have that both Gv and $G'v$ are connected, closed and have dimension equal to $\dim_{\mathbf{R}} Uv = \dim_{\mathbf{R}} U'v$, so the result follows. In the general case, fix a U-invariant positive-definite Hermitian form $(\cdot,\cdot)$ and choose $v_1 \in Gv$ of minimal length [DK85, p. 508]. Of course, $Gv_1 = Gv$ and v_1 is also of minimal length in $G'v_1$. It follows that $G'v_1$ is also closed [DK85, Thm. 1.1] and G_{v_1}, G'_{v_1} are θ-stable, where θ is the Cartan involution of G associated to U [DK85, Prop. 1.3]. Let $L = (G_{v_1})^\theta$ and $L' = (G'_{v_1})^\theta$. Now we can choose $w \in W$ such that $U_w = L$ by the same argument as in [Sch80, Prop. 5.8], and it easily follows that $U'_w = L'$. We have established that U_w (resp. U'_w) is a real form of G_{v_1} (resp. G'_{v_1}). Therefore

$$\begin{aligned}\dim Gv_1 &= \dim G - \dim G_{v_1}\\ &= \dim_{\mathbf{R}} U - \dim_{\mathbf{R}} U_w\\ &= \dim_{\mathbf{R}} Uw\end{aligned}$$

$$\begin{aligned} &= \dim_{\mathbf{R}} U'w \\ &= \dim_{\mathbf{R}} U' - \dim_{\mathbf{R}} U'_w \\ &= \dim G' - \dim G'_{v_1} \\ &= \dim G'v_1, \end{aligned}$$

which implies that $G'v_1 = Gv_1$. Since $Gv_1 = Gv$, we also have $G'v = Gv$, proving the claim.

Let $c \subset V$ be a Cartan subspace of τ. In view of the claim proved above, c consists of semisimple elements of $\tau|_{G'}$. Also, $\dim c = \dim V//G = \dim V//G'$, where the last equality follows from the fact that τ_u and $\tau_u|_{U'}$ have the same co-homogeneity in W. By [DK85, Prop. 2.2], every closed G'-orbit meets c, from which it follows that $\tau|_{G'}$ has the same closed orbits in V as τ. □

In order to prove Theorem 1, we will use the explicit lists of polar representations of compact Lie groups that have been obtained in [EH99] (irreducible case) and [Ber99,Ber01] (reducible case); see also [GT00] (both cases). For brevity, an isotropy representation of a semisimple symmetric space will be called an *s-representation*. Let $\rho : G_{\mathbf{R}} \to GL(V_{\mathbf{R}})$ be a polar orthogonal representation. Let $\tau = \rho^c : G \to GL(V)$, and suppose that ρ is given by $(\sigma, \tilde{\sigma})$ so that $G_{\mathbf{R}}$ is the identity component of G^σ. Let $(\theta, \tilde{\theta})$ be a Cartan pair as in Proposition 2, $U = G^\theta$, $W = V^{\tilde{\theta}}$, and $\tau_u : U \to GL(W)$ the associated real form. Then τ_u is a polar representation of a compact Lie group. By Dadok's theorem quoted in the introduction and the results in [EH99,Ber01], τ_u is either a Riemannian s-representation or one of the exceptions listed in those papers. We need the following fundamental lemma.

Lemma 9. *If τ_u is an irreducible Riemannian s-representation, then ρ is a pseudo-Riemannian s-representation.*

Proof. By assumption, $\hat{\mathfrak{u}} = \mathfrak{u} + W$ admits a real Lie algebra structure extending that of $\mathfrak{u}$ such that [HZ96, p. 182]

$$[X, w] = \tau_u(X)w \quad \text{and} \quad \big\langle X, [w, w']\big\rangle_{\mathfrak{u}} = \big\langle \tau_u(X)w, w'\big\rangle$$

for $X \in \mathfrak{u}$ and $w, w' \in W$, where

$$\langle X, Y\rangle_{\mathfrak{u}} = \operatorname{trace}_{\hat{\mathfrak{u}}}(\operatorname{ad}_X \operatorname{ad}_Y)$$

for $X, Y \in \mathfrak{u}$ and $\operatorname{ad}_X(Z) = [X, Z]$ for $X \in \mathfrak{u}$ and $Z \in \hat{\mathfrak{u}}$. Denote the Killing form of $\hat{\mathfrak{u}}$ by β; note that it is non-degenerate as $\hat{\mathfrak{u}}$ is semisimple. Also, it turns out that $\langle\cdot,\cdot\rangle_{\mathfrak{u}}$ is the restriction of β to $\mathfrak{u}$.

Now, since $\beta|_{W\times W}$ and $\langle\cdot,\cdot\rangle|_{W\times W}$ are both positive-definite real-valued symmetric bilinear forms which are $\mathfrak{u}$-invariant, and τ_u is irreducible, there exists $\lambda > 0$ such that $\beta(x, y) = \lambda\langle x, y\rangle$ for $x, y \in W$. By $\mathbf{C}$-bilinearity, $\beta^c(x, y) = \lambda\langle x, y\rangle$ for $x, y \in V$, where β^c is the Killing form of $\hat{\mathfrak{u}}^c = \mathfrak{g} + V$. It suffices to prove that $\hat{\mathfrak{g}}_{\mathbf{R}} := \mathfrak{g}_{\mathbf{R}} + V_{\mathbf{R}}$ is a real subalgebra of $\hat{\mathfrak{u}}^c$. It is clear that $[\mathfrak{g}_{\mathbf{R}}, \mathfrak{g}_{\mathbf{R}}] \subset \mathfrak{g}_{\mathbf{R}}$ and $[\mathfrak{g}_{\mathbf{R}}, V_{\mathbf{R}}] \subset V_{\mathbf{R}}$. We claim that also $[V_{\mathbf{R}}, V_{\mathbf{R}}] \subset \mathfrak{g}_{\mathbf{R}}$. In fact,

$$\beta^c(\tilde{\sigma}x, \tilde{\sigma}y) = \lambda\langle\tilde{\sigma}x, \tilde{\sigma}y\rangle = \lambda\overline{\langle x, y\rangle} = \overline{\lambda\langle x, y\rangle} = \overline{\beta^c(x, y)} \tag{5}$$

for $x, y \in V$. If $Z_1, Z_2 \in \mathfrak{g}$, then also

$$\begin{aligned}
\beta^c(\sigma Z_1, \sigma Z_2) &= \mathrm{trace}_{\hat{\mathfrak{u}}^c}(\mathrm{ad}_{\sigma Z_1}\,\mathrm{ad}_{\sigma Z_2}) \\
&= \mathrm{trace}_{\mathfrak{g}}(\mathrm{ad}_{\sigma Z_1}\,\mathrm{ad}_{\sigma Z_2}) + \mathrm{trace}_V(\mathrm{ad}_{\sigma Z_1}\,\mathrm{ad}_{\sigma Z_2}) \\
&= \mathrm{trace}_{\mathfrak{g}}(\sigma\,\mathrm{ad}_{Z_1}\,\mathrm{ad}_{Z_2}\,\sigma) + \mathrm{trace}_V(\tilde{\sigma}\,\mathrm{ad}_{Z_1}\,\mathrm{ad}_{Z_2}\,\tilde{\sigma}) \\
&= \overline{\mathrm{trace}_{\mathfrak{g}}(\mathrm{ad}_{Z_1}\,\mathrm{ad}_{Z_2})} + \overline{\mathrm{trace}_V(\mathrm{ad}_{Z_1}\,\mathrm{ad}_{Z_2})} \\
&= \overline{\mathrm{trace}_{\hat{\mathfrak{u}}^c}(\mathrm{ad}_{Z_1}\,\mathrm{ad}_{Z_2})} \\
&= \overline{\beta^c(Z_1, Z_2)},
\end{aligned} \tag{6}$$

where we used in the third equality that

$$\mathrm{ad}_{\sigma Z_1} x = \sigma Z_1 \cdot x = \tilde{\sigma}(Z_1 \cdot \tilde{\sigma} x) = \tilde{\sigma}\,\mathrm{ad}_{Z_1}\tilde{\sigma} x$$

for $x \in V$. Therefore

$$\begin{aligned}
\beta^c\big(Z, \sigma[x,y]\big) &= \overline{\beta^c\big(\sigma Z, [x,y]\big)} \quad \text{by (6)} \\
&= \overline{\beta^c(\sigma Z \cdot x, y)} \\
&= \beta^c\big(\tilde{\sigma}(\sigma Z \cdot x), \tilde{\sigma} y\big) \quad \text{by (5)} \\
&= \beta^c(Z \cdot \tilde{\sigma} x, \tilde{\sigma} y) \\
&= \beta^c\big(Z, [\tilde{\sigma} x, \tilde{\sigma} y]\big)
\end{aligned}$$

for all $Z \in \mathfrak{g}$ and $x, y \in V$. Hence $\sigma[x,y] = [\tilde{\sigma} x, \tilde{\sigma} y]$, proving the claim. Of course, ρ is now the isotropy representation of the pseudo-Riemannian symmetric space $\hat{G}_{\mathbf{R}}/G_{\mathbf{R}}$, where $\hat{G}_{\mathbf{R}} := \mathrm{Int}(\hat{\mathfrak{g}}_{\mathbf{R}})$ and $G_{\mathbf{R}}$ is the connected subgroup associated to $\mathfrak{g}_{\mathbf{R}}$. $\square$

Proof of Theorem 1. In view of Lemmas 6 and 7, it is enough to consider the following two cases:

(a) ρ is irreducible.
(b) ρ decomposes as $\rho_0 \oplus \rho_0^*$, where $\rho_0 : G_{\mathbf{R}} \to GL(V_0)$ is irreducible and non-orthogonalizable, $V_{\mathbf{R}} = V_0 \oplus V_0^*$, and the inner product on $V_{\mathbf{R}}$ is given by (4).

(a.1) Suppose first that ρ is absolutely irreducible. Then τ_u is an absolutely irreducible polar representation of a compact Lie group, so it is either a Riemannian s-representation and then the result follows from Lemma 9, or it is listed in [EH99]. In the latter case, it must be $(SO(3) \times Spin(7), \mathbf{R}^3 \otimes \mathbf{R}^8)$, where $\mathbf{R}^8$ denotes the spin representation; according to [Oni04, Table 5, p. 79], $G_{\mathbf{R}}$ equals $SO_0(1,2) \times Spin(7)$ (resp. $SO(3) \times Spin_0(3,4)$, $SO_0(1,2) \times Spin_0(3,4)$; here the subscript denotes the identity component), and $\rho : G_{\mathbf{R}} \to GL(\mathbf{R}^3 \otimes \mathbf{R}^8)$ is the tensor product of the standard representation and the spin representation. Since $Spin(7) \subset SO(8)$ and $Spin_0(3,4) \subset SO_0(4,4)$ [Har90, Thm. 14.2], and ρ extends to a pseudo-Riemannian s-representation ρ' of $SO_0(1,2) \times SO(8)$ (resp. $SO(3) \times SO_0(4,4)$, $SO_0(1,2) \times SO_0(4,4)$) on $\mathbf{R}^3 \otimes \mathbf{R}^8$, it follows from Lemma 8 that ρ has the same closed orbits as ρ', so this case is checked.

(a.2) Suppose now that ρ is irreducible but not absolutely irreducible. Then $V_{\mathbf{R}}$ admits an invariant complex structure.

(a.2.1) If τ_u is irreducible, then W admits a U-invariant complex structure, and by Lemma 9 we have only to consider the cases in which it is not an s-representation. According to [EH99], those are

$$\begin{gathered}\big(SO(2)\times G_2,\mathbf{R}^2\otimes\mathbf{R}^7\big),\\ \big(SO(2)\times Spin(7),\mathbf{R}^2\otimes\mathbf{R}^8\big),\\ \big(SU(p)\times SU(q),\big(\mathbf{C}^p\otimes\mathbf{C}^q\big)^r\big)\quad(p\neq q),\\ \big(SU(n),\big(\Lambda^2\mathbf{C}^n\big)^r\big)\quad(n\text{ odd}),\\ \big(Spin(10),\mathbf{C}^{16}\big).\end{gathered}\tag{7}$$

We do only the first and third cases, the others being similar in spirit. In the first case, $G_{\mathbf{R}}$ must be $SO(2)\times G_2^*$, where G_2^* is the automorphism group of the split octonions and ρ is the real tensor product of the standard representation of $SO(2)$ and the 7-dimensional representation of G_2^* since $V_{\mathbf{R}}$ admits an invariant complex structure. Now $G_2^*\subset SO_0(3,4)$ and there exists an obvious pseudo-Riemannian s-representation ρ' of $SO(2)\times SO_0(3,4)$ on $\mathbf{R}^2\otimes\mathbf{R}^7$. It follows from Lemma 8 that ρ and ρ' have the same closed orbits and we are done with this case. In the third case, viewing ρ as a complex representation, its conjugate representation $\bar\rho$ with respect to $G_{\mathbf{R}}$ must be equivalent to ρ^* because $\rho\oplus\bar\rho=(\tau_u)^c$. The only possibility is that ρ equals $(SU(r,p-r)\times SU(s,q-s),(\mathbf{C}^p\otimes\mathbf{C}^q)^r)$, which has the same closed orbits as the s-representation of the pseudo-Riemannian symmetric space

$$SU(r+s,p+q-r-s)/S\big(U(r,p-r)\times U(s,q-s)\big).$$

(a.2.2) If τ_u is not irreducible, then there exists a U-irreducible decomposition $W=W_1\oplus W_2$, where $(\tau_u)_i:U\to GL(W_i)$ is absolutely irreducible. Now $V=W_1^c\oplus W_2^c$ is a G-irreducible decomposition, where W_1^c and W_2^c are inequivalent by polarity (Lemma 5(c)) and W_2^c must be the conjugate representation to W_1^c with respect to $G_{\mathbf{R}}$. Denote $\tau_i=(\tau_u)_i^c:G\to GL(W_i^c)$. It follows that

$$\tilde\sigma W_1^c=W_2^c\quad\text{and}\quad\tau_2(g)=\tau_1\big(\sigma(g)\big)=\tilde\sigma\tau_1(g)\tilde\sigma\tag{8}$$

for $g\in G$. Since σ commutes with θ, we can view σ as an automorphism of U. Suppose first that τ_u is splitting, that is $U=U_1\times U_2$ and τ_u is the outer direct product of $(\tau_u)_1|_{U_1}$ and $(\tau_u)_2|_{U_2}$. On the level of Lie algebras, (8) implies that $\mathfrak{u}_1=\ker(\tau_u)_2=\sigma(\ker(\tau_u)_1)=\sigma(\mathfrak{u}_2)$. Now we can assume that $\mathfrak{u}_1=\mathfrak{u}_2$, $(\tau_u)_1=(\tau_u)_2$, $\mathfrak{g}=\mathfrak{g}_1\oplus\bar{\mathfrak{g}}_1$, where $\mathfrak{g}_1=\mathfrak{u}_1^c$ and $\bar{\mathfrak{g}}_1$ is the conjugate Lie algebra of $\mathfrak{g}_1$, and $\sigma:\mathfrak{g}\to\mathfrak{g}$ is given by $\sigma(Z',\bar Z'')=(Z'',\bar Z')$. Moreover, $V=W_1^c\oplus\overline{W}_1^c$ and $\tilde\sigma:V\to V$ is given by $\tilde\sigma(w',\bar w'')=(w'',\bar w')$. Hence $\mathfrak{g}_{\mathbf{R}}=\{(Z',\bar Z'')\in\mathfrak{g}_1\oplus\bar{\mathfrak{g}}_1\colon Z'=Z''\}$, $V_{\mathbf{R}}=\{(w',\bar w'')\in W_1^c\oplus\overline{W}_1^c\colon w'=w''\}$, and ρ is just the realification of the complexification of the real polar absolutely irreducible representation $(\tau_u)_1|_{U_1}:U_1\to GL(W_1)$. If $(\tau_u)_1|_{U_1}$ is an s-representation, this means that ρ is the s-representation of a complex symmetric space viewed as a real representation. The only other possibility is that $(\tau_u)_1|_{U_1}$ equals $(SO(3)\times Spin(7),\mathbf{R}^3\otimes\mathbf{R}^8)$. In this case, $\rho:SO(3,\mathbf{C})^r\times Spin(7,\mathbf{C})^r\to GL((\mathbf{C}^3\otimes\mathbf{C}^8)^r)$ has the same closed orbits as $SO(3,\mathbf{C})^r\times SO(8,\mathbf{C})^r\to GL((\mathbf{C}^3\otimes\mathbf{C}^8)^r)$, so we are done. Suppose now that τ_u is not splitting. Then $U=U_1\times U_0\times U_2$, where U_2 (resp. U_1) coincides with

$\ker(\tau_u)_1$ (resp. $\ker(\tau_u)_2$) up to some discrete part. Since $\sigma(\ker(\tau_u)_1) = \ker(\tau_u)_2$, the automorphism $\sigma : U \to U$ must restrict to isomorphisms $U_1 \to U_2$ and $U_0 \to U_0$. It follows that U_0 is essential for $(\tau_u)_1$ if and only if it is essential for $(\tau_u)_2$. Therefore τ_u is not almost splitting in the sense of [GT00, p. 58]; we use the classification given there: due to the facts that the $(\tau_u)_i$ admits no invariant complex structure and $\dim W_1 = \dim W_2$, we need only to consider the case in which $U_0 = Spin(8)$, $U_1 = U_2 = \{1\}$, and W_1, W_2 are two 8-dimensional inequivalent representations of $Spin(8)$. Referring to [Oni04, Table 5, p. 80], $G_{\mathbf{R}}$ must be either $Spin_0(3,5)$ or $Spin_0(1,7)$, and ρ must be the realification of an 8-dimensional complex representation of $G_{\mathbf{R}}$ which is not of real type (indeed, in each case there exist two such representations and they are conjugate to one another). Since τ is $(SO(8,\mathbf{C}), \mathbf{C}^8_+ \oplus \mathbf{C}^8_-)$ (where $\mathbf{C}^8_\pm$ denote the half-spin representations) with compact real form $(\mathbf{Spin}(8), \mathbf{R}^8_+ \oplus \mathbf{R}^8_-)$ having the same orbits as $(SO(8) \times SO(8), \mathbf{R}^8 \oplus \mathbf{R}^8)$, it follows that ρ has the same closed orbits as the standard action of $(SO(8,\mathbf{C}))^r$ on $(\mathbf{C}^8)^r$. Now the latter is a pseudo-Riemannian s-representation, so this case is also dealt with.

(b.1) Consider now the case in which $\rho = \rho_0 \oplus \rho_0^*$, where ρ_0 is absolutely irreducible and non-orthogonalizable. Then $\rho^c = \rho_0^c \oplus (\rho_0^*)^c$ is polar and ρ_0^c is irreducible. By polarity, ρ_0^c and $(\rho_0^*)^c = (\rho_0^c)^*$ are inequivalent, so ρ_0^c is not self-dual implying that it is not of real type with respect to U. Recall that $(\rho_0^c)^*$ is the conjugate representation $\hat{\rho}_0^c$ with respect to U. It follows that $\tilde{\theta}(v', \hat{v}'') = (v'', \hat{v}')$ for $(v', \hat{v}'') \in V_0^c \oplus \hat{V}_0^c$, the space $W = \{(v', \hat{v}'') \in V_0^c \oplus \hat{V}_0^c\colon\ v' = v''\}$, and $\tau_u : U \to GL(W)$ is irreducible, not absolutely irreducible, and equivalent to $(\rho_0^c)^r|_U : U \to GL((V_0^c)^r)$. In other words, W admits a U-invariant complex structure J and ρ_0 is just a real form of the holomorphic extension of $\tau_u : U \to GL(W, J)$ to a representation of G on (W, J). By Lemma 9, it suffices to consider the case in which τ_u is not an s-representation, namely, given in (7). We do only the case $(SU(n), (\Lambda^2\mathbf{C}^n)^r)$ for n odd, the others being similar in spirit. Since n is odd, [Oni04, Table 5] gives that $G_{\mathbf{R}} = SL(n,\mathbf{R})$ and ρ_0 is the representation on $\Lambda^2\mathbf{R}^n$. Now $\rho = \rho_0 \oplus \rho_0^*$ has the same closed orbits as $(GL^+(n,\mathbf{R}), \Lambda^2\mathbf{R}^n \oplus (\Lambda^2\mathbf{R}^n)^*)$, which turns out to be the s-representation of the pseudo-Riemannian symmetric space $SO(n,n)/GL^+(n,\mathbf{R})$ [Ber57, Tableau II].

(b.2) Finally, suppose that $\rho = \rho_0 \oplus \rho_0^*$, where ρ_0 is irreducible, not absolutely irreducible and non-orthogonalizable. Then ρ_0, ρ_0^* can be viewed as complex representations, and $\rho^c = \rho_0 \oplus \bar{\rho}_0 \oplus \hat{\rho}_0 \oplus \hat{\bar{\rho}}_0$ is an irreducible decomposition with pairwise inequivalent summands, where $\bar{\rho}_0$ (resp. $\hat{\rho}_0 = \rho_0^*$) is the conjugate representation to ρ_0 with respect to $G_{\mathbf{R}}$ (resp. U). We must have $\tau_u = (\tau_u)_1 \oplus (\tau_u)_2 : U \to GL(W_1 \oplus W_2)$, where $(\tau_u)_i$ is polar irreducible, not absolutely irreducible. Moreover, $\tau_1 = \rho_0 \oplus \hat{\rho}_0$ and $\tau_2 = \bar{\rho}_0 \oplus \hat{\bar{\rho}}_0$, where we have set $\tau_i = (\tau_u)_i^c$.

(b.2.1) Suppose τ_u is splitting. Then $U = U_1 \times U_2$ and τ_u is the outer direct product of $(\tau_u)_1|_{U_1}$ and $(\tau_u)_2|_{U_2}$, where each $(\tau_u)_i|_{U_i}$ is irreducible and not absolutely irreducible. The automorphism $\sigma : U \to U$ must take U_1 to U_2, so we can assume $U_1 = U_2$ and $(\tau_u)_1 = (\tau_u)_2$. Write $G = G_1 \times G_2$ where $\mathfrak{g}_i = \mathfrak{u}_i^c$. Then ρ is equivalent to the realification of $\tau_1|_{G_1} : G_1 \to GL(V_0^c \oplus V_0^{c*})$, and $\tau_1|_{G_1}$ is the complexification of a polar irreducible, not absolutely irreducible representation $(\tau_u)_1|_{U_1} : U_1 \to GL(W_1)$. We have only to consider the case in which it is not an s-representation, namely, given in (7). We do only the case $(Spin(10), (\mathbf{C}^{16})^r)$, the others being similar in spirit. Here τ_1 is $(Spin(10,\mathbf{C}), \mathbf{C}^{16} \oplus \mathbf{C}^{16*})$ and ρ is $(Spin(10,\mathbf{C})^r, (\mathbf{C}^{16})^r \oplus (\mathbf{C}^{16*})^r)$, which turns out to have the same closed orbits as the pseudo-Riemannian s-representation given by the realification of $(\mathbf{C}^\times \times Spin(10,\mathbf{C}), \mathbf{C}^{16} \oplus \mathbf{C}^{16*})$.

(b.2.2) Suppose τ_u is not splitting. Then it is not almost splitting by the same argument as in case (a.2.2). Owing to the fact that $(\tau_u)_i$ admits an invariant complex structure for $i = 1, 2$, we see from [GT00, p. 59] that this case is not possible. □

4. Isoparametric submanifolds

Let $V_{\mathbf{R}}$ be a finite-dimensional real vector space equipped with a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$. A submanifold M of $V_{\mathbf{R}}$ is called a *pseudo-Riemannian submanifold* if the restrictions of $\langle\cdot,\cdot\rangle$ to the tangent spaces of M are always non-degenerate. If M is a pseudo-Riemannian submanifold, the canonical flat connection D in $V_{\mathbf{R}}$ induces the Levi-Cività connection ∇ in M, the second fundamental form B of M, and the connection $\nabla^{\perp}$ in the normal bundle νM of M in the usual way. Namely,

$$D_X Y = \nabla_X Y + B(X, Y),$$

and

$$D_X \xi = -A_\xi X + \nabla^{\perp}_X \xi,$$

where X and Y are sections of TM and ξ is a section of νM, and the Weingarten operator $A_\xi : TM \to TM$ is defined by

$$\langle A_\xi X, Y\rangle = \big\langle B(X, Y), \xi\big\rangle.$$

For each $p \in M$, the map $A_\xi|_p : T_pM \to T_pM$ is a symmetric endomorphism with respect to the induced inner product in T_pM. Note that in the case in which this induced inner product is definite, the Weingarten operator is automatically diagonalizable over $\mathbf{R}$, whereas in the general case it may happen that $A_\xi|_p$ is not diagonalizable, not even over $\mathbf{C}$.

A properly embedded pseudo-Riemannian submanifold M of $V_{\mathbf{R}}$ will be called *isoparametric* if the following two conditions are satisfied:

(a) the normal connection is flat;
(b) the eigenvalues of the Weingarten operator along a locally defined parallel normal vector field together with their algebraic multiplicities are constant.

Isoparametric submanifolds of Euclidean spaces are very important in submanifold geometry and share a very rich history and an extensive literature, see [Ter85,Tho00,BCO03] and the references therein. On the other hand, isoparametric submanifolds of indefinite space forms are not as common, but have already been considered before in codimension one, see e.g. [Hah84, Mag85].

In this section, we will consider homogeneous isoparametric submanifolds whose Weingarten operators are everywhere diagonalizable over $\mathbf{C}$. We start with the following lemma.

Lemma 10. *Let $\tau : G \to O(V, \langle\cdot,\cdot\rangle)$ be a complex polar orthogonal representation of a complex reductive algebraic group.*

(a) *For $v \in V$, we have $c_v \subset (\mathfrak{g} \cdot v)^{\perp}$, and the equality holds if and only if v is regular.*
(b) *If $c \subset V$ is a Cartan subspace, then $\langle c, \mathfrak{g} \cdot c\rangle = 0$. In particular, the restrictions of $\langle\cdot,\cdot\rangle$ to c and $\mathfrak{g} \cdot v$ for regular v are non-degenerate.*

Proof. (a) If $x \in c_v$, then $\mathfrak{g} \cdot x \subset \mathfrak{g} \cdot v$, so

$$\langle x, \mathfrak{g} \cdot v\rangle = \langle \mathfrak{g} \cdot x, v\rangle = \langle \mathfrak{g} \cdot v, v\rangle = 0$$

by G-invariance of $\langle \cdot,\cdot\rangle$, proving the inclusion. If v is regular,

$$\begin{aligned}\dim c_v &= \dim V//G \\ &= \dim V - \max_{u\in V} \dim Gu \quad (\tau \text{ is stable}) \\ &= \dim V - \dim \mathfrak{g} \cdot v,\end{aligned}$$

and this shows that c_v is the orthocomplement of $\mathfrak{g} \cdot v$ in V.

(b) follows from (a). □

Before stating the next theorem, a couple of remarks are in order. Let $G_{\mathbf{R}}$ be a connected real form of a connected complex reductive algebraic group G, let $\rho : G_{\mathbf{R}} \to GL(V_{\mathbf{R}})$ be an arbitrary real representation, and let $\tau : G \to GL(V)$ be the complexification of ρ. If $v \in V$ is semisimple, then the isotropy subgroup G_v is reductive; hence, there exists a G_v-invariant subspace $N_v \subset V$ such that $V = \mathfrak{g} \cdot v \oplus N_v$. The restriction of τ to $G_v \to GL(N_v)$ is called the *slice representation* at v. If $v \in V_{\mathbf{R}}$, then G_v, N_v and the slice representation are defined over $\mathbf{R}$. There exists a Zariski-open and dense subset V_{pr} of V such that all isotropy subgroups G_v for semisimple $v \in V_{\mathrm{pr}}$ are in one conjugacy class [Sch80, Cor. 5.6]. A semisimple point $v \in V_{\mathrm{pr}}$ is called *principal*. Every principal point is regular. We have that $V_{\mathrm{pr}} \cap V_{\mathbf{R}}$ is dense in $V_{\mathbf{R}}$ [Bre93, 13.4]. If $v \in V_{\mathbf{R}}$, then $G_{\mathbf{R}}v$ is closed if and only if Gv is closed [Bir71], and $\dim_{\mathbf{R}} G_{\mathbf{R}}v = \dim Gv$; it follows that $\max_{v\in V_{\mathbf{R}}} \dim_{\mathbf{R}} G_{\mathbf{R}}v = \max_{v\in V} \dim Gv$. Suppose now that ρ is orthogonalizable; then so is τ, hence τ is stable; in this case, V_{pr} consists of semisimple elements only, and it follows from this discussion that $V_{\mathrm{pr}} \cap V_{\mathbf{R}}$ is an open and dense subset of $V_{\mathbf{R}}$ consisting of closed $G_{\mathbf{R}}$-orbits. Suppose now, in addition, that τ is polar. Since the slice representations of τ are the complexifications of the slice representations of the real form $\tau_u : U \to GL(W)$ [Sch80, Cor. 5.9], it follows from [BCO03, Cor. 5.4.3] that V_{pr} is precisely the set of regular points of τ.

Theorem 11. *Let $\rho : G_{\mathbf{R}} \to O(V_{\mathbf{R}}, \langle\cdot,\cdot\rangle)$ be an orthogonal representation. If ρ is polar then every orbit of ρ through a regular element $v \in V_{\mathbf{R}}$ is isoparametric with diagonalizable Weingarten operators. Conversely, if ρ is irreducible and there exists a regular element $v \in V_{\mathbf{R}}$ such that $G_{\mathbf{R}}v$ is isoparametric with diagonalizable Weingarten operators then ρ is polar.*

Proof. Suppose ρ is polar and $v \in V_{\mathbf{R}}$ is regular. Then $c_v = (\mathfrak{g} \cdot v)^{\perp}$ is a Cartan subspace of $\tau = \rho^c$ defined over $\mathbf{R}$. Denote the set of real points of c_v by $(c_v)_{\mathbf{R}}$ and let $M = G_{\mathbf{R}}v$. Then the normal space $\nu_v M = (c_v)_{\mathbf{R}}$. Since $\mathfrak{g}_v \cdot c_v = 0$ [DK85, Lem. 2.1(iii)], any $\xi \in \nu_v M$ extends to a locally defined equivariant normal vector field $\hat{\xi}$ along M given by $\hat{\xi}(gv) = g\xi$ for $g \in (G_{\mathbf{R}})^{\circ}$ (the connected component of the identity). For $X \in \mathfrak{g}_{\mathbf{R}}$, we have that $\nabla^{\perp}_{X\cdot v}\hat{\xi}$ is the orthogonal projection in $\nu_v M$ of $\frac{d}{dt}|_{t=0}(\exp tX)\xi = X \cdot \xi \in \mathfrak{g}_{\mathbf{R}} \cdot \xi$. Since $\mathfrak{g}_{\mathbf{R}} \cdot \xi \subset \mathfrak{g}_{\mathbf{R}} \cdot v$, it follows that $\nabla^{\perp}_{X\cdot v}\hat{\xi} = 0$. This proves that a locally defined equivariant normal vector field along M is parallel. By taking a basis of $\nu_v M$, we get a locally defined parallel normal frame along $\nu_v M$, which implies that $\nu_v M$ is flat. It is clear that the eigenvalues of the Weingarten operator along an equivariant normal vector field together with their algebraic multiplicities are constant, and that operator is diagonalizable over $\mathbf{C}$ by Example 12 below. Hence M is isoparametric with diagonalizable Weingarten operators.

Conversely, suppose ρ is irreducible and there exists a regular element $v \in V_{\mathbf{R}}$ such that $M = G_{\mathbf{R}}v$ is isoparametric with diagonalizable Weingarten operators. Irreducibility of ρ yields that M is full in $V_{\mathbf{R}}$, that is, not contained in a proper affine subspace. We first claim that a locally defined parallel normal vector field $\hat{\xi}$ along M is equivariant. Let U be a neighborhood of v in M where $\hat{\xi}$ is defined, and let $\hat{\xi}(v) = \xi$. Suppose that $g(t)$ is a continuous curve in $G_{\mathbf{R}}$ satisfying $g(0) = 1$ and $g(t)v \in U$. Consider the continuous curve $\xi(t) = g(t)^{-1}\hat{\xi}(g(t)v)$ in $\nu_v M$. By the isoparametric condition and the fact that the action of $G_{\mathbf{R}}$ is orthogonal, we have that $A_{\xi(t)}$ and A_{ξ} have the same complex eigenvalues with the same multiplicities. By connectedness of the domain interval of $g(t)$ and the facts that they are diagonalizable and commute, we get that $A_{\xi(t)} = A_{\xi}$ for all t. Fullness of M implies the injectivity of the map $\xi \mapsto A_{\xi}$, so $\xi(t) = \xi$ for all t. This proves the claim. Since the locally defined equivariant normal vector fields are parallel with respect to the normal connection,

$$X \cdot \xi = D_{X \cdot v}\hat{\xi} = -A_{\xi}(X \cdot v) + \nabla^{\perp}_{X \cdot v}\hat{\xi} = -A_{\xi}(X \cdot v) \in \mathfrak{g}_{\mathbf{R}} \cdot v,$$

where $\xi \in \nu_v M$ and $X \in \mathfrak{g}_{\mathbf{R}}$. This proves that $\nu_v M \subset (c_v)_{\mathbf{R}}$. Since

$$\dim_{\mathbf{R}} \nu_v M = \dim_{\mathbf{R}} V_{\mathbf{R}} - \dim_{\mathbf{R}} M = \dim V - \dim Gv = \dim V /\!/ G,$$

we get that $\dim c_v = \dim V/\!/G$ and hence $\tau = \rho^c$ (resp. ρ) is polar. □

Example 12. Let $\tau : G \to O(V, \langle\cdot,\cdot\rangle)$ be a complex polar orthogonal representation and fix an orthogonal real form $\rho : G_{\mathbf{R}} \to O(V_{\mathbf{R}}, \langle\cdot,\cdot\rangle)$ defined by $(\sigma, \tilde{\sigma})$. In this example, we compute the Weingarten operator of an orbit $M = G_{\mathbf{R}}v$ for a regular $v \in V_{\mathbf{R}} = V^{\tilde{\sigma}}$. Let c be a $\tilde{\theta}$- and $\tilde{\sigma}$-stable Cartan subspace of τ and consider the corresponding root space decomposition

$$\mathfrak{g} = \mathfrak{m} + \sum_{\alpha \in \mathscr{A}} \tilde{\mathfrak{g}}_{\alpha}$$

(see Subsection 5.2 for the notation and terminology used in this example). By Proposition 14 below, we may assume that $v \in c^{\tilde{\sigma}}$. Let ξ be a vector normal to M at v in $V_{\mathbf{R}}$. Then also $\xi \in c^{\tilde{\sigma}}$.

If α is a noncomplex root, $\tilde{\mathfrak{g}}_{\alpha}$ is σ-stable. We have (the superscript "$\top$" denotes the tangential component to the orbit)

$$A_{\xi}(X_{\alpha} \cdot v) = -(X_{\alpha} \cdot \xi)^{\top},$$

where $X_{\alpha} \in \tilde{\mathfrak{g}}_{\alpha}^{\sigma}$, and

$$X_{\alpha} \cdot v = \alpha(v) X_{\alpha} \cdot v_{\alpha}, \qquad X_{\alpha} \cdot \xi = \alpha(\xi) X_{\alpha} \cdot v_{\alpha},$$

so

$$A_{\xi}(X_{\alpha} \cdot v_{\alpha}) = \lambda X_{\alpha} \cdot v_{\alpha} \quad \bigl(\text{resp. } A_{\xi}(iX_{\alpha} \cdot v_{\alpha}) = \lambda i(X_{\alpha} \cdot v_{\alpha})\bigr)$$

where $\lambda = -\frac{\alpha(\xi)}{\alpha(v)}$ is a real eigenvalue and $X_\alpha \cdot v_\alpha$ (resp. $i(X_\alpha \cdot v_\alpha)$) is the associated eigenvector if α is real (resp. imaginary).

If α is a complex root, $\tilde{\mathfrak{g}}_\alpha$ is not σ-stable and $(\tilde{\mathfrak{g}}_\alpha \oplus \tilde{\mathfrak{g}}_{|\sigma\alpha|})^\sigma$ is spanned by $X_\alpha + \sigma X_\alpha$ and $i(X_\alpha - \sigma X_\alpha)$ for $X_\alpha \in \tilde{\mathfrak{g}}_\alpha^\theta$. The associated subspace of $T_v M$ is spanned by

$$\alpha(v) X_\alpha \cdot v_\alpha + \overline{\alpha(v)}\tilde{\sigma}(X_\alpha \cdot v_\alpha), \quad i\big(\alpha(v) X_\alpha \cdot v_\alpha - \overline{\alpha(v)}\tilde{\sigma}(X_\alpha \cdot v_\alpha)\big) \tag{9}$$

for $X_\alpha \in \tilde{\mathfrak{g}}_\alpha^\theta$.

Now $\lambda = -\frac{\alpha(\xi)}{\alpha(v)}$ is not real and the matrix of A_ξ in the basis (9) is given by

$$\begin{pmatrix} \Re\lambda & -\Im\lambda \\ \Im\lambda & \Re\lambda \end{pmatrix},$$

which is of course diagonalizable over $\mathbf{C}$.

5. Structural theory of polar representations of real reductive algebraic groups

Consider a semisimple pseudo-Riemannian symmetric space $\hat{G}_{\mathbf{R}}/G_{\mathbf{R}}$ and its complexification $\hat{G}/G$ as in the first two paragraphs of Section 3. Let $\hat{\sigma}$ denote the conjugation of $\hat{G}$ over $\hat{G}_{\mathbf{R}}$. We can choose a Cartan involution $\hat{\theta}$ of $\hat{G}_{\mathbf{R}}$ that commutes with $\hat{\tau}$ on $\hat{G}_{\mathbf{R}}$. Since $\hat{G}$ is simply-connected, we can extend $\hat{\theta}$ anti-holomorphically to a Cartan involution of $\hat{G}$ which will be denoted by the same letter. Note that $\hat{\theta}$ commutes with $\hat{\tau}$ and $\hat{\sigma}$ on $\hat{G}$. Set θ (resp. $\tilde{\theta}$) to be the restriction of $\hat{\theta}$ to G (resp. V), and set σ (resp. $\tilde{\sigma}$) to be the restriction of $\hat{\sigma}$ to G (resp. V). Then $\hat{U} = \hat{G}^{\hat{\theta}}$ (resp. $U = G^\theta$) is a compact real form of $\hat{G}$ (resp. G). Write $W = V^{\tilde{\theta}}$. Now we have the combined decomposition

$$\hat{\mathfrak{g}}_{\mathbf{R}} = \big(\underbrace{\mathfrak{g}_{\mathbf{R}} \cap \mathfrak{u}}_{:=\mathfrak{k}_{\mathbf{R}}} \overset{\theta}{+} \underbrace{\mathfrak{g}_{\mathbf{R}} \cap i\mathfrak{u}}_{:=\mathfrak{p}_{\mathbf{R}}}\big) \overset{\hat{\tau}}{+} \big(V_{\mathbf{R}} \cap W \overset{\tilde{\theta}}{+} V_{\mathbf{R}} \cap iW\big). \tag{10}$$

In this context, an element $v \in V_{\mathbf{R}}$ is called semisimple if ad_v is a semisimple endomorphism of $\hat{\mathfrak{g}}$, and a Cartan subspace of $\hat{\mathfrak{g}}_{\mathbf{R}}$ is a maximal Abelian subspace of $V_{\mathbf{R}}$ consisting of semisimple elements. It is known that the $\mathrm{Ad}_{G_{\mathbf{R}}}$-orbit of $v \in V_{\mathbf{R}}$ is closed if and only if v is semisimple [BH62, Cor. 10.3]; every semisimple element of $V_{\mathbf{R}}$ belongs to some Cartan subspace; every Cartan subspace of $V_{\mathbf{R}}$ is $\mathrm{Ad}_{(G_{\mathbf{R}})^\circ}$-conjugate to a $\tilde{\theta}$-stable Cartan subspace; there exist finitely many $\mathrm{Ad}_{(G_{\mathbf{R}})^\circ}$-conjugacy classes of $\tilde{\theta}$-stable Cartan subspaces in $V_{\mathbf{R}}$; two such $\tilde{\theta}$-stable Cartan subspaces are $\mathrm{Ad}_{(K_{\mathbf{R}})^\circ}$-conjugate if and only they are $\mathrm{Ad}_{(G_{\mathbf{R}})^\circ}$-conjugate if and only they are Ad_G-conjugate [HHNO99].

Throughout this section, we let $\tau : G \to GL(V)$ be a complex polar representation of a connected complex reductive algebraic group, consider a real form $\rho : G_{\mathbf{R}} \to GL(V_{\mathbf{R}})$ defined by $(\sigma, \tilde{\sigma})$, where $G_{\mathbf{R}}$ is the identity component of G^σ, and prove a collection of results for ρ similar to those stated in the previous paragraph for an s-representation. The first three results do not require that τ and ρ be orthogonalizable.

5.1. General facts about Cartan subspaces

A *Cartan subspace* of ρ is a subspace of $V^{\tilde{\sigma}}$ which is the $\tilde{\sigma}$-fixed point vector space of a $\tilde{\sigma}$-stable Cartan subspace of τ.

Lemma 13. *There exist $\tilde{\sigma}$-stable Cartan subspaces of τ.*

Proof. Owing to the remarks preceding Theorem 11, the set $V_{\mathrm{pr}} \cap V_{\mathbf{R}}$ is a nonempty open subset of $V_{\mathbf{R}} = V^{\tilde{\sigma}}$ consisting of regular elements of τ; it suffices to take c_v where v lies therein. □

We will use the following notion in the proof of the next proposition. The *rank* of τ is defined to be the difference $\dim c - \dim c^{\mathfrak{g}}$, where $c \subset V$ is a Cartan subspace and $c^{\mathfrak{g}}$ denotes the subspace of G-fixed points in c.

Proposition 14. *Given a semisimple $x \in V^{\tilde{\sigma}}$, there exists a Cartan subspace of $V^{\tilde{\sigma}}$ which contains x.*

Proof. Note that for a regular $x \in V^{\tilde{\sigma}}$, one can simply take $c = c_x$. In the general case, we proceed by induction on the rank of τ. Since x is semisimple, there exists a Cartan subspace c' such that $x \in c'$. If $x \in (c')^{\mathfrak{g}}$, then x belongs to any $\tilde{\sigma}$-stable Cartan subspace of τ. Suppose now $x \notin (c')^{\mathfrak{g}}$. Then the slice representation (G_x, N_x) is polar with rank strictly lower than τ, and $c' \subset N_x$ is a Cartan subspace of (G_x, N_x) [DK85, Thm. 2.4]. Without loss of generality, x is a minimal vector with respect to some U-invariant positive-definite Hermitian form $(\cdot,\cdot)$ which is real-valued on $V^{\tilde{\sigma}}$, and N_x is the orthocomplement of $\mathfrak{g} \cdot x$ with respect to $(\cdot,\cdot)$ [DK85, Rmk. 1.4]. Since $x \in V^{\tilde{\sigma}}$, it follows that G_x is σ-stable, N_x is $\tilde{\sigma}$-stable and (G_x, N_x) is defined over $\mathbf{R}$ with respect to $(\sigma, \tilde{\sigma})$. By the induction hypothesis, there exists a $\tilde{\sigma}$-stable Cartan subspace $c \subset N_x$ such that $x \in c$. Now c, c' are two Cartan subspaces of (G_x, N_x), so there exists $g \in G_x$ such that $g \cdot c' = c$. It follows that c is a Cartan subspace of τ. □

Theorem 15. *There exist only finitely many $G_{\mathbf{R}}$-conjugacy classes of Cartan subspaces of $V_{\mathbf{R}}$.*

Proof. According to the remarks preceding Theorem 11, the set of regular points of τ is a Zariski-open and dense subset V_{pr} of V. By a theorem of Whitney [Whi57], $V_{\mathrm{pr}} \cap V_{\mathbf{R}}$ has finitely many connected components.

Suppose now that $c^{\tilde{\sigma}}$ is a Cartan subspace of $V_{\mathbf{R}}$. Consider the map

$$G_{\mathbf{R}} \times c^{\tilde{\sigma}} \to V_{\mathbf{R}}, \quad (g, v) \mapsto g \cdot v;$$

it is easily seen to be a smooth submersion at v if v is a regular point of τ. It follows that $G_{\mathbf{R}} \cdot (c^{\tilde{\sigma}} \cap V_{\mathrm{pr}})$ is open in $V_{\mathbf{R}}$. But the sets $G_{\mathbf{R}} \cdot (c^{\tilde{\sigma}} \cap V_{\mathrm{pr}})$ for varying $c^{\tilde{\sigma}}$ obviously cover $V_{\mathrm{pr}} \cap V_{\mathbf{R}}$. Any two of them are not disjoint if and only if the corresponding Cartan subspaces are conjugate, in which case the sets coincide. The result follows. □

Consider the categorical quotient map $\pi : V \to V/\!/G$. Since G, V, and the action of G on V are defined over $\mathbf{R}$, so is the variety $V/\!/G$; denote its set of real points by $(V/\!/G)_{\mathbf{R}}$. By a theorem of Tarski and Seidenberg, $\pi(V_{\mathbf{R}})$ is a real semialgebraic subset of $(V/\!/G)_{\mathbf{R}}$. Recall that $\pi(V_{\mathrm{pr}} \cap V_{\mathbf{R}})$ is an open and dense subset of $\pi(V_{\mathbf{R}})$. We propose the following conjecture (compare [Rot71]).

Conjecture 16. *The map π sets up a one-to-one correspondence between the $G_{\mathbf{R}}$-conjugacy classes of Cartan subspaces of $V_{\mathbf{R}}$ and the connected components of the stratum $\pi(V_{\mathrm{pr}} \cap V_{\mathbf{R}})$.*

Henceforth we assume that ρ (and hence τ) is orthogonal with respect to $\langle\cdot,\cdot\rangle$.

Theorem 17.

(a) *Given a $\tilde{\sigma}$-stable Cartan subspace $c \subset V$, there exists a Cartan pair $(\eta, \tilde{\eta})$ commuting with $(\sigma, \tilde{\sigma})$ such that c is $\tilde{\eta}$-stable.*
(b) *Given a Cartan pair $(\theta, \tilde{\theta})$ commuting with $(\sigma, \tilde{\sigma})$, every $\tilde{\sigma}$-stable Cartan subspace $c \subset V$ is $(G^\sigma)^\circ$-conjugate to a $\tilde{\theta}$-stable one (hence also $\tilde{\sigma}$-stable).*

Proof. We begin by showing that there exists a Cartan pair $(\mu, \tilde{\mu})$ of τ such that $\tilde{\mu}(c) = c$. Indeed, suppose $(\theta', \tilde{\theta}')$ is any Cartan pair. We can select $v \in V^{\tilde{\theta}'}$ regular. Since c meets all the closed orbits, there exists $g \in G$ such that $g \cdot v \in c$. Define $\mu = \mathrm{Inn}_g \theta' \mathrm{Inn}_g^{-1}$ and $\tilde{\mu} = g\tilde{\theta}' g^{-1}$. Then $(\mu, \tilde{\mu})$ is a Cartan pair and $\tilde{\mu}(g \cdot v) = g \cdot v$. Hence $c = c_{g\cdot v}$ is $\tilde{\mu}$-stable.

The following construction of η is standard (compare [Oni04, §3, Prop. 6]). Set $\omega = \sigma\mu$. We can view ω as a complex linear automorphism of $\mathfrak{g}$. Consider the decomposition into the center and semisimple factor $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_{ss}$. Let β be the Killing form of $\mathfrak{g}_{ss}$. We can extend β to an ad-invariant symmetric bilinear form on $\mathfrak{g}$, denoted by the same letter, which is real-valued on $\mathfrak{g}^\mu$, $\mathfrak{g}^\sigma$ and negative-definite on $\mathfrak{g}^\mu$. Then one easily sees that ω is Hermitian with respect to the positive-definite Hermitian form $B_\mu(X, Y) = -\beta(X, \mu Y)$, where $X, Y \in \mathfrak{g}$. It follows that ω^2 is Hermitian and positive-definite, and hence belongs to a one-parameter family of Hermitian and positive-definite automorphisms of $\mathfrak{g}$. Therefore there exists a unique Hermitian, positive-definite automorphism φ of $\mathfrak{g}$ such that $\varphi^4 = \omega^2$. Since $\varphi|_{\mathfrak{g}_{ss}}$ belongs to a one-parameter group of automorphisms of $\mathfrak{g}$, we have that $\varphi|_{\mathfrak{g}_{ss}}$ is inner, that is, equals Ad_h for some $h \in G_{ss}$. Set $\eta = \mathrm{Inn}_h \mu \mathrm{Inn}_h^{-1}$. Then η is a Cartan involution of G. Also, on the Lie algebra level, $\mu\omega\mu = \omega^{-1}$, so $\mu\omega^2\mu = \omega^{-2}$ and $\mu\varphi\mu = \varphi^{-1}$. Of course, $\omega\omega^2\omega^{-1} = \omega^2$, so $\omega\varphi\omega^{-1} = \varphi$ and $\omega\varphi^2\omega^{-1} = \varphi^2$. Now we have

$$\eta\sigma = \varphi\mu\varphi^{-1}\sigma = \varphi^2\mu\sigma = \varphi^2\omega^{-1} = \omega^{-1}\varphi^2,$$
$$\sigma\eta = \sigma\varphi\mu\varphi^{-1} = \sigma\mu\varphi^{-2} = \omega\varphi^{-2},$$

so $\varphi^4 = \omega^2$ implies that $\eta\sigma = \sigma\eta$ on $\mathfrak{g}$, and also on G.

For the next step, define $\tilde{\omega} = \tilde{\sigma}\tilde{\mu}$. Then $\tilde{\omega}$ is a G-equivariant complex automorphism of V. Further, $\tilde{\omega}$ is Hermitian with respect to the positive-definite Hermitian form $B_{\tilde{\mu}}(x, y) = -\langle x, \tilde{\mu} y\rangle$ on V. It follows that $\tilde{\omega}^2$ is Hermitian and positive-definite, so as above there is a unique Hermitian and positive-definite automorphism $\tilde{\varphi}$ of V such that $\tilde{\varphi}^4 = \tilde{\omega}^2$. Setting $\tilde{\eta} = \tilde{\varphi}\tilde{\mu}\tilde{\varphi}^{-1}$, we have that $\tilde{\eta}\tilde{\sigma} = \tilde{\sigma}\tilde{\eta}$ by a computation similar to that in the previous paragraph. Moreover, $\tilde{\eta}(c) = c$, because $\tilde{\sigma}(c) = c$ and $\tilde{\mu}(c) = c$. We also have $(x, y \in V)$

$$\begin{aligned}
\langle \tilde{\eta}x, \tilde{\eta}y\rangle &= \big\langle \tilde{\varphi}\tilde{\mu}\tilde{\varphi}^{-1}x, \tilde{\varphi}\tilde{\mu}\tilde{\varphi}^{-1}y\big\rangle \\
&= \big\langle \tilde{\mu}\tilde{\varphi}^{-1}x, \tilde{\mu}\tilde{\varphi}^{-1}y\big\rangle \\
&= \overline{\big\langle \tilde{\varphi}^{-1}x, \tilde{\varphi}^{-1}y\big\rangle} \\
&= \overline{\langle x, y\rangle}
\end{aligned}$$

and, if $0 \neq x \in V^{\tilde{\eta}}$,

$$\begin{aligned}\langle x, x\rangle &= \langle \tilde\varphi^{-1}x, \tilde\varphi^{-1}x\rangle \\ &< 0 \quad \left(\tilde\varphi^{-1}x \in V^{\tilde\eta}\right),\end{aligned}$$

where we have used that $(x, y \in V)$

$$\begin{aligned}\langle \tilde\varphi x, \tilde\varphi y\rangle &= -B_{\tilde\mu}(\tilde\varphi x, \tilde\mu\tilde\varphi y) \\ &= -B_{\tilde\mu}\left(\tilde\varphi x, \tilde\varphi^{-1}\tilde\mu y\right) \quad \left(\tilde\mu\tilde\varphi\tilde\mu = \tilde\varphi^{-1}\right) \\ &= -B_{\tilde\mu}(x, \tilde\mu y) \quad (\tilde\varphi \text{ is Hermitian}) \\ &= \langle x, y\rangle \\ &= \langle \tilde\varphi^{-1}x, \tilde\varphi^{-1}y\rangle.\end{aligned}$$

In order to see that $(\eta, \tilde\eta)$ is a Cartan pair, it only remains to check that $\tilde\eta(g \cdot v) = \eta(g) \cdot \tilde\eta(v)$ for $g \in G$, $v \in V$. It suffices to prove that $\tilde\varphi = \tau(h)$. Denote the induced representation $\mathfrak{g} \to \mathfrak{gl}(V)$ by $d\tau$. Since Ad_h is Hermitian, positive-definite with respect to B_μ, the element h can be taken of the form $\exp Y$, where $Y \in i\mathfrak{g}_{ss}^{\mu}$. Then $\tau(h) = e^{d\tau(Y)}$. This implies that $\tau(h)$ is Hermitian, positive-definite with respect to $B_{\tilde\mu}$. Since $(X \in \mathfrak{g})$

$$\tilde\sigma d\tau(X)\tilde\sigma^{-1} = d\tau(\sigma X) \quad \text{and} \quad \tilde\mu d\tau(X)\tilde\mu^{-1} = d\tau(\mu X),$$

we also have that

$$\tilde\omega d\tau(X)\tilde\omega^{-1} = d\tau(\omega X). \tag{11}$$

Since the irreducible summands of V must be pairwise inequivalent by polarity, each one of them is $\tilde\omega$-invariant. Let V_0 be an irreducible summand of V and suppose that the action of $\mathfrak{z}$ on V_0 is given by a linear functional $\Lambda : \mathfrak{z} \to \mathbf{C}$. Eq. (11) implies that $\Lambda(X) = \Lambda(\omega X)$ for $X \in \mathfrak{z}$. Now, if $X \in \mathfrak{z}$ and $v \in V_0$, we have

$$\tau(h)^4 d\tau(X)\tau(h)^{-4}v = \Lambda(X)v = \Lambda\left(\omega^2(X)\right)v = d\tau\left(\omega^2(X)\right)v, \tag{12}$$

and if $X \in \mathfrak{g}_{ss}$,

$$\begin{aligned}\tau(h)^4 d\tau(X)\tau(h)^{-4} &= d\tau\left(\mathrm{Ad}_h^4 X\right) \\ &= d\tau\left(\varphi^4(X)\right) \\ &= d\tau\left(\omega^2(X)\right).\end{aligned} \tag{13}$$

Eqs. (11), (12) and (13) imply that $\tilde\omega^2$ and $\tau(h)^4$ are two intertwining maps between the representations $d\tau$ and $d\tau \circ \omega^2$. It follows that they are multiples of each other on each irreducible summand. Since both maps are positive-definite, $\tau(h)^4 = \lambda\tilde\omega^2$ for $\lambda \in \mathbf{R}$, $\lambda > 0$. Since both are isometries with respect to $\langle\cdot,\cdot\rangle$, one has $\lambda = 1$. Now (a) is proved. For proving (b), construct $(\eta, \tilde\eta)$ as in (a) and note that it is conjugate to $(\theta, \tilde\theta)$ by an element $g' \in (G^\sigma)^\circ$ by Corollary 4. Now $c' = g' \cdot c$ is a $\tilde\theta$-stable Cartan subspace. □

In case a Cartan pair $(\theta, \tilde{\theta})$ commuting with $(\sigma, \tilde{\sigma})$ is fixed, a $\tilde{\theta}$-stable Cartan subspace of ρ will sometimes be called *standard*.

Corollary 18. *If $(\theta, \tilde{\theta})$ is a Cartan pair commuting with $(\sigma, \tilde{\sigma})$, then every closed $(G^\sigma)^\circ$-orbit in $V^{\tilde{\sigma}}$ intersects a standard Cartan subspace of $V^{\tilde{\sigma}}$.*

Proof. Suppose that $(G^\sigma)^\circ x$ is a closed orbit in $V^{\tilde{\sigma}}$. By Proposition 14, there exists a $\tilde{\sigma}$-stable Cartan subspace $c \subset V$ such that $x \in c^{\tilde{\sigma}}$. By Theorem 17, there exists $g \in (G^\sigma)^\circ$ such that $g \cdot c$ is a $\tilde{\sigma}$- and $\tilde{\theta}$-stable Cartan subspace. Of course, $(G^\sigma)^\circ x$ meets $g \cdot c$. □

5.2. Roots and co-roots

In the rest of the paper, we assume that ρ is orthogonal with respect to $\langle \cdot, \cdot \rangle$ and a Cartan pair $(\theta, \tilde{\theta})$ commuting with $(\sigma, \tilde{\sigma})$ has been fixed according to Proposition 2. We also recall the Hermitian form $(\cdot, \cdot)$ that was introduced in that proposition and satisfies Eq. (1).

For a given Cartan subspace $c \subset V$, the set of singular elements $c_{\text{sing}} \subset c$ is by definition the complement of the set of regular elements in c. If the rank of τ is not zero, it is known that c_{sing} is a union of finitely many complex hyperplanes

$$c_{\text{sing}} = \bigcup_{\alpha \in \mathscr{A}} c_\alpha,$$

where $\mathscr{A}$ is a finite index set [DK85, Lem. 2.11]. Fix a $\tilde{\sigma}$- and $\tilde{\theta}$-stable Cartan subspace $c \subset V$, set $\mathfrak{g}_\alpha$ to be the centralizer of c_α in $\mathfrak{g}$ and G_α to be the corresponding connected subgroup of G.

Lemma 19. *We have that $\langle \mathfrak{g}_\alpha \cdot c, \mathfrak{g}_\beta \cdot c \rangle = 0$ for $\alpha \neq \beta$.*

Proof. It follows from Lemma 10 that $c \subset (\mathfrak{g} \cdot v)^\perp$ for $v \in c$. Since $(\mathfrak{g} \cdot v)^\perp$ is $\mathfrak{g}_v$-invariant, this implies $\langle \mathfrak{g}_v \cdot c, \mathfrak{g} \cdot v \rangle = 0$. In particular, if $v \in c_\alpha \setminus \bigcup_{\beta \neq \alpha} c_\beta$, then $\mathfrak{g}_v = \mathfrak{g}_\alpha$ [DK85, p. 516], so $\langle \mathfrak{g}_\alpha \cdot c, \mathfrak{g} \cdot v \rangle = 0$ implying that $\langle \mathfrak{g}_\alpha \cdot c, \mathfrak{g} \cdot c_\alpha \rangle = 0$. Since $\mathfrak{g}_\beta \cdot c \subset \mathfrak{g} \cdot c_\alpha$ for $\alpha \neq \beta$ [DK85, p. 517], the desired result follows. □

Lemma 20. *Each c_α meets $V^{\pm\tilde{\theta}}$ in a real hyperplane.*

Proof. It is equivalent to prove that each c_α is $\tilde{\theta}$-stable. Of course, $(\cdot, \cdot)$ is non-degenerate on $c_\alpha \times c_\alpha$ as $(\cdot, \cdot)$ is positive-definite. Choose $v_\alpha \in c$ to be $(\cdot, \cdot)$-orthogonal to c_α. We claim that the decomposition $c = c_\alpha \oplus \mathbf{C} v_\alpha$ is $\langle \cdot, \cdot \rangle$-orthogonal. Since $\langle x, \tilde{\theta} y \rangle = -(x, y)$ for $x, y \in V$, this will prove the desired result. In order to prove the claim, note that $c \oplus \mathfrak{g}_\alpha \cdot c$ is a G_α-invariant subspace [DK85, Thm. 2.12(ii)] and $\langle \cdot, \cdot \rangle$ is non-degenerate on $c \oplus \mathfrak{g}_\alpha \cdot c$ by Lemmas 10 and 19. Since G_α acts trivially on c_α and $\mathfrak{g}_\alpha \cdot c = \mathfrak{g}_\alpha \cdot v_\alpha$, it follows that $c_\alpha \oplus \mathbf{C} v_\alpha \oplus \mathfrak{g}_\alpha \cdot v_\alpha$ is $\langle \cdot, \cdot \rangle$-orthogonal. □

Since $\langle \cdot, \cdot \rangle$ is positive-definite on $V^{-\tilde{\theta}} \times V^{-\tilde{\theta}}$, the vector v_α in the proof of Lemma 20 can be chosen to satisfy

$$v_\alpha \in V^{-\tilde{\theta}}, \quad \langle v_\alpha, c_\alpha^{-\tilde{\theta}} \rangle = 0 \quad \text{and} \quad \langle v_\alpha, v_\alpha \rangle = 1,$$

and then it is uniquely defined up to a sign. We select a connected component of $c^{-\tilde{\theta}} - \bigcup_{\alpha\in\mathscr{A}} c_\alpha^{-\tilde{\theta}}$ once and for all, and then v_α is uniquely defined (but the sign of v_α will not actually matter for our purposes). The vector v_α is called a (*unnormalized*) *co-root*. The associated *root* is the linear functional $\alpha : c \to \mathbf{C}$ obtained by setting

$$\alpha(v) = \langle v, v_\alpha\rangle \in \mathbf{R}$$

for $v \in c^{-\tilde{\theta}}$ and then considering its complex-linear extension to c. A root is called: *real* (resp. *imaginary*) if α is real-valued (resp. purely imaginary-valued) on $c^{\tilde{\sigma}}$, and it is called *complex* otherwise. It follows from the $\langle\cdot,\cdot\rangle$-orthogonality of the decomposition $c^{-\tilde{\theta}} = c^{\tilde{\sigma}} \cap c^{-\tilde{\theta}} \oplus c^{-\tilde{\sigma}} \cap c^{-\tilde{\theta}}$ that α is real (resp. imaginary) if and only if it vanishes on $c^{-\tilde{\sigma}} \cap c^{-\tilde{\theta}}$ (resp. $c^{\tilde{\sigma}} \cap c^{-\tilde{\theta}}$), in which case v_α belongs to $c^{\tilde{\sigma}} \cap c^{-\tilde{\theta}}$ (resp. $c^{-\tilde{\sigma}} \cap c^{-\tilde{\theta}}$). It follows that α in noncomplex if and only if c_α is $\tilde{\sigma}$-invariant if and only if it is $\tilde{\omega}$-invariant, where $\tilde{\omega} = \tilde{\sigma}\tilde{\theta} = \tilde{\theta}\tilde{\sigma}$. Recall that $\tilde{\theta}$ gets replaced by its opposite by changing the sign of $\langle\cdot,\cdot\rangle$, so the choice of some signs above does not have intrinsic meaning, as compared to the case of an s-representation in which the sign of $\langle\cdot,\cdot\rangle$ is fixed by the Killing form of $\hat{\mathfrak{g}}_{\mathbf{R}}$ (see (10)).

Let $\mathfrak{m}$ be the centralizer of c in $\mathfrak{g}$. Then $\mathfrak{m}$ is σ-, θ-stable. Since $\mathfrak{m}$ is a reductive subalgebra of $\mathfrak{g}_\alpha$, there exists a θ- and $\mathrm{ad}_{\mathfrak{m}}$-stable splitting

$$\mathfrak{g}_\alpha = \mathfrak{m} \oplus \tilde{\mathfrak{g}}_\alpha,$$

where $\tilde{\mathfrak{g}}_\alpha$ is a subspace, which is called a *root space*. Now assume α is noncomplex. Then $\tilde{\mathfrak{g}}_\alpha$ can be taken ω-stable, so that $\tilde{\mathfrak{g}}_\alpha = \tilde{\mathfrak{g}}_\alpha^\omega \oplus \tilde{\mathfrak{g}}_\alpha^{-\omega}$. An imaginary root $\alpha \in \mathscr{A}$ is called *noncompact imaginary* if $\tilde{\mathfrak{g}}_\alpha^{-\omega} \neq 0$ and *compact imaginary* otherwise. A real root $\alpha \in \mathscr{A}$ is called *compact real* if $\tilde{\mathfrak{g}}_\alpha^{\omega} \neq 0$ and *noncompact real* otherwise. Finally, define

$$\tilde{\sigma}\alpha(v) = \overline{\alpha(\tilde{\sigma}v)},$$

where $v \in c$. Since $\tilde{\sigma}$ takes singular orbits to singular orbits and maps hyperplanes of c to hyperplanes of c, this defines an action on $\mathscr{A} \cup (-\mathscr{A})$. Also, $\tilde{\sigma}\alpha = \alpha$ (resp. $\tilde{\sigma}\alpha = -\alpha$) if and only if α is real (resp. imaginary). We can choose the root spaces so that $\sigma\tilde{\mathfrak{g}}_\alpha = \tilde{\mathfrak{g}}_{|\tilde{\sigma}\alpha|}$ for all $\alpha \in \mathscr{A}$, where $|\cdot| : \mathscr{A} \cup (-\mathscr{A}) \to \mathscr{A}$ has its obvious meaning.

5.3. Cayley transforms

By Corollary 18, every closed $G_{\mathbf{R}}$-orbit in $V_{\mathbf{R}}$ meets some standard Cartan subspace of $V_{\mathbf{R}}$. We want to study standard Cartan subspaces of $V_{\mathbf{R}}$, so consider a $\tilde{\sigma}$- and $\tilde{\theta}$-stable Cartan subspace $c \subset V$. Note that

$$c^{\tilde{\sigma}} = c^{\tilde{\sigma}} \cap c^{\tilde{\theta}} \oplus c^{\tilde{\sigma}} \cap c^{-\tilde{\theta}},$$

and $\dim_{\mathbf{R}} c^{\tilde{\sigma}} \cap c^{\tilde{\theta}}$ (resp. $\dim_{\mathbf{R}} c^{\tilde{\sigma}} \cap c^{-\tilde{\theta}}$) is an invariant of the $G_{\mathbf{R}}$-conjugacy class of $c^{\tilde{\sigma}}$, called the *compact dimension* (resp. *noncompact dimension*) of $c^{\tilde{\sigma}}$. We call a standard Cartan subspace $c^{\tilde{\sigma}}$ *maximally compact* (resp. *maximally noncompact*) if its compact dimension (resp. *noncompact dimension*) is as large as possible. Note that the compact and noncompact dimensions of $c^{\tilde{\sigma}}$ are interchanged if we replace $\langle\cdot,\cdot\rangle$ and $\tilde{\theta}$ by their opposites. A maximally compact or maximally noncompact standard Cartan subspace will also be called *extremal*. Cayley transforms are used to pass from one $G_{\mathbf{R}}$-conjugacy class of Cartan subspaces to another one, namely, to increase or decrease its compact dimension by one (and correspondingly decrease or increase its noncompact dimension by one). In general, an element $g \in G$ maps a $\tilde{\sigma}$- and $\tilde{\theta}$-stable Cartan subspace c of V to another $\tilde{\sigma}$- and $\tilde{\theta}$-stable Cartan subspace if and only if $\sigma(g)g^{-1}$ and $\theta(g)g^{-1}$ belong to the normalizer $N_G(c)$ of c in G, as is easily seen. Recall that the *Weyl group* of c is the finite group [DK85, p. 513]

$$W(c) = N_G(c)/Z_G(c),$$

where $Z_G(c)$ denotes the centralizer of c in G. We will construct a special kind of Cayley transform. We first consider the case of a rank one polar orthogonal irreducible representation $\tau : G \to O(V, \langle\cdot,\cdot\rangle)$. Fix a standard Cartan subspace c which is extremal, say maximally compact. Here $c^{\tilde{\sigma}} = c^{\tilde{\sigma}} \cap c^{\tilde{\theta}}$ and $\mathscr{A} = \{\alpha\}$. Assume that α is an imaginary root. We will show how one can pass from $c^{\tilde{\sigma}}$ to a Cartan subspace $\hat{c}$ in another $G_{\mathbf{R}}$-conjugacy class which in this case, by dimensional reasons, must be maximally noncompact, namely, $\hat{c}^{\tilde{\sigma}} = \hat{c}^{\tilde{\sigma}} \cap \hat{c}^{-\tilde{\theta}}$. Since the rank is one, $\tau_u : U \to O(W, \langle\cdot,\cdot\rangle)$ is a co-homogeneity one action of a compact Lie group. Let $v = iv_\alpha \in c^{\tilde{\sigma}} \cap c^{\tilde{\theta}}$. Then $\langle v, v\rangle = -1$ and $U(v)$ is a round sphere $S^{n-1} \approx U/U_v$ in W. Introduce the following notation: $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the decomposition into ± 1-eigenspaces of ω, $\mathfrak{k}_{\mathbf{R}} = \mathfrak{k}^\sigma = \mathfrak{k}^\theta$, $\mathfrak{p}_{\mathbf{R}} = \mathfrak{p}^\sigma$, and $K_{\mathbf{R}} = U^\sigma = U \cap G_{\mathbf{R}}$; note that $K_{\mathbf{R}}$ is a maximal compact subgroup of $G_{\mathbf{R}}$ and hence it is connected since $G_{\mathbf{R}}$ is so.

Claim 21. *We have that α is compact imaginary if and only if $K_{\mathbf{R}} \subset U$ is transitive on S^{n-1}.*

In fact, here we have $\mathfrak{g}_\alpha = \mathfrak{m} \oplus \tilde{\mathfrak{g}}_\alpha$ where $\mathfrak{g}_\alpha = \mathfrak{g}$, $\mathfrak{m} = \mathfrak{g}_v$, and $\mathfrak{g} = \mathfrak{g}_v \oplus \tilde{\mathfrak{g}}_\alpha$ is θ-stable. Taking θ-fixed points, we get $\mathfrak{u} = \mathfrak{u}_v \oplus \tilde{\mathfrak{g}}_\alpha^\theta$. Now $K_{\mathbf{R}}$ is transitive on S^{n-1} if and only if $\mathfrak{u}_v + \mathfrak{k}_{\mathbf{R}} = \mathfrak{u}$ if and only if $\mathfrak{k}_{\mathbf{R}} \supset \tilde{\mathfrak{g}}_\alpha^\theta$ if and only if $\mathfrak{k} \supset \tilde{\mathfrak{g}}_\alpha$ if and only if $\tilde{\mathfrak{g}}_\alpha^{-\omega} = \{0\}$.

Claim 22. *If $K_{\mathbf{R}}$ is not transitive on S^{n-1}, then we can take $g \in G$ such that $\theta(g) = g$, $\sigma(g) = g^{-1}$ and $g^2 = -\operatorname{id} \in W(c)$.*

Indeed, the assumption is equivalent to $\tilde{\mathfrak{g}}_\alpha^\theta \cap \tilde{\mathfrak{g}}_\alpha^{-\sigma} \neq \{0\}$; take a nonzero X therein. We can choose X so that $\gamma(t) = \exp tX \cdot v$ is a unit speed geodesic of S^{n-1} connecting $\gamma(0) = v$ to $\gamma(\pi) = -v$. Set $g = \exp \frac{\pi}{2} X \in U$. Clearly, $\theta(g) = g$. Also, $\sigma(g) = g^{-1}$, and $g^2 \cdot v = \exp \pi X \cdot v = -v$, so $g^2 = -\operatorname{id}$ on $\mathbf{C}v = c$.

Claim 23. *If g is as in the previous claim and $\hat{c} = g \cdot c$, then $\hat{c}^{\tilde{\sigma}}$ is a maximally noncompact Cartan subspace of $V^{\tilde{\sigma}}$.*

In fact, $\theta(g)g^{-1} = \operatorname{id}$ and $\sigma(g)g^{-1} = g^{-2} = -\operatorname{id}$ both belong to $W(c)$, so $\hat{c}$ is $\tilde{\sigma}$- and $\tilde{\theta}$-stable. Also,

$$\tilde{\sigma}(gv) = \sigma(g)\tilde{\sigma}(v) = g^{-1}v = -g^{-1}g^2 v = -gv,$$

so

$$\hat{c}^{\tilde{\sigma}} = \mathbf{R}(igv) \quad \text{and} \quad \tilde{\theta}(igv) = -i\theta(g)\tilde{\theta}(v) = -igv.$$

We have shown that in the rank one case, associated to a noncompact imaginary root α, a Cayley transformation $\mathbf{c}_\alpha = \tau(g)$ can be constructed so that it maps a given $\tilde{\sigma}$- and $\tilde{\theta}$-stable Cartan subspace c to a $\tilde{\sigma}$- and $\tilde{\theta}$-stable Cartan subspace $\hat{c} = \mathbf{c}_\alpha(c)$ such that the noncompact dimension of $\hat{c}^{\tilde{\sigma}}$ is one higher than that of $c^{\tilde{\sigma}}$. In the sequel, we want to generalize this construction to an arbitrary polar orthogonal representation $\tau : G \to O(V, \langle\cdot,\cdot\rangle)$.

Indeed, suppose now that the rank of τ is arbitrary, let c be an arbitrary $\tilde{\sigma}$- and $\tilde{\theta}$-stable Cartan subspace and assume there exists a noncompact imaginary root $\alpha \in \mathscr{A}$ which we suppose fixed. Write $c = c_\alpha \oplus \mathbf{C}v_\alpha$ where $v_\alpha \in i(c^{\tilde{\sigma}} \cap c^{\tilde{\theta}}) = c^{-\tilde{\sigma}} \cap c^{-\tilde{\theta}}$ is the co-root. Note that

$$c^{\tilde{\sigma}} = c_\alpha^{\tilde{\sigma}} \oplus \mathbf{R}(iv_\alpha),$$

and $iv_\alpha \in c^{\tilde{\theta}}$. Now $(\mathfrak{g}_\alpha, c \oplus \mathfrak{g}_\alpha \cdot c)$ is a rank one polar action [DK85, Thm. 2.12]. Since $V = c \oplus \bigoplus_{\alpha \in \mathscr{A}} \mathfrak{g}_\alpha \cdot c$ is a $\langle\cdot,\cdot\rangle$-orthogonal direct sum, $(\mathfrak{g}_\alpha, c \oplus \mathfrak{g}_\alpha \cdot c)$ is orthogonal with respect to the restriction of $\langle\cdot,\cdot\rangle$; we restrict it to $(\mathfrak{g}_\alpha, \mathbf{C}v_\alpha \oplus \tilde{\mathfrak{g}}_\alpha \cdot v_\alpha)$ to get an irreducible polar orthogonal action of rank one. Since $X \in \mathfrak{g}_\alpha \mapsto X \cdot v_\alpha$ is injective on $\tilde{\mathfrak{g}}_\alpha$, the kernel of this representation is contained in $\mathfrak{m}$. Let $Z \in \mathfrak{m}$. Then $Z \cdot v_\alpha = 0$. If $Z \cdot \tilde{\mathfrak{g}}_\alpha \cdot v_\alpha = 0$, then $[Z, \tilde{\mathfrak{g}}_\alpha] \cdot v_\alpha = 0$. Since $[Z, \tilde{\mathfrak{g}}_\alpha] \subset \tilde{\mathfrak{g}}_\alpha$, we get that $[Z, \tilde{\mathfrak{g}}_\alpha] = 0$, so $Z \in Z_{\mathfrak{m}}(\tilde{\mathfrak{g}}_\alpha)$. Now $(\mathfrak{g}'_\alpha, V_\alpha)$ is an effective irreducible polar orthogonal action of rank one, where we have set

$$\mathfrak{g}'_\alpha = \mathfrak{g}_\alpha / Z_{\mathfrak{m}}(\tilde{\mathfrak{g}}_\alpha) \quad \text{and} \quad V_\alpha = \mathbf{C}v_\alpha \oplus \tilde{\mathfrak{g}}_\alpha \cdot v_\alpha.$$

Note that α can also be considered as a root of $(\mathfrak{g}'_\alpha, V_\alpha)$, and then it is a noncompact imaginary root, so by the previous discussion we can find $g \in G_\alpha$ as above and perform a Cayley transform $\mathbf{c}_\alpha = \tau(g)$ as follows:

$$\hat{c} = \mathbf{c}_\alpha(c) = c_\alpha \oplus \mathbf{C}(gv_\alpha).$$

Note that

$$\hat{c}^{\tilde{\sigma}} = c_\alpha^{\tilde{\sigma}} \oplus \mathbf{R}(gv_\alpha),$$

and $gv_\alpha \in c^{-\tilde{\theta}}$, so the noncompact dimension of $\hat{c}^{\tilde{\sigma}}$ is one higher than that of $c^{\tilde{\sigma}}$. In a completely analogous way, one can define a Cayley transform that increases the compact dimension of $c^{\tilde{\sigma}}$ by one by using a compact real root.

5.4. Uniqueness of extremal Cartan subspaces

The Cayley transform allows us to derive some important properties of extremal Cartan subspaces.

Theorem 24. *We have that* $(K_{\mathbf{R}}, V_{\mathbf{R}} \cap iW)$ (*resp.* $(K_{\mathbf{R}}, V_{\mathbf{R}} \cap W)$) *is a polar representation. The sections are given by* $c^{\tilde{\sigma}} \cap c^{-\tilde{\theta}}$ (*resp.* $c^{\tilde{\sigma}} \cap c^{\tilde{\theta}}$), *where* $c^{\tilde{\sigma}}$ *is a maximally noncompact* (*resp. compact*) *Cartan subspace of* $V_{\mathbf{R}} = V^{\tilde{\sigma}}$.

Proof. It suffices to treat the case of $(K_{\mathbf{R}}, V_{\mathbf{R}} \cap iW)$. Let $c^{\tilde{\sigma}}$ be a maximally noncompact Cartan subspace. Then there are no noncompact imaginary roots, for otherwise a Cayley transform could be performed increasing the noncompact dimension of $c^{\tilde{\sigma}}$. We claim that there exists $v_2 \in c^{\tilde{\sigma}} \cap c^{-\tilde{\theta}}$ such that

$$\mathfrak{k}_{\mathbf{R}}(v_2) \oplus c^{\tilde{\sigma}} \cap c^{-\tilde{\theta}} = V^{\tilde{\sigma}} \cap V^{-\tilde{\theta}} = V_{\mathbf{R}} \cap iW.$$

In order to prove this claim, we first remark that [DK85, Thm. 2.12]

$$\mathfrak{g}_v = \mathfrak{m} + \sum_{\alpha(v)=0} \tilde{\mathfrak{g}}_\alpha$$

for $v \in c$,

$$\mathfrak{g}_v = \begin{cases} \mathfrak{m} + \sum_{\alpha \text{ imag}} \tilde{\mathfrak{g}}_\alpha & \text{for generic } v \in c^{-\tilde{\omega}}, \\ \mathfrak{m} + \sum_{\alpha \text{ real}} \tilde{\mathfrak{g}}_\alpha & \text{for generic } v \in c^{\tilde{\omega}}, \end{cases}$$

and

$$\mathfrak{g}_{v_1} = \underbrace{\left(\mathfrak{m}^{\omega} + \sum_{\alpha \text{ real}} \tilde{\mathfrak{g}}_\alpha^{\omega}\right)}_{\subset \mathfrak{k}} \oplus \underbrace{\left(\mathfrak{m}^{-\omega} + \sum_{\alpha \text{ real}} \tilde{\mathfrak{g}}_\alpha^{-\omega}\right)}_{\subset \mathfrak{p}} \quad \text{for generic } v_1 \in c^{\tilde{\sigma}} \cap c^{\tilde{\theta}},$$

$$\mathfrak{g}_{v_2} = \underbrace{\left(\mathfrak{m}^{\omega} + \sum_{\alpha \text{ imag}} \tilde{\mathfrak{g}}_\alpha^{\omega}\right)}_{\subset \mathfrak{k}} \oplus \underbrace{\mathfrak{m}^{-\omega}}_{\subset \mathfrak{p}} \quad \text{for generic } v_2 \in c^{\tilde{\sigma}} \cap c^{-\tilde{\theta}},$$

where in the last line we have used the nonexistence of noncompact imaginary roots. Select generic $v_1 \in c^{\tilde{\sigma}} \cap c^{\tilde{\theta}}$, $v_2 \in c^{\tilde{\sigma}} \cap c^{-\tilde{\theta}}$ and set $v = v_1 + v_2 \in c^{\tilde{\sigma}}$. For each $\alpha \in \mathscr{A}$,

$$\alpha(v) = \underbrace{\alpha(v_1)}_{\in i\mathbf{R}} + \underbrace{\alpha(v_2)}_{\in \mathbf{R}},$$

where at least one of the two summands on the right-hand side in not zero by the choice of v_1, v_2. This shows that v is regular for (G, V). By polarity, $\mathfrak{g} \cdot v \oplus c = V$. Taking real parts in $V_{\mathbf{R}}$ yields

$$\mathfrak{g}_{\mathbf{R}}(v) \oplus c^{\tilde{\sigma}} = V_{\mathbf{R}},$$

which is the same as

$$\big(\mathfrak{k}_{\mathbf{R}}(v_1) + \mathfrak{p}_{\mathbf{R}}(v_2)\big) \oplus \big(\mathfrak{k}_{\mathbf{R}}(v_2) + \mathfrak{p}_{\mathbf{R}}(v_1)\big) \oplus c^{\tilde{\sigma}} \cap c^{\tilde{\theta}} \oplus c^{\tilde{\sigma}} \cap c^{-\tilde{\theta}} = V_{\mathbf{R}} \cap W \oplus V_{\mathbf{R}} \cap iW.$$

In particular,

$$\big(\mathfrak{k}_{\mathbf{R}}(v_2) + \mathfrak{p}_{\mathbf{R}}(v_1)\big) \oplus c^{\tilde{\sigma}} \cap c^{-\tilde{\theta}} = V_{\mathbf{R}} \cap iW.$$

The claim will follow if we show that $\mathfrak{k}_{\mathbf{R}}(v_2) \supset \mathfrak{p}_{\mathbf{R}}(v_1)$. This is to be a consequence of $\mathfrak{k} \cdot v_2 \supset \mathfrak{p} \cdot v_1$, as $\mathfrak{k} \cdot v_2$ and $\mathfrak{p} \cdot v_1$ are $\tilde{\sigma}$-stable and $\mathfrak{k}_{\mathbf{R}}(v_2) = (\mathfrak{k} \cdot v_2)^{\tilde{\sigma}}$, $\mathfrak{p}_{\mathbf{R}}(v_1) = (\mathfrak{p} \cdot v_1)^{\tilde{\sigma}}$.

Now $\mathfrak{p} \cdot v_1$ is spanned by

$$\underbrace{\tilde{\mathfrak{g}}_\alpha^{-\omega}}_{=0} \cdot v_1 \quad \text{for } \alpha \text{ imaginary, and}$$

$$\begin{aligned}(X_\alpha - \omega X_\alpha) \cdot v_1 &= X_\alpha \cdot v_1 - \tilde{\omega}(X_\alpha \cdot v_1) \\ &= \underbrace{\alpha(v_1)}_{\neq 0}(1-\tilde{\omega})(X_\alpha \cdot v_\alpha) \quad \text{for } \alpha \text{ complex and } X_\alpha \in \tilde{\mathfrak{g}}_\alpha.\end{aligned}$$

On the other hand, $\mathfrak{k}(v_2)$ is spanned by

$$\tilde{\mathfrak{g}}_\alpha^{\omega} \cdot v_2 \quad \text{for } \alpha \text{ real, and}$$

$$\begin{aligned}(X_\alpha + \omega X_\alpha) \cdot v_2 &= X_\alpha \cdot v_2 - \tilde{\omega}(X_\alpha \cdot v_2) \\ &= \underbrace{\alpha(v_2)}_{\neq 0}(1-\tilde{\omega})(X_\alpha \cdot v_\alpha) \quad \text{for } \alpha \text{ complex and } X_\alpha \in \tilde{\mathfrak{g}}_\alpha.\end{aligned}$$

This proves that $\mathfrak{p} \cdot v_1 \subset \mathfrak{k} \cdot v_2$, and hence that $\mathfrak{k}_{\mathbf{R}}(v_2) \oplus c^{\tilde{\sigma}} \cap c^{-\tilde{\theta}} = V_{\mathbf{R}} \cap iW$. Since $\langle \mathfrak{g} \cdot c, c\rangle = 0$, we get that $c^{\tilde{\sigma}} \cap c^{-\tilde{\theta}}$ is the $\langle \cdot,\cdot \rangle$-orthogonal complement of $K_{\mathbf{R}}(v_2)$ in $V_{\mathbf{R}} \cap iW$. Since $K_{\mathbf{R}}$ is compact and $\langle \cdot,\cdot \rangle$ is positive-definite on $V_{\mathbf{R}} \cap iW$, it easily follows that $c^{\tilde{\sigma}} \cap c^{-\tilde{\theta}}$ meets all the $K_{\mathbf{R}}$-orbits in $V_{\mathbf{R}} \cap iW$. Again by $\langle \mathfrak{g} \cdot c, c\rangle = 0$, one has that $c^{\tilde{\sigma}} \cap c^{-\tilde{\theta}}$ meets all the other $K_{\mathbf{R}}$-orbits orthogonally. This finishes the proof. □

Corollary 25. *Two maximally noncompact* (*resp. compact*) *Cartan subspaces* $c_1^{\tilde{\sigma}}$ *and* $c_2^{\tilde{\sigma}}$ *of* $V^{\tilde{\sigma}} = V_{\mathbf{R}}$ *are* $K_{\mathbf{R}}$*-conjugate. As a consequence, there exists a unique* $G_{\mathbf{R}}$*-conjugacy class of maximally noncompact* (*resp. compact*) *Cartan subspaces of* $V_{\mathbf{R}}$.

Proof. Again, it suffices to treat the case of maximally noncompact Cartan subspaces. By Theorem 24, we may assume that

$$c_1^{\tilde{\sigma}} \cap c_1^{-\tilde{\theta}} = c_2^{\tilde{\sigma}} \cap c_2^{-\tilde{\theta}}.$$

Take a generic point v_2 lying therein. Since $v_2 \in V_{\mathbf{R}} \cap iW$, we have that $\mathfrak{u}_{v_2} = (\mathfrak{k}_{\mathbf{R}})_{v_2} + (i\mathfrak{p}_{\mathbf{R}})_{v_2}$, and this is a decomposition into the ± 1-eigenspaces of σ on $\mathfrak{u}_{v_2}$, so

$$(U_{v_2})^\circ = (K_{\mathbf{R}})_{v_2} \exp\bigl[(i\mathfrak{p}_{\mathbf{R}})_{v_2}\bigr].$$

Consider the slice of the polar action $(U, V^{-\tilde{\theta}})$ at v_2; it is also polar with the same sections:

$$c_i^{-\tilde{\theta}} = c_i^{-\tilde{\sigma}} \cap c_i^{-\tilde{\theta}} \oplus c_i^{\tilde{\sigma}} \cap c_i^{-\tilde{\theta}}$$

for $i = 1, 2$. Now $c_1^{-\tilde{\theta}}$ and $c_2^{-\tilde{\theta}}$ must be conjugate by an element of $(U_{v_2})^\circ$. Since $\exp[(i\mathfrak{p}_{\mathbf{R}})_{v_2}]$ centralizes c (for $(i\mathfrak{p}_{\mathbf{R}})_{v_2} = \mathfrak{m}^{-\sigma} \cap \mathfrak{m}^{\theta}$), they must indeed be conjugate by an element of $(K_{\mathbf{R}})_{v_2}$ (which necessarily fixes $c_1^{\tilde{\sigma}} \cap c_1^{-\tilde{\theta}} = c_2^{\tilde{\sigma}} \cap c_2^{-\tilde{\theta}}$ since this is a section of $(K_{\mathbf{R}}, V_{\mathbf{R}} \cap iW)$ and

v_2 is a regular point of that action). Hence, so are $c_1^{\tilde{\sigma}} \cap c_1^{\tilde{\theta}} = i(c_1^{-\tilde{\sigma}} \cap c_1^{-\tilde{\theta}})$ and $c_2^{\tilde{\sigma}} \cap c_2^{\tilde{\theta}} = i(c_2^{-\tilde{\sigma}} \cap c_2^{-\tilde{\theta}})$. $\square$

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