THE ADAPTED HYPER-KÄHLER STRUCTURE ON THE CROWN DOMAIN

LAURA GEATTI AND ANDREA IANNUZZI

Abstract. Let Ξ be the crown domain associated with a non-compact irreducible hermitian symmetric space G/K. We give an explicit description of the unique G-invariant *adapted hyper-Kähler structure* on Ξ, i. e. compatible with the adapted complex structure J_{ad} and with the G-invariant Kähler structure of G/K. We also compute invariant potentials of the involved Kähler metrics and the associated moment maps.

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1. Introduction

A quaternionic complex structure on a 4n-dimensional real manifold consists of three complex structures $I,\ J,\ K$ such that $IJK = -Id$. It is called hyper-Kähler if there exist 2-forms $\omega_I,\ \omega_J,\ \omega_K$ which are Kähler for $I,\ J,\ K$, respectively, and define the same Riemannian metric given by

$$g(\,\cdot\,,\,\cdot\,) = \omega_I(\,\cdot\,, I\,\cdot\,) = \omega_J(\,\cdot\,, J\,\cdot\,) = \omega_K(\,\cdot\,, K\,\cdot\,)\,.$$

A hyper-Kähler manifold is holomorphic symplectic with respect to any of its complex structures, e.g. the complex symplectic form $\omega_J + i\omega_K$ is holomorphic with respect to I.

Let $(G/K,\, g_0)$ be an irreducible Hermitian symmetric space. In [BiGa96a], O. Biquard and B. Gauduchon proved that in the compact case the holomorphic cotangent bundle $T^*G/K^{1,0}$, endowed with its canonical holomorphic symplectic form $\omega^{\mathbb{C}}_{can}$, carries a unique G-invariant hyper-Kähler metric whose restriction to G/K, identified with the zero section, coincides with g_0 (see also [Cal79]). They also showed that in the non-compact case such a hyper-Kähler metric only exists on an appropriate tubular neighbourhood of G/K in $T^*G/K^{1,0}$.

Identify $T^*G/K^{1,0} \cong T^*G/K$ with the tangent bundle TG/K via the metric g_0. For G/K a classical Hermitian symmetric space, A. S. Dancer and R. Szöke ([DaSz97]) have shown that the hyper-Kähler metric constructed in [BiGa96a] is determined by $\omega^{\mathbb{C}}_{can}$ and the pull-back of the so-called adapted complex structure

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J_{ad} (see [LeSz91] and [GuSt91]) via a G-equivariant fiber preserving diffeomorphism of the tangent bundle TG/K. This suggests that, on the maximal domain of existence of J_{ad}, there exists a G-invariant hyper-Kähler structure which includes J_{ad}.

In this paper we consider an arbitrary non-compact Hermitian symmetric space G/K. We regard the maximal domain of existence of the adapted complex structure as a G-invariant domain Ξ in the Lie group complexification $G^{\mathbb{C}}/K^{\mathbb{C}}$ of G/K, where J_{ad} coincides with the complex structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$ (cf. [BHH03]). In the literature Ξ is referred to as the crown domain associated with G/K.

We show that indeed Ξ admits a unique G-invariant *adapted hyper-Kähler structure*, i.e. such that $J = J_{ad}$ and the restriction of (I, ω_I) to G/K coincides with the Kähler structure defined by g_0. The adapted hyper-Kähler structure coincides with the pull-back of the hyper-Kähler structure determined by O. Biquard and P. Gauduchon. However, it satisfies different initial conditions and its uniqueness does not follow from their arguments. Moreover, from the condition $J = J_{ad}$ it is easy to deduce that the forms ω_I and ω_K are locally $G^{\mathbb{C}}$-invariant (Lemma 7.2), a fact which was not evident from the previous investigations.

For all the quantities involved in the adapted hyper-Kähler structure, we provide explicit formulas in Lie theoretical terms. In the case of G/K compact, one can adopt a similar strategy to obtain a unique invariant adapted hyper-Kähler structure on the whole complexification $G^{\mathbb{C}}/K^{\mathbb{C}} \cong TG/K$.

In order to state our main result we need to fix some notation. Let $\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G with respect to K. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$ and denote by Σ the associated restricted root system. The crown domain associated with G/K in $G^{\mathbb{C}}/K^{\mathbb{C}}$ is by definition

$$\Xi = G \exp i\Omega K^{\mathbb{C}}/K^{\mathbb{C}}\,,$$

where $\Omega := \{H \in \mathfrak{a} \;:\; |\alpha(H)| < \frac{\pi}{2},\ \forall \alpha \in \Sigma\}$ is the cell defined by D. N. Akhiezer and S. G. Gindikin in [AkGi90]. The closed subset $\exp i\Omega K^{\mathbb{C}}/K^{\mathbb{C}}$ is a G-slice of Ξ.

Let I_0 be the G-invariant complex structure of G/K. On $\mathfrak{p} \cong T_{eK}G/K$, it coincides with the adjoint action of a central element of $\mathfrak{k}$ (see (5)). Its $\mathbb{C}$-linear extension to $\mathfrak{p}^{\mathbb{C}}$ is also denoted by I_0. The conjugation with respect to $\mathfrak{p}$ of an element Z in $\mathfrak{p}^{\mathbb{C}}$ is indicated by $\overline{Z}$.

The Killing form of $\mathfrak{g}^{\mathbb{C}}$, as well as its restrictions to $\mathfrak{p}^{\mathbb{C}}$ and to $\mathfrak{p}$, is denoted by B. The standard G-invariant Kähler structure (I_0, ω_0) on G/K is uniquely determined by its restriction to $\mathfrak{p}$, namely $\omega_0(\,\cdot\,,\,\cdot\,) = B(I_0\,\cdot\,,\,\cdot\,)$. Finally, for $z \in G^{\mathbb{C}}/K^{\mathbb{C}}$ and $Z \in \mathfrak{g}^{\mathbb{C}}$, let

$$\widetilde{Z}_z := \tfrac{d}{ds}\big|_{s=0} \exp(sZ)\cdot z$$

be the vector field induced by the holomorphic $G^{\mathbb{C}}$-action on $G^{\mathbb{C}}/K^{\mathbb{C}}$. Our main result is as follows.

Theorem. *Let G/K be an irreducible, non-compact Hermitian symmetric space endowed with its standard G-invariant Kähler structure, and let Ξ be the associated crown domain. There exists a unique G-invariant adapted hyper-Kähler structure $(I, J, K, \omega_I, \omega_J, \omega_K)$ on Ξ, i.e. such that $J = J_{ad}$ and the Kähler structure (I, ω_I) coincides with (I_0, ω_0) when restricted to G/K.*

(**a**) *The symplectic J-holomorphic form $\omega_I - i\omega_K$ is the restriction of a $G^{\mathbb{C}}$-invariant form on $G^{\mathbb{C}}/K^{\mathbb{C}}$ and is uniquely determined by*

$$(\omega_I - i\omega_K)(Z, W) = B(I_0 Z,\, W)\,,$$

for $Z, W \in \mathfrak{p}^{\mathbb{C}} \cong T_{eK^{\mathbb{C}}} G^{\mathbb{C}}/K^{\mathbb{C}}$.

(**b**) *For $z = aK^{\mathbb{C}}$ on the G-slice $\exp i\Omega K^{\mathbb{C}}/K^{\mathbb{C}}$ of Ξ, the G-invariant complex structure I is given by*

$$I\widetilde{Z}_z = \widetilde{I_0 Z}_z\,.$$

Let $\{A_1, \cdots, A_r\}$ be the basis of $\mathfrak{a}$ defined in (2) and (3) of Section 2 and $C := B(A_1,\, A_1) = \cdots = B(A_r,\, A_r)$. Write $a = \exp iH$, where $H = \sum_{j=1}^{r} t_j A_j$.

(**c**) *Let ρ_0 be a potential of ω_0 and let $p : \Xi \to G/K$ the G-equivariant projection given by $p(gaK^{\mathbb{C}}) = gK$. A potential of ω_I is given by $\rho_0 \circ p + \rho_I$, where the G-invariant function ρ_I is defined by*

$$\rho_I(gaK^{\mathbb{C}}) := -\tfrac{C}{4} \textstyle\sum_j f_I(2t_j)\,,$$

with f_I a real valued function satisfying $\frac{\sin x}{\cos x} f_I'(x) = \cos x - 1$.

(**d**) *A G-invariant potential of ω_J is given by*

$$\rho_J(gaK^{\mathbb{C}}) := -\tfrac{C}{4} \textstyle\sum_{j=1}^{r} \cos(2t_j)\,.$$

The moment map $\mu_J : \Xi \to \mathfrak{g}^*$ associated with ρ_J is given by

$$\mu_J(gaK^{\mathbb{C}})(X) = B(\mathrm{Ad}_{g^{-1}} X, \Psi(H))\,,$$

where $\Psi(\sum_{j=1}^{r} t_j A_j) = \frac{1}{2} \sum_{j=1}^{r} \sin(2t_j) A_j$.

We sketch the strategy of the proof. If such a G-invariant hyper-Kähler structure exists, then the forms ω_I and ω_K are necessarily restrictions of $G^{\mathbb{C}}$-invariant forms on $G^{\mathbb{C}}/K^{\mathbb{C}}$ (Lemma 7.2). It follows that they coincide with the forms given in (a) (Rem. 7.4). A standard argument also shows that they are closed (Lemma 5.2(iii)).

The forms ω_I, ω_K, the complex structure $J = J_{ad}$, and the almost complex structure I defined in (b), determine a G-invariant quaternionic almost complex structure. Then, by a result of N. J. Hitchin, the integrability of I and $K := IJ$ follows from the closeness of $\omega_J(\,\cdot\,,\,\cdot\,) := \omega_I(J\,\cdot\,, I\,\cdot\,)$ ([Hit87], Lemma 6.8). This property is proved by showing that the G-invariant function ρ_J defined in (d) is a potential of ω_J by means of restricted root theory and moment map techniques (Prop. 6.2). As a result, $(I,\ J,\ K,\ \omega_I,\ \omega_J,\ \omega_K)$ is a G-invariant adapted hyper-Kähler structure, as claimed. A similar strategy is used to obtain the G-invariant potential ρ_I of $\omega_I - p^*\omega_0$ indicated in (c) (Prop. 8.2). Such potential is expressed in terms of a real function f_I satisfying a simple trigonometric differential equation (cf. [BiGa96a], Thm. 1). A proof of uniqueness of the adapted hyper-Kähler structure is outlined in Section 7. In the case of $G = SL_2(\mathbb{R})$, all details are given in Appendix A.

As a further application of the above techniques, we also provide a Lie theoretical formulation of the pull-back to Ξ of the canonical real symplectic form on the cotangent bundle T^*G/K (Appendix B).

The exposition is organized as follows. In Section 2 we collect the basic facts which are needed in the sequel. In Section 3 we introduce the (almost) complex structure I. In Section 4 we express in a Lie theoretical fashion the inverse of the G-equivariant diffeomorphism introduced in [DaSz97]. This leads to a useful expression of I which is exploited in the computation of $2i\partial\bar{\partial}_J\rho_J$. In Section 5 we introduce the forms ω_I, ω_J, ω_K and study their basic properties. In Section 6 we show that the G-invariant function ρ_J is a potential of ω_J and compute the associated moment map. In Section 7 we prove the main theorem by assembling the results obtained in other sections. In Section 8 we show that the G-invariant function ρ_I is a potential of $\omega_I - p^*\omega_0$.

2. Preliminaries

Let $\mathfrak{g}$ be a non-compact semisimple Lie algebra and let $\mathfrak{k}$ be a maximal compact subalgebra of $\mathfrak{g}$. Denote by θ the Cartan involution of $\mathfrak{g}$ with respect to $\mathfrak{k}$, with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$. The dimension of $\mathfrak{a}$ is by definition the *rank* of G/K. The adjoint action of $\mathfrak{a}$ decomposes $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}^\alpha,$$

where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, the joint eigenspace $\mathfrak{g}^\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X, \text{ for every } H \in \mathfrak{a}\}$ is the α-restricted root space and Σ consists of those $\alpha \in \mathfrak{a}^*$ for which $\mathfrak{g}^\alpha \neq \{0\}$. Denote by B the Killing form of $\mathfrak{g}$, as well as its holomorphic extension to $\mathfrak{g}^{\mathbb{C}}$ (which coincides with the Killing form of $\mathfrak{g}^{\mathbb{C}}$).

For $\alpha \in \Sigma$, consider the θ-stable space $\mathfrak{g}[\alpha] := \mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}$, and denote by $\mathfrak{k}[\alpha]$ and $\mathfrak{p}[\alpha]$ the projections of $\mathfrak{g}[\alpha]$ onto $\mathfrak{k}$ and $\mathfrak{p}$, respectively. Then

$$\mathfrak{k} = \mathfrak{m} \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{k}[\alpha] \qquad \text{and} \qquad \mathfrak{p} = \mathfrak{a} \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{p}[\alpha] \tag{1}$$

are B-orthogonal decompositions of $\mathfrak{k}$ and $\mathfrak{p}$, respectively.

Lemma 2.1. *Every element X in $\mathfrak{p}$ decomposes in a unique way as*

$$X_{\mathfrak{a}} + \textstyle\sum_{\alpha\in\Sigma^+} P^\alpha,$$

where $X_{\mathfrak{a}} \in \mathfrak{a}$ *and* $P^\alpha \in \mathfrak{p}[\alpha]$. *The vector* P^α *can be written uniquely as* $P^\alpha = X^\alpha - \theta X^\alpha$, *where* X^α *is the component of* X *in the root space* $\mathfrak{g}^\alpha$. *Moreover,* $[H, P^\alpha] = \alpha(H) K^\alpha$, *where* K^α *is the element in* $\mathfrak{k}[\alpha]$ *uniquely defined by* $K^\alpha = X^\alpha + \theta X^\alpha$.

Proof. The proof of this lemma is an easy exercise. □

The restricted root system of a Lie algebra $\mathfrak{g}$ of Hermitian type is either of type C_r (if G/K is of tube type) or of type BC_r (if G/K is not of tube type), i.e. there exists a basis $\{e_1, \ldots, e_r\}$ of $\mathfrak{a}^*$ for which $\Sigma = \Sigma^+ \cup -\Sigma^+$, with

$$\Sigma^+ = \{2e_j,\ 1 \leqslant j \leqslant r,\ e_k \pm e_l,\ 1 \leqslant k < l \leqslant r\}, \qquad \text{for type } C_r,$$

$$\Sigma^+ = \{e_j,\ 2e_j,\ 1 \leqslant j \leqslant r,\ e_k \pm e_l,\ 1 \leqslant k < l \leqslant r\}, \qquad \text{for type } BC_r\,.$$

With the above choice of a positive system Σ^+, the roots

$$\lambda_1 := 2e_1,\ \ldots\ ,\ \lambda_r := 2e_r$$

form a maximal set of long strongly orthogonal positive restricted roots (i.e. such that $\lambda_k \pm \lambda_l \notin \Sigma$, for $k \neq l$).

For every $j = 1, \ldots, r$, the root space $\mathfrak{g}^{\lambda_j}$ is one-dimensional. Fix $E^j \in \mathfrak{g}^{\lambda_j}$ such that the $\mathfrak{sl}(2)$-triples $\{E^j,\ \theta E^j,\ A_j := [\theta E^j,\ E^j]\}$ are normalized as follows

$$[A_j,\ E^j] = 2E^j, \quad \text{for} \quad j = 1, \ldots, r. \tag{2}$$

The vectors $\{A_1, \ldots, A_r\}$ form a B-orthogonal basis of $\mathfrak{a}$ and

$$[E^k,\ E^l] = [E^k,\ \theta E^l] = 0, \quad [A_k,\ E^l] = \lambda_l(A_k) E^l = 0, \quad \text{for } k \neq l\,. \tag{3}$$

For $j = 1, \ldots, r$, define

$$K^j := E^j + \theta E^j \quad \text{ and } \quad P^j := E^j - \theta E^j. \tag{4}$$

Denote by I_0 the G-invariant complex structure of G/K. On $\mathfrak{p} \cong T_{eK}G/K$, it coincides with the adjoint action of the element $Z_0 \in Z(\mathfrak{k})$ given by

$$Z_0 = S + \tfrac{1}{2} \textstyle\sum_{j=1}^r K^j\,, \tag{5}$$

for some element $S \in \mathfrak{m}$ (see Lemma 2.4 in [GeIa13]). The complex structure I_0 permutes the blocks of the decomposition (1) of $\mathfrak{p}$. Indeed, from the normalizations (2) and (3), one sees that

$$I_0 P^j = [Z_0,\ P^j] = A_j \qquad \text{and} \qquad I_0 A_j = [Z_0,\ A_j] = -P^j\,, \tag{6}$$

for $j = 1, \ldots, r$. In particular $I_0 \mathfrak{a} = \bigoplus_{j=1}^r \mathfrak{p}[\lambda_j]$. Moreover, one can easily check that

$$I_0 \mathfrak{p}[e_k + e_l] = \mathfrak{p}[e_k - e_l]\,, \qquad I_0 \mathfrak{p}[e_j] = \mathfrak{p}[e_j] \text{ (non-tube case)}. \tag{7}$$

For $a = \exp iH$, with $H \in \mathfrak{a}$, define a $\mathbb{C}$-linear operator $F_a : \mathfrak{p}^{\mathbb{C}} \to \mathfrak{p}^{\mathbb{C}}$ by

$$F_a := \pi_{\#} \circ \mathrm{Ad}_{a^{-1}}|_{\mathfrak{p}^{\mathbb{C}}}\,, \tag{8}$$

where $\pi_{\#} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{p}^{\mathbb{C}}$ be the linear projection along $\mathfrak{k}^{\mathbb{C}}$. One easily checks that

$$F_a A = A \qquad \text{and} \qquad F_a P^\alpha = \cos\alpha(H) P^\alpha\,, \tag{9}$$

for all $A \in \mathfrak{a}$ and $P^\alpha \in \mathfrak{p}[\alpha]$. In particular, for every $H \in \Omega$, the operator F_a is an isomorphism and $F_a(\mathfrak{p}) = \mathfrak{p}$.

For $z \in G^{\mathbb{C}}/K^{\mathbb{C}}$ and $Z \in \mathfrak{g}^{\mathbb{C}}$, let

$$\widetilde{Z}_z := \tfrac{d}{ds}\big|_{s=0} \exp(sZ) \cdot z \tag{10}$$

be the vector field induced by the holomorphic $G^{\mathbb{C}}$-action on $G^{\mathbb{C}}/K^{\mathbb{C}}$. For $z = aK^{\mathbb{C}}$ on the slice of Ξ and $Z \in \mathfrak{p}^{\mathbb{C}}$, one has

$$\widetilde{Z}_z = a_* F_a Z, \qquad a_* Z = \widetilde{F_a^{-1} Z}_z. \tag{11}$$

Denote by Ξ' the G-invariant subdomain of Ξ defined by

$$\Xi' := G \exp i\Omega' K^{\mathbb{C}}\,, \tag{12}$$

where $\Omega' := \{H \in \Omega \ : \ \alpha(H) \neq 0,\ \forall \alpha \in \Sigma\}$ is the regular subset of Ω. Note that Ω' is dense in Ω and Ξ' is dense in Ξ.

Later on, in the computation of the potentials of the various Kähler forms, we need the identities contained in the next lemma.

Lemma 2.2. *Fix $z = aK^{\mathbb{C}}$, with $a = \exp iH$ and $H \in \Omega'$. Decompose $Y \in \mathfrak{p}$ as $Y = Y_{\mathfrak{a}} + \sum_\alpha Q^\alpha$, where $Y_{\mathfrak{a}} \in \mathfrak{a}$ and $Q^\alpha = Y^\alpha - \theta Y^\alpha \in \mathfrak{p}[\alpha]$ (see Lemma 2.1). Then*

(i) $\widetilde{iY}_z = \frac{d}{ds}\big|_{s=0} \exp sC \exp i(H + sY_{\mathfrak{a}})K^{\mathbb{C}}$, *where*

$$C := -\textstyle\sum_\alpha \frac{\cos\alpha(H)}{\sin\alpha(H)} K^\alpha \qquad \textit{and} \qquad K^\alpha := Y^\alpha + \theta Y^\alpha.$$

(ii) $\widetilde{iQ^\alpha}_z = a_* i F_a Q^\alpha = -\frac{\cos\alpha(H)}{\sin\alpha(H)} \widetilde{K^\alpha}_z$.

(iii) $\widetilde{K^\alpha}_z = -\sin\alpha(H) a_* i Q^\alpha$.

Proof. (i) By (11) one has

$$\widetilde{iY}_z = a_* F_a iY = a_*\big(iY_{\mathfrak{a}} + \textstyle\sum_\alpha \cos\alpha(H) iQ^\alpha\big)\,. \tag{13}$$

On the other hand, for $K^\alpha = Y^\alpha + \theta Y^\alpha$ and $C = \sum_\alpha c_\alpha K^\alpha$, Campbell-Hausdorff formula yields

$$\exp sC \exp i(H + sY_{\mathfrak{a}})K^{\mathbb{C}} = a \exp sAd_{a^{-1}}C \exp siY_{\mathfrak{a}} K^{\mathbb{C}} =$$

$$= a \exp(sAd_{a^{-1}}C + siY_{\mathfrak{a}} + \frac{s^2}{2}[Ad_{a^{-1}}C, iY_{\mathfrak{a}}] + \ldots)K^{\mathbb{C}}.$$

Differentiating the above expression at 0, one obtains

$$a_*\pi_\#(Ad_{a^{-1}}C + iY_{\mathfrak{a}}) = a_*\big(iY_{\mathfrak{a}} - \textstyle\sum_\alpha c_\alpha \sin\alpha(H) iQ^\alpha\big).$$

Then the required identity follows by taking $c_\alpha = -\frac{\cos\alpha(H)}{\sin\alpha(H)}$.

(ii) This is a special case of (i).

(iii) By setting $C = K^\alpha$ and $Y_{\mathfrak{a}} = 0$ in the proof of (i), one obtains

$$\widetilde{K^\alpha}_z = a_*\pi_\# Ad_{a^{-1}}K^\alpha = -a_* \sin\alpha(H) iQ^\alpha\,.$$

□

3. The complex structure I

In this section we introduce a new G-invariant almost complex structure I on Ξ. Its integrability will be settled in Section 7. Eventually, I, $J = J_{ad}$ and $K := IJ$ will be the three complex structures of our hyper-Kähler structure on Ξ.

Definition 3.1. *For $gaK^{\mathbb{C}}$ in Ξ and $Z \in \mathfrak{p}^{\mathbb{C}}$ the G-invariant (almost) complex structure I is defined by*

$$Ig_*\widetilde{Z}_{aK^{\mathbb{C}}} = g_*\widetilde{I_0 Z}_{aK^{\mathbb{C}}}\,.$$

We claim that the above definition is well posed. Suppose that $z = gaK^{\mathbb{C}} = g'a'K^{\mathbb{C}}$, for some $g,\, g' \in G$ and $a, a' \in \exp i\Omega$, and that $g_*\widetilde{Z}_{aK^{\mathbb{C}}} = g'_*\widetilde{U}_{a'K^{\mathbb{C}}}$, for some $Z, U \in \mathfrak{p}^{\mathbb{C}}$. This is equivalent to $g = g'wk$ and $a = w^{-1}a'w$, for some $w \in N_K(\mathfrak{a})$ and $z \in Z_K(a)$ (see [KrSt05], Prop.4.1), and $Ad_{wk}Z = U$. Then

$$Ig'_*\widetilde{U}_{a'K^{\mathbb{C}}} = g'_*\widetilde{I_0 U}_{a'K^{\mathbb{C}}} = g'_*\widetilde{I_0 Ad_{wk} Z}_{waw^{-1}K^{\mathbb{C}}} = g'_*(wk)_*\widetilde{I_0 Z}_{aK^{\mathbb{C}}} =$$
$$= g_*\widetilde{I_0 Z}_{aK^{\mathbb{C}}} = Ig_*\widetilde{Z}_{aK^{\mathbb{C}}},$$

as claimed.

By equations (11) one has

$$Ia_*Z = I\widetilde{F_a^{-1}Z}_z = \widetilde{I_0 F_a^{-1}Z}_z = a_*F_aI_0F_a^{-1}\overline{Z} \tag{14}$$

for every $z = aK^{\mathbb{C}}$ on the slice of Ξ. In particular, for $\alpha \in \Sigma^+ \cup \{0\}$ and $P^\alpha \in \mathfrak{p}[\alpha]$ with $I_0P^\alpha \in \mathfrak{p}[\beta]$, one has

$$Ia_*P^\alpha = a_*\tfrac{\cos\beta(H)}{\cos\alpha(H)}I_0P^\alpha\,.$$

From Definition 3.1 it is also clear that $I^2 = -Id$ and $IJ = -JI$. Then, by defining $K := IJ$, one obtains a quaternionic (almost) complex structure $(I,\, J,\, K)$ on Ξ.

Proposition 3.2. *The G-equivariant projection*

$$p : \Xi \to G/K \qquad gaK^{\mathbb{C}} \to gK\,,$$

is holomorphic with respect to the G-invariant complex structures I on Ξ and I_0 on G/K.

Proof. Since p is G-equivariant, it is sufficient to consider its restriction to the slice. Let $Z = X + iY$ be an element of $\mathfrak{p}^{\mathbb{C}}$, with $X, Y \in \mathfrak{p}$. We claim that the differential $p_*: T\Xi \to TG/K$ at $z = aK^{\mathbb{C}}$, is given by

$$p_*(\widetilde{Z}_z) = p_*(\widetilde{X}_z) + p_*(\widetilde{iY}_z) = X.$$

It is straightforward to check that $p_*(\widetilde{X}_z) = X$. In order to verify that $p_*(\widetilde{iY}_z) = 0$, write $Y = Y_{\mathfrak{a}} + \sum_\alpha Q^\alpha$, according to Lemma 2.1. By Lemma 2.2(i) one has $\widetilde{iY}_z = \frac{d}{ds}\big|_{s=0} \exp sC \exp siY_{\mathfrak{a}} aK^{\mathbb{C}}$, for an appropriate element $C \in \mathfrak{k}$. Then from the definition of p it follows that $p_*(\widetilde{iY}_z) = 0$. Now for $Z \in \mathfrak{p}^{\mathbb{C}}$ one has

$$p_*(I\widetilde{Z}_z) = p_*(\widetilde{I_0 Z}_z) = p_*(\widetilde{I_0 X}_z) - p_*(\widetilde{iI_0 Y}_z) = I_0 X = I_0 p_*(\widetilde{Z}_z),$$

which concludes the proof of the statement. □

4. The inverse of Dancer-Szöke's deformation vs. the complex structure I

In this section we define a G-equivariant diffeomorphism ψ of the tangent bundle TG/K with the property that our complex structure I is the pull-back via ψ of the natural complex structure of the holomorphic cotangent bundle $T^*G/K^{1,0} \cong T^*G/K \cong TG/K$ (see also Rem. 10.2). The map ψ is the inverse of the diffeomorphism introduced in [DaSz97], Sect. 4. However, here ψ and I are expressed in a Lie theoretical fashion, a fact that will be repeatedly exploited in the sequel.

By identifying the tangent bundle TG/K with the homogeneous vector bundle $G \times_K \mathfrak{p}$, the map ψ is completely determined by its restriction $\Psi: \mathfrak{p} \to \mathfrak{p}$, namely $\psi[g, X] = [g, \Psi(X)]$, for $g \in G$ and $X \in \mathfrak{p}$. Note that ψ maps every fiber into itself.

Let $Z_0 \in Z(\mathfrak{k})$ be the element inducing the complex structure I_0 on $\mathfrak{p}$ and let $\pi_\# : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{p}^{\mathbb{C}}$ be the linear projection along $\mathfrak{k}^{\mathbb{C}}$.

Lemma 4.1. *Let $\Psi : \mathfrak{p} \to \mathfrak{p}$ be the map defined by*

$$\Psi(Y) := -I_0 \circ J \circ \pi_\# \circ Ad_{\exp iY} Z_0 \,.$$

Then

(i) *Ψ is Ad_K-equivariant,*

(ii) *for $H = \sum_j t_j A_j$ in $\mathfrak{a}$ one has*

$$\Psi(H) = \tfrac{1}{2} \textstyle\sum_{j=1}^r \sin \lambda_j(H) A_j = \tfrac{1}{2} \sum_{j=1}^r \sin(2t_j) A_j \,.$$

Proof. (i) Since Z_0 lies in the center of $\mathfrak{k}$, for every $k \in K$ one has $Ad_{\exp i\mathrm{Ad}_k Y} Z_0 = Ad_{k \exp iY} Z_0 = Ad_k \circ Ad_{\exp iY} Z_0$. Now the statement follows from the Ad_K-equivariance of all the remaining maps in the composition defining Ψ.

(ii) One has

$$Ad_{\exp iH} Z_0 = e^{ad_{iH}} Z_0 = \cos ad_H Z_0 + i \sin ad_H Z_0 =$$
$$= \sum_{n \geqslant 0} \tfrac{(-1)^n}{(2n)!} ad_H^{2n} Z_0 + i \sum_{n \geqslant 0} \tfrac{(-1)^n}{(2n+1)!} ad_H^{2n+1} Z_0.$$

Lemma 2.1 and relations (5) and (6), imply that

$$ad_H^{2n} Z_0 = \tfrac{1}{2} \sum_j \lambda_j^{2n}(H) K^j, \quad ad_H^{2n+1} Z_0 = \tfrac{1}{2} \sum_j \lambda_j^{2n+1}(H) P^j.$$

It follows that the $\mathfrak{p}^{\mathbb{C}}$-component of $Ad_{\exp iH} Z_0$ is given by

$$\pi_{\#} Ad_{\exp iH} Z_0 = \tfrac{i}{2} \sum_j \sin \lambda_j(H) P^j = \tfrac{i}{2} \sum_j \sin(2t_j) P^j \,,$$

and

$$-I_0 \circ J \circ \pi_{\#} \circ Ad_{\exp iH} Z_0 =$$
$$= -I_0 \circ J\big(\tfrac{i}{2} \sum_j \sin \lambda_j(H) P^j\big) = \tfrac{1}{2} I_0 \sum_j \sin \lambda_j(H) P_j = \tfrac{1}{2} \sum_j \sin(2t_j) A_j,$$

as claimed. □

For $a = \exp iH$, with $H \in \Omega$, consider the $\mathbb{C}$-linear map $E_a \colon \mathfrak{p}^{\mathbb{C}} \to \mathfrak{p}^{\mathbb{C}}$ uniquely defined by

$$E_a := \pi_{\#} \circ \widetilde{E}_a|_{\mathfrak{p}^{\mathbb{C}}},$$

where $\widetilde{E}_a \colon \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ is given by $\widetilde{E}_a = \sum_{n \geqslant 0} \frac{(-1)^n}{(n+1)!} \mathrm{ad}_{iH}^n$, and has the property that (cf. [Var84], Thm. 2.14.3, p. 108) $(\exp_*)_{iH} = (\exp iH)_* \widetilde{E}_a$. One can verify that

$$E_a A = A, \qquad E_a P^\alpha = \tfrac{\sin \alpha(H)}{\alpha(H)} P^\alpha, \tag{15}$$

for all $A \in \mathfrak{a}$ and $P^\alpha \in \mathfrak{p}[\alpha]$, with $\alpha \in \Sigma$.

Lemma 4.2. *Fix $a = \exp iH$, with $H \in \Omega'$, and $P^\alpha \in \mathfrak{p}[\alpha]$. Then*

$$\tfrac{d}{ds}\big|_{s=0} \mathrm{Ad}_{\exp i(H + sP^\alpha)} Z_0 = \tfrac{d}{ds}\big|_{s=0} \mathrm{Ad}_{\exp -s \frac{K^\alpha}{\alpha(H)} \exp iH} Z_0 \,.$$

Proof. Since $\mathrm{Ad}_{G^{\mathbb{C}}} Z_0 \cong G^{\mathbb{C}}/K^{\mathbb{C}}$ the statement can be reformulated as

$$\tfrac{d}{ds}\big|_{s=0} \exp i(H + sP^\alpha) K^{\mathbb{C}} = \tfrac{d}{ds}\big|_{s=0} \exp -s \tfrac{K^\alpha}{\alpha(H)} \exp iH K^{\mathbb{C}} \,.$$

By the definition of E_a, the left-hand side is $a_* E_a i P^\alpha$; likewise the right-hand side is

$$-a_* \pi_{\#} Ad_{a^{-1}} \tfrac{1}{\alpha(H)} K^\alpha = a_* \tfrac{\sin \alpha(H)}{\alpha(H)} i P^\alpha = a_* E_a i P^\alpha.$$

□

Fix $H \in \mathfrak{a}$. Identify as usual $T_H\mathfrak{p}$ and $T_{\Psi(H)}\mathfrak{p}$, the tangent spaces to $\mathfrak{p}$ at H and at $\Psi(H)$, with $\mathfrak{p}$. Consider the differential $(\Psi_*)_H : \mathfrak{p} \to \mathfrak{p}$ of Ψ at H.

Lemma 4.3. (cf. [DaSz97], Lemma 2.4) *Fix $a = \exp iH$, with $H = \sum_j t_j A_j$ in Ω'. Then*

(i) $(\Psi_*)_H A_j = \cos\lambda_j(H) A_j$, *for all* $j = 1, \dots, r$,

(ii) $(\Psi_*)_H P^\alpha = \frac{\alpha(\Psi(H))}{\alpha(H)} P^\alpha$, *for all* $P^\alpha \in \mathfrak{p}[\alpha]$ *with* $\alpha \in \Sigma^+$.

In particular $(\Psi_*)_H$ *is self-adjoint with respect to the Killing form* B.

Proof. Part (i) follows from the definition of Ψ.
(ii) By Lemma 4.2 and the Ad_K-equivariance of Ψ one has

$$(\Psi_*)_H P^\alpha = \tfrac{d}{ds}\big|_{s=0}\Psi(H + sP^\alpha) = -I_0 \circ J \circ \pi_\#\big(\tfrac{d}{ds}\big|_{s=0}\mathrm{Ad}_{\exp i(H+sP^\alpha)} Z_0\big) =$$
$$= -I_0 \circ J \circ \pi_\#\big(\tfrac{d}{ds}\big|_{s=0}\mathrm{Ad}_{\exp -s\frac{K^\alpha}{\alpha(H)}} \exp iH Z_0\big) = \tfrac{d}{ds}\big|_{s=0}\mathrm{Ad}_{\exp -s\frac{K^\alpha}{\alpha(H)}} \Psi(H)$$
$$= -\big[\tfrac{K^\alpha}{\alpha(H)},\, \Psi(H)\big] = \tfrac{\alpha(\Psi(H))}{\alpha(H)} P^\alpha\,.$$

□

Extend $\mathbb{C}$-linearly $(\Psi_*)_H \colon \mathfrak{p}^\mathbb{C} \to \mathfrak{p}^\mathbb{C}$, and define a G-invariant, real analytic map $L : \exp i\Omega' \to \mathrm{GL}(\mathfrak{p}^\mathbb{C})$ by

$$a \to L_a := I_0 F_a^{-1} (\Psi_*)_H E_a^{-1}\,.$$

Lemma 4.4. *Given* $a \in \exp i\Omega'$ *one has* $L_a = F_a I_0 F_a^{-1}$. *In particular* L *extends real-analytically to* $\exp i\Omega$ *and*

$$I a_* Z = a_* L_a \overline{Z}\,,$$

for every $Z \in \mathfrak{p}^\mathbb{C}$.

Proof. Since the maps F_a, E_a and $(\Psi_*)_H$ are $\mathbb{C}$-linear and commute, the statement of the lemma is equivalent to

$$(\Psi_*)_H E_a^{-1} = -I_0 F_a I_0 \tag{16}$$

on $\mathfrak{p}$. Recall that I_0 permutes the blocks of decomposition (1), namely

$$I_0 A_j \in \mathfrak{p}[2e_j], \qquad I_0\mathfrak{p}[e_j] = \mathfrak{p}[e_j], \qquad I_0\mathfrak{p}[e_k + e_l] = \mathfrak{p}[e_k - e_l]\,.$$

As $\lambda_j = 2e_j$, by (6) and Lemma 4.3, one easily verifies that

$$(\Psi_*)_H E_a^{-1} A_j = \cos\lambda_j(H) A_j = -I_0 F_a I_0 A_j$$

and that

$$(\Psi_*)_H E_a^{-1} I_0 A_j = \tfrac{\lambda_j(\Psi(H))}{\sin\lambda_j(H)} I_0 A_j = I_0 A_j = -I_0 F_a I_0\, I_0 A_j.$$

If $I_0\mathfrak{p}[\alpha] = \mathfrak{p}[\beta]$, with $\alpha, \beta \neq 0$, then we have that $(\Psi_*)_H E_a^{-1} P^\alpha = -I_0 F_a I_0 P^\alpha$ if and only if the following identity holds true

$$\alpha(\Psi(H)) = \sin\alpha(H)\cos\beta(H). \tag{17}$$

For $\alpha = \beta = e_k$, equation (17) becomes $\frac{1}{2}\sin 2t_k = \sin t_k \cos t_k$, which is obviously verified. For $\alpha = e_k \pm e_l$ and $\beta = e_k \mp e_l$, equation (17) becomes

$$\tfrac{1}{2}\left(\sin 2t_k \pm \sin 2t_l\right) = \sin(t_k \pm t_l)\cos(t_k \mp t_l),$$

which can be easily checked. □

Remark 4.5. *By using the identity*

$$L_a = E_a(\Psi_*)_H^{-1} I_0 (\Psi_*)_H E_a^{-1}$$

one can also verify that the complex structure I is the pull-back via ψ of the natural complex structure on the holomorphic cotangent bundle $T^*G/K^{1,0} \cong T^*G/K \cong TG/K$ of G/K (see also Rem. 10.2) .

5. The hyper-kähler structure

In this section we introduce three G-invariant differential 2-forms ω_I, ω_J and ω_K on the crown domain Ξ in $G^{\mathbb{C}}/K^{\mathbb{C}}$ and study their basic properties. The forms ω_I and ω_K are restrictions of $G^{\mathbb{C}}$-invariant forms on $G^{\mathbb{C}}/K^{\mathbb{C}}$ and therefore closed; ω_J will be shown to be closed in Proposition 6.2. The forms ω_I, ω_J and ω_K are invariant under the (almost) complex structures I, J and K, respectively. Eventually, they will be the three Kähler forms of our hyper-Kähler structure.

Definition 5.1. *For $g \cdot z \in \Xi$, with $z = aK^{\mathbb{C}}$, and $Z \in \mathfrak{p}^{\mathbb{C}}$ define G-invariant real-analytic forms by*

$$\begin{aligned}
\omega_I(g_*\widetilde{Z}_z,\, g_*\widetilde{W}_z) &:= \quad \mathrm{Re}B(I_0F_aZ,\, F_aW)\,,\\
\omega_J(g_*\widetilde{Z}_z,\, g_*\widetilde{W}_z) &:= \omega_I(J\widetilde{Z}_z, I\widetilde{W}_z) = -\mathrm{Im}B(I_0F_aZ, F_aI_0\overline{W}),\\
\omega_K(g_*\widetilde{Z}_z,\, g_*\widetilde{W}_z) &:= -\mathrm{Im}B(I_0F_aZ,\, F_aW),
\end{aligned}$$

where $F_a = \pi_\# Ad_{a^{-1}}|_{\mathfrak{p}^{\mathbb{C}}}$ (see (8)).

We claim that the forms ω_I, ω_J and ω_K are well defined. Indeed, assume that $gaK^{\mathbb{C}} = g'a'K^{\mathbb{C}}$, for some $g,\, g' \in G$ and $a,\, a' \in \exp i\Omega$, and that $g_*\widetilde{Z}_{aK^{\mathbb{C}}} = h_*\widetilde{U}_{a'K^{\mathbb{C}}}$ and $g_*\widetilde{W}_{aK^{\mathbb{C}}} = g'_*\widetilde{V}_{a'K^{\mathbb{C}}}$, for some $Z,\, U,\, W,\, V \in \mathfrak{p}^{\mathbb{C}}$. This is equivalent to $g = g'wk$ and $a = w^{-1}a'w$, for some $w \in N_K(\mathfrak{a})$ and $z \in Z_K(a)$ (see [KrSt05], Prop.4.1), and in addition $Ad_{wk}Z = U$ and $Ad_{wk}W = V$. Then, from the definition of the operator F_a, the Ad_K-equivariance of $\pi_\#$ and the Ad_K-invariance of B, it follows that

$$\begin{aligned}
B(I_0F_{a'}U,\, F_{a'}V) = B(I_0\pi_\# Ad_{wa^{-1}w^{-1}}Ad_{wk}Z, \pi_\# Ad_{wa^{-1}w^{-1}}Ad_{wk}W) = \\
= B(I_0F_aZ, F_aW).
\end{aligned}$$

As a result,

$$\omega_I(g'_*\widetilde{U}_{a'K^{\mathbb{C}}}, g'_*\widetilde{V}_{a'K^{\mathbb{C}}}) = \mathrm{Re}B(I_0F_{a'}U,\, F_{a'}V) = \mathrm{Re}B(I_0F_aZ, F_aW) = \\ = \omega_I(g_*\widetilde{Z}_{aK^{\mathbb{C}}}, g_*\widetilde{W}_{aK^{\mathbb{C}}}),$$

which says that ω_I is well defined. A similar reasoning applies to ω_K.

The form ω_J is well defined, since the complex structure I and the form ω_I are. For $Z, W \in \mathfrak{p}^{\mathbb{C}}$, one has

$$\omega_J(g_*\widetilde{Z}_z,\, g_*\widetilde{W}_z) = \omega_I(Jg_*\widetilde{Z}_z,\, Ig_*\widetilde{W}_z) = \omega_I(g_*\widetilde{iZ}_z,\, g_*\widetilde{I_0\overline{W}}_z) = \\ = \mathrm{Re}B(I_0F_aiZ, F_aI_0\overline{W}) = -\mathrm{Im}B(I_0F_aZ, F_aI_0\overline{W}).$$

Lemma 5.2.

(i) *On* Ξ' *one has*

$$\omega_J(g_*\widetilde{Z}_{aK^{\mathbb{C}}},\, g_*\widetilde{W}_{aK^{\mathbb{C}}}) = -\mathrm{Im}B\big(Z,\, (\Psi_*)_H E_a^{-1}F_a\overline{W}\big).$$

(ii) *For* $gaK^{\mathbb{C}} \in \Xi$ *and* $Z, W \in \mathfrak{p}^{\mathbb{C}}$*, one has*

$$\begin{aligned} \omega_I((ga)_*Z, (ga)_*W) &= \mathrm{Re}B(I_0Z,\, W);\\ \omega_J((ga)_*Z, (ga)_*W) &= -\mathrm{Im}B(I_0Z,\, F_aI_0F_a^{-1}\overline{W});\\ \omega_K((ga)_*Z, (ga)_*W) &= -\mathrm{Im}B(I_0Z,\, W). \end{aligned}$$

(iii) *The forms* ω_I *and* ω_K *are locally* $G^{\mathbb{C}}$*-invariant and are closed.*

Proof. (i) By (16), the quantity on the right-hand side equals

$$-\mathrm{Im}B(Z, (\Psi_*)_H E_a^{-1}F_a\overline{W}) = -\mathrm{Im}B(F_aZ,\, (\Psi_*)_H E_a^{-1}\overline{W}) = \\ \mathrm{Im}B(F_aZ,\, I_0F_aI_0\overline{W}) = -\mathrm{Im}B(I_0F_aZ,\, F_aI_0\overline{W})\,,$$

as claimed.

(ii) One has $a_*Z = \widetilde{F_a^{-1}Z}_{aK^{\mathbb{C}}}$. The statements about ω_I and ω_K are immediate. For ω_J, one has

$$\omega_J((ga)_*Z, (ga)_*W) = \omega_I(J(ga)_*Z,\, I(ga)_*W) = \omega_I((ga)_*iZ,\, (ga)_*L_a\overline{W}) = \\ \omega_I((ga)_*iZ,\, (ga)_*F_aI_0F_a^{-1}\overline{W}) = \mathrm{Re}\, iB(I_0Z, F_aI_0F_a^{-1}W) = -\mathrm{Im}B(I_0Z, F_aI_0F_a^{-1}\overline{W}).$$

(iii) We first show that the complex form $\omega_I - i\omega_K$ is locally $G^{\mathbb{C}}$-invariant. To this aim, consider the map $\mathfrak{p} \times i\mathfrak{p} \times K^{\mathbb{C}} \to G^{\mathbb{C}}$, given by $(X,\, iY,\, k) \to \exp X \exp iYk$, which sends a neighborhood of $(0,\, 0,\, e)$ in $\mathfrak{p} \times i\mathfrak{p} \times K^{\mathbb{C}}$ onto a neighborhood of e in $G^{\mathbb{C}}$. For every $Y \in \mathfrak{p}$ one can write $\exp iY = h \exp iHh^{-1}$, for some $h \in K$ and $H \in \mathfrak{a}$. Then, due to the $K^{\mathbb{C}}$-equivariance of I_0, the $G^{\mathbb{C}}$-invariance of B and the formulas proved in (ii), for $\exp X \exp iYk \in \exp\mathfrak{p} \exp i\mathfrak{p}K^{\mathbb{C}}$ one has

$$(\omega_J - i\omega_K)((\exp X \exp iYk)_*Z,\, (\exp X \exp iYk)_*W) = \\ = (\omega_J - i\omega_K)((\exp Xh \exp iHh^{-1}k)_*Z,\, (\exp Xh \exp iHh^{-1}k)_*W) =$$

$$= B(I_0 \mathrm{Ad}_{h^{-1}k} Z,\, \mathrm{Ad}_{h^{-1}k} W) = B(I_0 Z,\, W) = \,(\omega_J - i\omega_K)(Z,\, W)\,.$$

By letting $\exp X \exp iYk$ vary in a neighborhood of e in $G^{\mathbb{C}}$, one concludes that the forms are locally $G^{\mathbb{C}}$-invariant near $eK^{\mathbb{C}}$.

Since ω_I and ω_K are real-analytic, they are restrictions of $G^{\mathbb{C}}$-invariant forms on $G^{\mathbb{C}}/K^{\mathbb{C}}$. In order to prove that they are closed, we adapt the proof in [Wol84], Thm. 8.5.6, p. 250, to our complex setting.

Let ω be an arbitrary $G^{\mathbb{C}}$-invariant 2-form on $G^{\mathbb{C}}/K^{\mathbb{C}}$. A similar argument as in [Wol84], Lemma 8.5.5, shows that ω is closed if and only if so is its pull-back $\pi^*\omega$ to $G^{\mathbb{C}}$. Since π is $G^{\mathbb{C}}$-equivariant, the form $\pi^*\omega$ is $G^{\mathbb{C}}$-invariant. Given left invariant vector fields $\widehat{X}$, $\widehat{Y}$, $\widehat{W}$ on $G^{\mathbb{C}}$ such that at least one of them lies in the kernel $\mathfrak{k}^{\mathbb{C}}$ of the differential π_* at e, one has

$$d\pi^*\omega(\widehat{X},\, \widehat{Y},\, \widehat{W}) \equiv d\pi^*\omega(\widehat{X}_e,\, \widehat{Y}_e,\, \widehat{W}_e) = \pi^* d\omega(X,\, Y,\, W) =$$

$$= d\omega(\pi_* X,\, \pi_* Y,\, \pi_* W) = 0\,.$$

Hence we may assume that $X,\, Y,\, W \in \mathfrak{p}^{\mathbb{C}}$. From Cartan's formula for the external derivation and the $G^{\mathbb{C}}$-invariance of $\pi^*\omega$ one has

$$d\pi^*\omega(\widehat{X},\, \widehat{Y},\, \widehat{W}) = \pi^*\omega(\widehat{X},\, \widehat{[Y,\, W]}) - \pi^*\omega(\widehat{Y},\, \widehat{[X,\, W]}) + \pi^*\omega(\widehat{W},\, \widehat{[X,\, Y]})\,,$$

which vanishes due to the inclusion $[\mathfrak{p}^{\mathbb{C}},\, \mathfrak{p}^{\mathbb{C}}] \subset \mathfrak{k}^{\mathbb{C}}$. □

Lemma 5.3. *The form* ω_I *is* I*-invariant, namely*

$$\omega_I(\,\cdot\,,\,\cdot\,) = \omega_I(I\,\cdot\,, I\,\cdot\,)\,.$$

Likewise, ω_J *is* J*-invariant and* ω_K *is* K*-invariant.*

Proof. By the G-invariance of the forms, it is sufficient to prove the statements for $z = aK^{\mathbb{C}}$ in the slice $\exp i\Omega K^{\mathbb{C}}$.
One has

$$\omega_I(Ia_*Z,\, Ia_*W) = \omega_I(a_*L_a\overline{Z},\, a_*L_a\overline{W}) = \mathrm{Re}B(I_0 L_a\overline{Z},\, L_a\overline{W}) =$$

$$-\mathrm{Re}B(F_a^{-1}(\Psi_*)_H E_a^{-1}\overline{Z}, L_a\overline{W}) = -\mathrm{Re}B(\overline{Z},\, F_a^{-1}(\Psi_*)_H E_a^{-1} L_a\overline{W}) =$$

$$-\mathrm{Re}B(I_0\overline{Z},\, L_a^2\overline{W}) = \mathrm{Re}B(I_0\overline{Z},\, \overline{W}) = \mathrm{Re}B(I_0 Z,\, W) = \omega_I(a_*Z,\, a_*W)\,.$$

Similarly, one obtains the J-invariance of ω_J and the K-invariance of ω_K. □

6. An invariant potential for ω_J and the associated moment map

In this section we exhibit a G-invariant potential for ω_J, i.e. a G-invariant, smooth function ρ_J such that $2i\partial\bar{\partial}_J\rho_J = \omega_J$. The fact that ρ_J is J-strictly plurisubharmonic implies that ω_J is a Kähler form with respect to J. We prove that $2i\partial\bar{\partial}_J\rho_J = \omega_J$ by applying moment map techniques, namely we use the following reformulation of Lemma 7.1 in [HeGe07].

Let $\mathcal{G}$ be a real Lie group acting by holomorphic transformations on a manifold M with a complex structure $\mathcal{I}$. For $X \in Lie(\mathcal{G})$, denote by $\widetilde{X}$ the vector field on M induced by the $\mathcal{G}$-action, namely $\widetilde{X}_z := \frac{d}{ds}\big|_{s=0} \exp sX \cdot z$. Let $\rho : M \to \mathbb{R}$ be a G-invariant, smooth $\mathcal{I}$-strictly plurisubharmonic function. Set $d^c_{\mathcal{I}}\rho := d\rho \circ \mathcal{I}$, so that $2i\partial\bar{\partial}_{\mathcal{I}}\rho = -dd^c_{\mathcal{I}}\rho$. Then $-dd^c_{\mathcal{I}}\rho$ is a $\mathcal{G}$-invariant Kähler form with respect to $\mathcal{I}$ and the map $\mu : M \to Lie(\mathcal{G})^*$, defined by

$$\mu(z)(X) = d^c_{\mathcal{I}}\rho(\widetilde{X}_z)\,,$$

for $X \in Lie(\mathcal{G})$, is a moment map. It is referred to as the moment map associated with ρ.

In order to compute the G-invariant form $-dd^c_{\mathcal{I}}\rho$, also in the case when ρ is not known to be $\mathcal{I}$-strictly plurisubharmonic (as we do in Section 8), one can apply the following lemma.

Lemma 6.1. *Let $\rho\colon M \to \mathbb{R}$ be a smooth $\mathcal{G}$-invariant function. For $X \in Lie(\mathcal{G})$, define $\mu^X : M \to \mathbb{R}$ by $\mu^X(z) = d^c_{\mathcal{I}}\rho(\widetilde{X}_z)$. Then*

$$d\mu^X = -\iota_{\widetilde{X}} dd^c_{\mathcal{I}}\rho\,.$$

Proof. The same proof as the one of Lemma 7.1 in [HeSc07] applies to our situation. Indeed their argument needs neither the compactness of $\mathcal{G}$ nor the plurisubharmonicity of ρ. □

Proposition 6.2. *Let $z = gaK^{\mathbb{C}}$, with $a = \exp iH$, be an element in Ξ. The G-invariant function $\rho_J : \Xi \to \mathbb{R}$ defined by*

$$\rho_J(gaK^{\mathbb{C}}) := -\tfrac{1}{4}\textstyle\sum_{j=1}^r \cos\lambda_j(H)B(A_j, A_j)\,,$$

is a strictly plurisubharmonic potential for ω_J. The associated moment map $\mu_J : \Xi \to \mathfrak{g}^*$ is given by

$$\mu_J(gaK^{\mathbb{C}})(X) = B\big(\mathrm{Ad}_{g^{-1}}(X),\, \Psi(H)\big)\,,$$

for $X \in \mathfrak{g}$.

Note that $B(A_1, A_1) = \cdots = B(A_r, A_r)$ is a constant depending only on the symmetric space G/K.

Proof. The map $\Omega \to \mathbb{R}$, given by $H \to \rho_J(\exp iHK^{\mathbb{C}})$, is strictly convex. Then Theorem 10 in [BHH03] implies that ρ_J is strictly plurisubharmonic.

The form $-dd^c_J\rho_J$ is computed by applying Lemma 6.1. For this we first determine $\mu^X_J(z) := d^c_J\rho_J(\widetilde{X}_z)$. In particular we obtain the moment map μ_J associated with ρ_J.

Fix $z = aK^{\mathbb{C}}$, with $a = \exp iH$ and $H \in \Omega'$ (see (12)). Start with $X \in \mathfrak{p}$ and write $X_{\mathfrak{a}} = \frac{1}{2}\sum_j \lambda_j(X_{\mathfrak{a}})A_j$ for the $\mathfrak{a}$-component of X. By Lemma 2.2(i) and the G-invariance of ρ_J, one has

$$\mu^X_J(z) = d^c_J\rho_J(\widetilde{X}_z) = d\rho_J(\widetilde{iX}_z) = \tfrac{d}{ds}\big|_{s=0}\rho_J(\exp i(H + sX_{\mathfrak{a}})K^{\mathbb{C}}) =$$
$$= -\tfrac{1}{4}\tfrac{d}{ds}\big|_{s=0}\textstyle\sum_{j=1}^r \cos\lambda_j(H + sX_{\mathfrak{a}})B(A_j,\, A_j) =$$
$$= \tfrac{1}{4}\textstyle\sum_{j=1}^r \sin\lambda_j(H)\lambda_j(X_{\mathfrak{a}})B(A_j,\, A_j) = B\big(X_{\mathfrak{a}},\, \Psi(H)\big) = B\big(X,\, \Psi(H)\big)\,.$$

The G-invariance of ρ_J and of the complex structure J then implies

$$\mu^X_J(g\cdot z) = d^c_J\rho_J(\widetilde{X}_{g\cdot z}) = d^c_J\rho_J(\widetilde{\mathrm{Ad}_{g^{-1}}X}_z) = B\big(\mathrm{Ad}_{g^{-1}}X,\, \Psi(H)\big),$$

for all $g \in G$ and $X \in \mathfrak{p}$. In order to show that such an identity holds true also for $X \in \mathfrak{k}$, write $X = M + \sum_\alpha K^\alpha$, with $M \in \mathfrak{m}$ and $K^\alpha = X^\alpha + \theta X^\alpha$. Set $P^\alpha = X^\alpha - \theta X^\alpha$. One has

$$d^c_J\rho_J(\widetilde{X}_z) = \tfrac{d}{ds}\big|_{s=0}\rho_J(\exp siXaK^{\mathbb{C}}) =$$
$$\tfrac{d}{ds}\big|_{s=0}\rho_J\big(a\exp si\mathrm{Ad}_{a^{-1}}XK^{\mathbb{C}}\big) = \tfrac{d}{ds}\big|_{s=0}\rho_J\big(a\exp\big(s\textstyle\sum_\alpha \sin\alpha(H)P^\alpha\big)K^{\mathbb{C}}\big) =$$
$$\tfrac{d}{ds}\big|_{s=0}\rho_J\big(\exp\big(s\textstyle\sum_\alpha \frac{\sin\alpha(H)}{\cos\alpha(H)}P^\alpha\big)aK^{\mathbb{C}}\big) = 0\,,$$

where the last equality follows from the G-invariance of ρ_J. Since $\mathfrak{a} \perp_B \mathfrak{k}$, it follows that $d^c_J\rho_J(\widetilde{X}_z) = B\big(X,\, \Psi(H)\big)$, as wished.

Next we need to show that $-dd^c_J\rho_J(\widetilde{Z}_z, \widetilde{W}_z) = \omega_J(\widetilde{Z}_z, \widetilde{W}_z)$, for all $Z,\, W \in \mathfrak{p}^{\mathbb{C}}$. Since both forms are J-invariant, it is enough to show that

$$-dd^c_J\rho_J(\widetilde{X}_z,\, \widetilde{W}_z) = \omega_J(\widetilde{X}_z,\, \widetilde{W}_z)$$

for all $X \in \mathfrak{p}$ and $W \in \mathfrak{p}^{\mathbb{C}}$. Moreover, since both forms are G-invariant it is sufficient to prove the above identity at points $z = aK^{\mathbb{C}}$, with $a = \exp iH$ and $H \in \Omega'$ (see (12)). Write $W = U + iV$, with $U,\, V \in \mathfrak{p}$. Then by Lemma 6.1, one has

$$-dd^c\rho_J(\widetilde{X}_z, \widetilde{W}_z) = d\mu^X_J(\widetilde{W}_z) = d\mu^X_J(\widetilde{U}_z) + d\mu^X_J(\widetilde{iV_z}).$$

The first summand on the right-hand side is zero:

$$d\mu^X_J(\widetilde{U}_z) = \tfrac{d}{ds}\big|_{s=0}B\big(\mathrm{Ad}_{\exp -sU}X,\, \Psi(H)\big) = \tfrac{d}{ds}\big|_{s=0}B\big(X - s[U,X],\, \Psi(H)\big) = 0\,,$$

since $[U,X] \in \mathfrak{k}$ and $\mathfrak{k} \perp_B \mathfrak{a}$.

For the second summand, write $V = V_{\mathfrak{a}} + \sum_\alpha Q^\alpha$, where $V_{\mathfrak{a}} \in \mathfrak{a}$ and $Q^\alpha = V^\alpha - \theta V^\alpha \in \mathfrak{p}[\alpha]$. From Lemma 2.2(i), we get

$$d\mu^X_J(\widetilde{iV}_z) = \tfrac{d}{ds}\big|_{s=0}\mu^X_J(\exp sC\exp i(H + sV_{\mathfrak{a}})K^{\mathbb{C}}) =$$

$$= \frac{d}{ds}\Big|_{s=0} B\big(X - s[C,\, X],\, \Psi(H + sV_{\mathfrak{a}})\big) = -B\big([C,\, X],\, \Psi(H)\,\big) + B\big(X,\, (\Psi_*)_H V_{\mathfrak{a}}\big) =$$
$$= B\big(X,\, [C,\, \Psi(H)]\,\big) + B\big(X,\, (\Psi_*)_H V_{\mathfrak{a}}\big)\,,$$

where

$$C = -\textstyle\sum_\alpha \frac{\cos\alpha(H)}{\sin\alpha(H)} K^\alpha \quad \text{and} \quad K^\alpha = V^\alpha + \theta V^\alpha.$$

Hence

$$[C,\, \Psi(H)] = \textstyle\sum_\alpha \frac{\alpha(\Psi(H))\cos\alpha(H)}{\sin\alpha(H)} Q^\alpha\,.$$

In addition, by (9), (15) and Lemma 4.3 (ii), one obtains

$$(\Psi_*)_H V_{\mathfrak{a}} + [C,\, \Psi(H)] = (\Psi_*)_H E_a^{-1} F_a V.$$

As a result,

$$-dd^c_J \rho_J(\widetilde{X}_z, \widetilde{W}_z) = B\big(X\,, (\Psi_*)_H E_a^{-1} F_a V\big)\,,$$

and, for $Z = X + iY$, one has

$$-dd^c_J \rho_J(\widetilde{Z}_z,\, \widetilde{W}_z) = -dd^c_J \rho_J(\widetilde{X}_z,\, \widetilde{W}_z) + dd^c_J \rho_J(\widetilde{Y}_z,\, \widetilde{iW}_z) =$$
$$= -\mathrm{Im} B\big(Z,\, (\Psi_*)_H E_a^{-1} F_a \overline{W}\big)\,.$$

From Lemma 5.2(i), it follows that $-dd^c_J \rho_J = \omega_J$, as desired. □

The plurisubharmonicity of ρ_J will also be proved in Proposition 7.1 where we show that $\omega_J(\,\cdot\,, J\,\cdot\,)$ is positive definite.

7. Proof of the theorem

In this section we carry out the proof of the main theorem, mainly by collecting results proved in the previous sections. As a preliminary step we show that the forms defined in Section 5 are Kähler with respect to the corresponding (almost) complex structures.

Proposition 7.1. *The G-invariant forms ω_I, ω_J, ω_K are Kähler with respect to the corresponding (almost) complex structures I, J, K. Moreover, they define the same Riemannian metric*

$$g(\,\cdot\,,\,\cdot\,) = \omega_I(\,\cdot\,, I\,\cdot\,) = \omega_J(\,\cdot\,, J\,\cdot\,) = \omega_K(\,\cdot\,, K\,\cdot\,)\,.$$

Proof. The invariance of ω_I, ω_J, ω_K with respect to the corresponding almost complex structures was shown in Lemma 5.3. Their closeness was proved in Lemma 5.2(iii) and Proposition 6.2. From Definition 5.1 it is easy to check that $\omega_I(\,\cdot\,, I\,\cdot\,) = \omega_J(\,\cdot\,, J\,\cdot\,) = \omega_K(\,\cdot\,, K\,\cdot\,)$. In order to prove that the forms define a Riemannian metric, note that

$$\omega_J((ga)_*X,\, J(ga)_* iV) = \mathrm{Im} B(I_0 X, F_a I_0 F_{a^{-1}} V) = 0\,,$$

for every $X \in \mathfrak{p}$ and $iV \in i\mathfrak{p}$. As ω_J is J-invariant, it is enough to check that

$$\omega_J((ga)_*X, J(ga)_*X) = \mathrm{Re} B(I_0 X, F_a I_0 F_{a^{-1}} X) > 0$$

for every $X \in \mathfrak{p}\backslash\{0\}$. Since the blocks of decomposition (1) are eigenspaces of the map F_a (see (9)) and are permuted by the complex structure I_0 (see (6)), it is sufficient to compute $\mathrm{Re}B(I_0P^\alpha, F_aI_0F_{a^{-1}}P^\alpha)$, for $\alpha \in \Sigma^+ \cup \{0\}$. For this, note that if $I_0P^\alpha \in \mathfrak{p}[\beta]$ (with the convention $\mathfrak{p}[0] = \mathfrak{a}$), one obtains

$$\mathrm{Re}B(I_0P^\alpha,\, F_aI_0F_a^{-1}P^\alpha) = \tfrac{\cos\beta(H)}{\cos\alpha(H)}B(I_0P^\alpha,\, I_0P^\alpha) = \tfrac{\cos\beta(H)}{\cos\alpha(H)}B(P^\alpha,\, P^\alpha)\,,$$

which is strictly positive for all $H \in \Omega$. $\square$

The next two lemmas are concerned with the uniqueness question for an arbitrary G-invariant hyper-Kähler structure $(\mathcal{I},\, \mathcal{J},\, \mathcal{K},\, \omega_\mathcal{I},\, \omega_\mathcal{J},\, \omega_\mathcal{K})$ with the property that $\mathcal{J} = J_{ad}$ and the restriction of the Kähler structure $(\mathcal{I},\, \omega_\mathcal{I})$ to $\mathfrak{p} \cong T_{eK^\mathbb{C}}G/K$ coincides with the standard Kähler structure $(I_0,\, \omega_0)$ of G/K.

Lemma 7.2. *Let G/K be an irreducible non-compact Hermitian symmetric space. Assume that $\omega_\mathcal{I}$ and $\omega_\mathcal{K}$ are elements of a G-invariant, hyper-Kähler structure on Ξ such that $\mathcal{J} = J_{ad}$. Then the $\mathcal{J}$-holomorphic symplectic form $\omega_\mathcal{I} - i\omega_\mathcal{K}$ is locally $G^\mathbb{C}$-invariant.*

Proof. Recall that the local action of $G^\mathbb{C}$ on Ξ is $\mathcal{J}$-holomorphic and the complex form $\omega_\mathcal{I} - i\omega_\mathcal{K}$ is G-invariant and $\mathcal{J}$-holomorphic. Since G is a real form of $G^\mathbb{C}$, the result follows from the analytic continuation principle. $\square$

Denote by $I_0 : \mathfrak{p}^\mathbb{C} \to \mathfrak{p}^\mathbb{C}$ and by $\bar{I}_0 : \mathfrak{p}^\mathbb{C} \to \mathfrak{p}^\mathbb{C}$ the linear and the anti-linear extension of $I_0 : \mathfrak{p} \to \mathfrak{p}$, respectively. Then $\bar{I}_0Z = I_0\overline{Z}$.

Lemma 7.3. *Assume that $\omega_\mathcal{I}$, $\omega_\mathcal{K}$ and $\mathcal{J}$ are elements of a G-invariant, hyper-Kähler structure on Ξ with $\mathcal{J} = J_{ad}$ and such that the Kähler structure $(\mathcal{I},\, \omega_\mathcal{I})$ coincides with $(I_0,\, \omega_0)$ when restricted to G/K. Then the restrictions of $\mathcal{I}$ and $\omega_\mathcal{I}$ to $\mathfrak{p}^\mathbb{C} \cong T_{eK^\mathbb{C}}G^\mathbb{C}/K^\mathbb{C}$ coincide with $\bar{I}_0$ and $\mathrm{Re}B(I_0\,\cdot\,,\,\cdot\,)$, respectively.*

Proof. Recall that the restriction of $\mathcal{J} = J_{ad}$ to $\mathfrak{p}^\mathbb{C}$ is multiplication by i. Since $\mathcal{I}\mathcal{J} = -\mathcal{J}\mathcal{I}$, for every $X \in \mathfrak{p}$ one has $\mathcal{I}iX = -i\mathcal{I}X = -iI_0X$. This says that the restriction of $\mathcal{I}$ to $\mathfrak{p}^\mathbb{C}$ coincides with $\bar{I}_0$.

In a hyper-Kähler structure the form $\omega_\mathcal{I}$ is anti-$\mathcal{J}$-invariant. This determines its restriction to $i\mathfrak{p} = \mathcal{J}\mathfrak{p}$.

Then, in order to show that its restriction to $\mathfrak{p}^\mathbb{C}$ coincides with $\mathrm{Re}B(I_0\,\cdot\,,\,\cdot\,)$, we are left to show that $\mathfrak{p}$ and $\mathcal{J}\mathfrak{p}$ are $\omega_\mathcal{I}$-orthogonal.

For this recall that, by Lemma 7.2, for every $t \in \mathbb{R}$ the form $\omega_\mathcal{I}$ is $\exp itZ_0$-invariant. Since for $X,\, Y \in \mathfrak{p}$ one has

$$\mathrm{Ad}_{\exp itZ_0}X = \cosh t\, X + \sinh t\, iI_0X \quad \text{and} \quad \mathrm{Ad}_{\exp itZ_0}Y = \cosh t\, Y + \sinh t\, iI_0Y\,,$$

it follows that

$$\omega_{\mathcal{I}}(X,\,Y)=\omega_{\mathcal{I}}(\cosh tX+\sinh t iI_0X,\,\cosh tY+\sinh t iI_0Y)=$$
$$\cosh^2 t\,\omega_{\mathcal{I}}(X,\,Y)-\sinh^2 t\,\omega_{\mathcal{I}}(I_0X,\,I_0Y)+$$
$$\cosh t\sinh t\,\omega_{\mathcal{I}}(X,\,\mathcal{J}I_0Y)+\cosh t\sinh t\,\omega_{\mathcal{I}}(\mathcal{J}I_0X,\,Y)\,.$$

Thus $\omega_{\mathcal{I}}(X,\,\mathcal{J}I_0Y)+\omega_{\mathcal{I}}(\mathcal{J}I_0X,\,Y)=0$ which, by

$$\omega_{\mathcal{I}}(X,\,\mathcal{J}I_0Y)=\omega_{\mathcal{I}}(\mathcal{J}X,\,I_0Y)=\omega_{\mathcal{I}}(\mathcal{J}I_0X,\,Y)\,,$$

is equivalent to $\omega_{\mathcal{I}}(\mathcal{J}X,\,I_0Y)=0$ for every $X,\,Y$ in $\mathfrak{p}$. That is, $\mathfrak{p}$ and $\mathcal{J}\mathfrak{p}$ are $\omega_{\mathcal{I}}$-orthogonal, as wished. □

Remark 7.4. *Under the assumptions of Lemma 7.3, the local $G^{\mathbb{C}}$-invariance of $\omega_{\mathcal{I}}$ and $\omega_{\mathcal{K}}$ (Lemma 7.2) implies that*

$$\omega_{\mathcal{I}}((ga)_*Z,\,(ga)_*W)=\mathrm{Re}B(I_0Z,\,W)\,,$$

and

$$\omega_{\mathcal{K}}((ga)_*Z,\,(ga)_*W)=\omega_{\mathcal{K}}(Z,\,W)=\omega_{\mathcal{I}}(\mathcal{I}Z,\,\mathcal{I}\mathcal{J}W)=\omega_{\mathcal{I}}(Z,\,\mathcal{J}W)=-\mathrm{Im}B(I_0Z,\,W)\,.$$

Proof of the main Theorem. Let $J=J_{ad}$ be the adapted complex structure on Ξ, let I be the almost complex structure defined in Section 3 and $K:=IJ$. Then the usual algebraic properties

$$I^2=J^2=K^2=-Id=IJK$$

follow directly from Definition 3.1. By Proposition 7.1, the 2-forms ω_I, ω_J and ω_K defined in Section 5 are Kähler with respect to the corresponding almost complex structures and all define the same Riemannian metric. In addition, they are closed in view of Lemma 5.2(iii) and Proposition 6.2. Now Lemma 6.8 in [Hit87] implies that I, J and K are integrable. This concludes the proof of the existence of a hyper-Kähler structure on Ξ with the required properties. The proof of Part (c) in the main theorem can be found in Section 8.

Finally, we outline a proof of uniqueness of the adapted hyper-Kähler structure. Let

$$(\mathcal{I},\,\mathcal{J},\,\mathcal{K},\,\omega_{\mathcal{I}},\,\omega_{\mathcal{J}},\,\omega_{\mathcal{K}})$$

be an arbitrary G-invariant hyper-Kähler structure with the property that $\mathcal{J}=J_{ad}$ and the restriction of the Kähler structure $(\mathcal{I},\,\omega_{\mathcal{I}})$ to $\mathfrak{p}$ coincides with the standard Kähler structure $(I_0,\,\omega_0)$ of G/K. By Lemma 7.2 and Lemma 7.3, the restriction of $(\mathcal{I},\,\omega_{\mathcal{I}})$ to $\mathfrak{p}^{\mathbb{C}}$ necessarily coincides with $(\bar{I}_0,\,\mathrm{Re}B(I_0\,\cdot\,,\,\cdot))$. Moreover, the forms $\omega_{\mathcal{I}}$ and $\omega_{\mathcal{K}}$, being locally $G^{\mathbb{C}}$-invariant, are uniquely determined everywhere on Ξ (see Rem. 7.4). Then the relations $\mathcal{K}=\mathcal{I}\mathcal{J}$ and $\omega_{\mathcal{J}}(\cdot,\cdot)=\omega_{\mathcal{I}}(\mathcal{I}\cdot,\cdot)$ show that the hyper-Kähler structure is uniquely determined if the complex structure $\mathcal{I}$ is. In turn, because of its G-invariance, $\mathcal{I}$ is uniquely determined by the map $\overline{L} : \Omega \to GL_{\mathbb{R}}(\mathfrak{p}^{\mathbb{C}})$, given by $H \to \overline{L}_H$, which describes $\mathcal{I}$ along the slice. That is,

$$\mathcal{I}a_* Z = a_* \overline{L}_H Z.$$

In our hyper-Kähler setting, the operator $\overline{L}_H$ is an anti-linear anti-involution and the form $\mathrm{Re}B(I_0\,\cdot,\,\cdot,\ )$ is $\overline{L}_H$-invariant. Moreover, the condition $d\omega_{\mathcal{J}} = 0$ yields a system of first order differential equations in the real analytic components of $\overline{L}_H$ with initial conditions $\overline{L}_0 = \bar{I}_0$. Then the uniqueness of the solution can be obtained by applying Cauchy-Kowaleskaya theorem. In the case of $G = SL_2(\mathbb{R})$ the details of this strategy are carried out in Appendix A. $\square$

8. A potential for ω_I.

In this section we determine a G-invariant function ρ_I with the property that $\omega_I = -dd^c_I\rho_I + p^*\omega_0$, where ω_0 is the standard G-invariant Kähler form on G/K. Since the projection $p : \Xi \to G/K$ is holomorphic (Prop. 3.2), a potential of $p^*\omega_0$ is given by the pull-back of a potential ρ_0 of ω_0. Then $\rho_0 \circ p + \rho_I$ is a potential of ω_I.

We adopt the same strategy used in Section 6. As a preliminary step, for a class of smooth G-invariant functions $\rho : \Xi \to \mathbb{R}$, we determine the associated function $\mu^X : \Xi \to \mathbb{R}$ defined by $\mu^X(z) := d^c_I\rho(\widetilde{X}_z)$, for $X \in \mathfrak{g}$ (cf. Sect. 6).

Lemma 8.1. *Given a smooth function $f : \mathbb{R} \to \mathbb{R}$, consider the G-invariant function $\rho : \Xi \to \mathbb{R}$ defined by*

$$\rho(gaK^{\mathbb{C}}) := -\tfrac{1}{4}\textstyle\sum_{j=1}^r f(\lambda_j(H))B(A_j, A_j)\,.$$

For X in $\mathfrak{g}$ one has

$$\mu^X(gaK^{\mathbb{C}}) = \tfrac{1}{2}\textstyle\sum_{j=1}^r \widetilde{f}(\lambda_j(H))B(\mathrm{Ad}_{g^{-1}}X,\, [I_0A_j,\, H])\,,$$

where $\widetilde{f}(t) = \frac{\sin t}{t\cos t}f'(t)$.

Proof. The G-invariance of ρ and I implies that,

$$d^c_I\rho(\widetilde{X}_{g\cdot z}) = d^c_I\rho(\widetilde{\mathrm{Ad}_{g^{-1}}X}_z)$$

for every $z \in \Xi$ and $g \in G$. So it is sufficient to prove the lemma for $g = e$ and $z = aK^{\mathbb{C}}$, with $a = \exp iH$ and $H \in \Omega'$ (see (12)).

Take $X \in \mathfrak{p}$. Since $I\widetilde{X}_z = \widetilde{I_0X}_z$, the G-invariance of ρ implies that $d^c_I\rho(\widetilde{X}_z) = d\rho(\widetilde{I_0X}_z) = 0$.

Next, take $X \in \mathfrak{k}$ and decompose it as $X = M + \sum_\alpha K^\alpha$, as in (1), with $M \in \mathfrak{m}$ and $K^\alpha = X^\alpha + \theta X^\alpha$ in $\mathfrak{k}[\alpha]$. Since $\widetilde{M}_z = 0$ for $z = aK^{\mathbb{C}}$, one has $d^c_I\rho(\widetilde{M}_z) = 0$.

For $\alpha \neq \lambda_1, \ldots, \lambda_r$ and $K^\alpha = X^\alpha + \theta X^\alpha$ in $\mathfrak{k}[\alpha]$, set $P^\alpha = X^\alpha - \theta X^\alpha$ in $\mathfrak{p}[\alpha]$. Also set $I_0 P^\alpha =: P^\beta = X^\beta - \theta X^\beta$ in $\mathfrak{p}[\beta]$ and $K^\beta = X^\beta + \theta X^\beta$. Note that $\beta \neq 0$. Then by (11), Lemma 2.2(iii) and (7) one has

$$d^c_I \rho(\widetilde{K^\alpha}_z) = -d\rho(I a_* \sin\alpha(H) i P^\alpha) = d\rho(a_* \sin\alpha(H) i F_a I_0 F_a^{-1} P^\alpha) =$$
$$d\rho\big(a_* \tfrac{\sin\alpha(H)\cos\beta(H)}{\cos\alpha(H)} i P^\beta\big) = -d\rho\big(\tfrac{\sin\alpha(H)\cos\beta(H)}{\sin\beta(H)\cos\alpha(H)} \widetilde{K^\beta}_z\big) = 0\,.$$

For $\alpha = \lambda_l$, with $l = 1, \ldots r$, one has

$$d^c_I \rho(\widetilde{K^l}_z) = d\rho\big(a_* \sin\lambda_l(H) i F_a I_0 F_a^{-1} P_l\big) = d\rho\big(a_* \tfrac{\sin\lambda_l(H)}{\cos\lambda_l(H)} i A_l\big) =$$
$$-\tfrac{1}{4} \tfrac{d}{ds}\big|_{s=0} \textstyle\sum_j f\big(\lambda_j\big(H + s\tfrac{\sin\lambda_l(H)}{\cos\lambda_l(H)} A_l\big)\big) B(A_j, A_j) =$$
$$-\tfrac{1}{2} \tfrac{\sin\lambda_l(H)}{\cos\lambda_l(H)} f'(\lambda_l(H)) B(A_l,\, A_l)\,.$$

Since

$$\begin{aligned} B(A_l,\, A_l) =& B(I_0 A_l,\, I_0 A_l) = \\ -\tfrac{1}{\lambda_l(H)} B([H,\, K^l],\ I_0 A_l) =& -\tfrac{1}{\lambda_l(H)} B(K^l,\, [I_0 A_l,\, H])\,, \end{aligned} \tag{18}$$

the above computation yields

$$d^c_I \rho(\widetilde{X}_{aK^{\mathbb{C}}}) = \tfrac{1}{2} \textstyle\sum_j \widetilde{f}(\lambda_j(H)) B(X,\, [I_0 A_j,\, H])\,,$$

as desired. □

Proposition 8.2. *Fix a function* $f_I : \mathbb{R} \to \mathbb{R}$ *with the property that* $\frac{\sin t}{\cos t} f_I'(t) = \cos t - 1$. *The* G*-invariant function* $\rho_I : \Xi \to \mathbb{R}$ *defined by*

$$\rho_I(gaK^{\mathbb{C}}) := -\tfrac{1}{4} \textstyle\sum_{j=1}^r f_I(\lambda_j(H)) B(A_j, A_j)\,,$$

is a potential for $\omega_I - p^*\omega_0$. *In particular the form* $\omega_I = -dd^c_I \rho_I + p^*\omega_0$ *is closed (cf. Prop.* 5.2 *(iii)).*

Proof. We are going to show that $-dd^c_I \rho_I = \omega_I - p^*\omega_0$. By the G-invariance of the forms involved it is sufficient to prove the statement for $z = aK^{\mathbb{C}}$, with $a = \exp iH$ and $H \in \Omega'$ (see (12)). Observe that

$$p^*\omega_0(\widetilde{Z}_z,\, \widetilde{W}_z) = \omega_0(X,\, U) = B(I_0 X,\, U)\,, \tag{19}$$

for every $Z = X + iY$ and $W = U + iV$ in $\mathfrak{p}^{\mathbb{C}}$. Also recall that by Lemma 6.1, for every $X \in \mathfrak{p}$ and $W \in \mathfrak{p}^{\mathbb{C}}$ one has

$$-dd^c_I \rho_I(\widetilde{X}_z,\, \widetilde{W}_z) = d\mu_I^X(\widetilde{W}_z)\,,$$

where the map μ_I^X can be computed as in Lemma 8.1.

Fix a point $z = aK^{\mathbb{C}}$ in the slice in Ξ'. We perform the computation exploiting the block decomposition of $\mathfrak{p}$ given in (1) with the convention $\mathfrak{p}[0] = \mathfrak{a}$.

- $\boldsymbol{-dd^c_I \rho_I = \omega_I - \pi^*\omega_0}$ **on** $\boldsymbol{a_*\mathfrak{p} \times a_*\mathfrak{p}}$**.**

For $X,\, U \in \mathfrak{p}$, by Lemma 6.1 and Lemma 8.1 one has

$$-dd^c_I\rho_I(\widetilde{X}_z,\,\widetilde{U}_z)=\tfrac{d}{ds}\big|_{s=0}\mu_I^X(\exp sUaK^{\mathbb{C}})=$$
$$=\tfrac{1}{2}\tfrac{d}{ds}\big|_{s=0}\textstyle\sum_j\widetilde{f}_I(\lambda_j(H))B(X-s[U,\,X],\,[I_0A_j,\,H])=$$
$$-\tfrac{1}{2}\textstyle\sum_{j=1}^r\widetilde{f}_I(\lambda_j(H))B(X,\,[[I_0A_j,\,H],U])\,.$$

Moreover

$$B(X,\,[[I_0A_j,\,H],U])=-B(X,\,[[U,\,I_0A_j],\,H]+[[H,\,U],\,I_0A_j])=$$
$$-B([H,\,X],\,[U,\,I_0A_j])-B([I_0A_j,\,X],\,[H,U])=$$
$$-B([H,\,X],\,[A_j,\,I_0U])+B([A_j,\,I_0X],\,[H,U])=$$
$$B([A_j,\,[H,\,X]],\,I_0U)-B(I_0X,\,[A_j,\,[H,\,U]])$$

which, for $X=P^\alpha\in\mathfrak{p}[\alpha]$ and $U=Q^\beta\in\mathfrak{p}[\beta]$, becomes

$$-\big(\alpha(A_j)\alpha(H)+\beta(A_j)\beta(H)\big)B(I_0P^\alpha,\,Q^\beta)\,.$$

Thus one obtains

$$-dd^c_I\rho_I(\widetilde{P^\alpha}_z,\,\widetilde{Q^\beta}_z)=\tfrac{1}{2}\textstyle\sum_j\widetilde{f}_I(\lambda_j(H))\big(\alpha(A_j)\alpha(H)+\beta(A_j)\beta(H)\big)B(I_0P^\alpha,\,Q^\beta)\,.$$

It is clear that $-dd^c_I\rho_I(\widetilde{A}_z,\,\widetilde{A'}_z)=0$, for all $A,\,A'$ in $\mathfrak{a}$. In view of relations (1), (6) and (7), we are left to check the following cases.

Case $\boldsymbol{\alpha=0}$ and $\boldsymbol{\beta=\lambda_k}$. The above computation and our assumption on f_I imply

$$-dd^c_I\rho_I(\widetilde{A}_z,\,\widetilde{P^k}_z)=\tfrac{1}{2}\widetilde{f}_I(\lambda_k(H))2\lambda_k(H)B(I_0A,\,P^k)=(\cos\lambda_k(H)-1)B(I_0A,\,P^k)=$$
$$=\omega_I(\widetilde{A}_z,\,\widetilde{P^k}_z)-p^*\omega_0(\widetilde{A}_z,\,\widetilde{P^k}_z)\,,$$

where the last identity follows from (19) and

$$\cos\lambda_k(H)B(I_0A,\,P^k)=B(I_0F_aA,\,F_aP^k)=\omega_I(\widetilde{A}_z,\,\widetilde{P^k}_z)\,.$$

Case $\boldsymbol{\alpha=e_k+e_l}$ and $\boldsymbol{\beta=e_k-e_l}$. Since $\lambda_k=\alpha+\beta$ and $\lambda_l=\alpha-\beta$, one has

$$-dd^c_I\rho_I(\widetilde{P^\alpha}_z,\,\widetilde{Q^\beta}_z)=\tfrac{1}{2}\big(\widetilde{f}_I(\lambda_k(H))(\alpha(H)+\beta(H))+\widetilde{f}_I(\lambda_l(H))(\alpha(H)-\beta(H))\big)B\big(I_0P^\alpha,\,Q^\beta\big)=$$
$$\tfrac{1}{2}\big(\widetilde{f}_I(\lambda_k(H))\lambda_k(H)+\widetilde{f}_I(\lambda_l(H))\lambda_l(H)\big)B(I_0P^\alpha,\,Q^\beta)=$$
$$\big(\tfrac{\cos\lambda_k(H)+\cos\lambda_l(H)}{2}-1\big)B(I_0P^\alpha,\,Q^\beta)\,.$$

By the trigonometric identity

$$\cos 2t_k+\cos 2t_l=2\cos(t_k+t_l)\cos(t_k-t_l)$$

one obtains

$$\tfrac{1}{2}(\cos\lambda_k(H)+\cos\lambda_l(H))B(I_0P^\alpha,\,Q^\beta)=B(I_0F_aP^\alpha,\,F_aQ^\beta)=\omega_I(\widetilde{P^\alpha}_z,\,\widetilde{Q^\beta}_z)\,.$$

Hence

$$-dd^c_I\rho_I(\widetilde{P^\alpha}_z,\,\widetilde{Q^\beta}_z)=\omega_I(\widetilde{P^\alpha}_z,\,\widetilde{Q^\beta}_z)-p^*\omega_0(\widetilde{P^\alpha}_z,\,\widetilde{Q^\beta}_z)\,.$$

Case $\boldsymbol{\alpha} = \boldsymbol{e_k} = \boldsymbol{\beta}$. Since $\lambda_k = 2\alpha$, one has

$$-dd^c_I\rho_I(\widetilde{P^\alpha}_z,\, \widetilde{Q^\alpha}_z) = \tfrac{1}{2}\widetilde{f}_I(\lambda_k(H))\lambda_k(H)B(I_0P^\alpha,\, Q^\alpha) = \tfrac{1}{2}\big(\cos\lambda_k(H){-}1\big)B(I_0P^\alpha,\, Q^\alpha) =$$
$$(\cos^2\alpha(H) - 1)B(I_0P^\alpha,\, Q^\alpha) = \omega_I(\widetilde{P^\alpha}_z,\, \widetilde{Q^\alpha}_z) - p^*\omega_0(\widetilde{P^\alpha}_z,\, \widetilde{Q^\alpha}_z)$$

where in the last equality one uses

$$\cos^2\alpha(H)B(I_0P^\alpha,\, Q^\alpha) = B(I_0F_aP^\alpha,\, F_aQ^\alpha) = \omega_I(\widetilde{P^\alpha}_z,\, \widetilde{Q^\alpha}_z)\,.$$

- $\boldsymbol{-\,dd^c_I\rho_I = 0}$ **on** $\boldsymbol{a_*\mathfrak{p} \times a_*i\mathfrak{p}}$.

Take $X \in \mathfrak{p}$ and $iV \in i\mathfrak{p}$. By Lemma 2.2(i) one has $\widetilde{iV}_z = \frac{d}{ds}\big|_{s=0}\exp sC\exp i(H+sV_\mathfrak{a})K^\mathbb{C}$, with $C \in \mathfrak{k}$. Thus

$$-dd^c\rho_I(\widetilde{X}_z,\, \widetilde{iV}_z) = \tfrac{d}{ds}\big|_{s=0}\mu_I^X(\exp sC\exp i(H+sV_\mathfrak{a})K^\mathbb{C}) =$$
$$\tfrac{1}{2}\tfrac{d}{ds}\big|_{s=0}\textstyle\sum_j \widetilde{f}(\lambda_j(H+sV_\mathfrak{a}))B(Ad_{\exp -sC}X, [I_0A_j,\, H+sV_\mathfrak{a}]) =$$
$$\tfrac{1}{2}\textstyle\sum_j \widetilde{f}'(\lambda_j(H))\lambda_j(V_\mathfrak{a})B(X,\, [I_0A_j,\, H]) + \widetilde{f}(\lambda_j(H))\big(B([X,C],[I_0A_j,\, H]) + B(X,[I_0A_j,\, V_\mathfrak{a}])\big)\,.$$

All the above summands are zero, since $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ and $B(\mathfrak{p},\mathfrak{k}) \equiv 0$.

- $\boldsymbol{-\,dd^c_I\rho_I = \omega_I}$ **on** $\boldsymbol{a_*i\mathfrak{p} \times a_*i\mathfrak{p}}$.

Case $\boldsymbol{\alpha} = \boldsymbol{0} = \boldsymbol{\beta}$. We show that $-dd^c_I\rho_I(\widetilde{iA}_z,\, \widetilde{iA'}_z) = 0$ for every $A,\, A' \in \mathfrak{a}$. Since $[\widetilde{iA},\, \widetilde{iA'}] = [\widetilde{iA,\, iA'}] = 0$, the usual formula for the exterior derivation applied to the 1-form $d^c_I\rho_I$ yields

$$dd^c_I\rho_I(\widetilde{iA}_z,\, \widetilde{iA'}_z) = \widetilde{iA}_z(d^c_I\rho_I(\widetilde{iA'})) - \widetilde{iA'}_z(d^c_I\rho_I(\widetilde{iA})) =$$
$$= \tfrac{d}{dt}\big|_{t=0}d\rho_I\big(I\tfrac{d}{ds}\big|_{s=0}\exp siA'\exp tiAaK^\mathbb{C} - I\tfrac{d}{ds}\big|_{s=0}\exp siA\exp tiA'aK^\mathbb{C}\big) =$$
$$= \tfrac{d}{dt}\big|_{t=0}d\rho_I\big(\tfrac{d}{ds}\big|_{s=0}\exp siI_0A'a\exp tiAK^\mathbb{C} - \tfrac{d}{ds}\big|_{s=0}\exp siI_0Aa\exp tiA'K^\mathbb{C}\big)\,.$$

By Lemma 2.2(ii), for $A = A_k$ and $A' = A_l$ the above expression equals to

$$-\tfrac{d}{dt}\big|_{t=0}\tfrac{d}{ds}\big|_{s=0}\rho_I\big(\exp s\tfrac{\cos\lambda_l(H+tA_k)}{\sin\lambda_l(H+tA_k)}K^l\exp i(H+tA_k)K^\mathbb{C}\big)$$
$$+\tfrac{d}{dt}\big|_{t=0}\tfrac{d}{ds}\big|_{s=0}\rho_I\big(\exp s\tfrac{\cos\lambda_k(H+tA_l)}{\sin\lambda_k(H+tA_l)}K^k\exp i(H+tA_l)K^\mathbb{C}\big)\,,$$

which vanishes by the G-invariance of ρ_I.

Case $\boldsymbol{\alpha} \neq \boldsymbol{0}$ and $\boldsymbol{\beta} = \boldsymbol{0}$. Take $iP^\alpha \in i\mathfrak{p}$, with $\alpha \neq 0$, and $A_k \in \mathfrak{a}$. Then, by Lemma 2.2(iii)

$$-dd^c_I\rho_I(a_*iP^\alpha,\, a_*iA_k) = \tfrac{1}{\sin\alpha(H)}dd^c_I\rho_I((\widetilde{K^\alpha})_z, (\widetilde{A_k})_z) =$$
$$-\tfrac{1}{\sin\alpha(H)}\tfrac{d}{ds}\big|_{s=0}\mu_I^{K^\alpha}(\exp isA_k\, aK^\mathbb{C}) =$$
$$-\tfrac{1}{2\sin\alpha(H)}\tfrac{d}{ds}\big|_{s=0}\textstyle\sum_j \widetilde{f}(\lambda_j(H+sA_k))B(K^\alpha,[I_0A_j,\, H+sA_k]) =$$
$$-\tfrac{1}{2\sin\alpha(H)}\textstyle\sum_j \widetilde{f}'(\lambda_j(H))\lambda_j(A_k)B(K^\alpha,[I_0A_j,\, H]) + \widetilde{f}(\lambda_j(H))B(K^\alpha,[I_0A_j,\, A_k])\,.$$

Since $\lambda_j(A_k) = 0 = B(K^\alpha, [I_0A_j, A_k])$ for $j \neq k$, the above sum reduces to

$$-\tfrac{1}{2\sin\lambda_k(H)}\big(\widetilde{f}'(\lambda_k(H))2B(K^\alpha, [I_0A_k\, H]) + \widetilde{f}(\lambda_k(H))\big)B(K^\alpha, [I_0A_k\, A_k]) =$$

$$= \tfrac{1}{2\sin\lambda_k(H)}\big(\widetilde{f}'(\lambda_k(H))2\alpha(H) + \widetilde{f}(\lambda_k(H))\alpha(A_k)\big)B(I_0P^\alpha,\ A_k).$$

This expression vanishes for $\alpha \neq \lambda_k$, while for $P^\alpha = P^k \in \mathfrak{p}[\lambda_k]$ becomes

$$\tfrac{1}{\sin\lambda_k(H)}\big(\widetilde{f}'(\lambda_k(H))\lambda_k(H) + \widetilde{f}(\lambda_k(H))\big)B(I_0P^k,\ A_k).$$

Under our assumption $\widetilde{f}(t) = \frac{1}{t}(\cos t - 1)$, one has $\widetilde{f}'(t) = -\frac{1}{t^2}(\cos t - 1) - \frac{\sin t}{t}$, and consequently

$$\tfrac{1}{\sin t}\big(\widetilde{f}'(t)t + \widetilde{f}(t)\big) = -1\,.$$

Thus

$$-dd^c_I\rho_I(a_*iP^\alpha,\, a_*iA_k) = -B(I_0P^k,\, A_k) = \omega_I(a_*iP^k,\, a_*iA_k)\,.$$

Case $\boldsymbol{\alpha \neq 0 \neq \beta}$. Next, consider $iP^\alpha,\ iQ^\beta \in i\mathfrak{p}$, with $\alpha \neq 0 \neq \beta$. By Lemma 2.2(i) one has $a_*iP^\alpha = -\frac{1}{\sin\alpha(H)}\widetilde{K}^\alpha$ and $a_*iQ^\beta = -\frac{1}{\sin\beta(H)}\widetilde{C}^\beta$, for appropriate $K^\alpha \in \mathfrak{k}[\alpha]$ and $C^\beta \in \mathfrak{k}[\beta]$. Then

$$-dd^c_I\rho_I(a_*iP^\alpha,\, a_*iQ^\beta) = \tfrac{1}{\sin\alpha(H)\sin\beta(H)}\tfrac{d}{ds}\Big|_{s=0}\mu_I^{K^\alpha}(\exp sC^\beta\, aK^{\mathbb{C}}) =$$

$$-\tfrac{1}{2\sin\alpha(H)\sin\beta(H)}\textstyle\sum_j \widetilde{f}(\lambda_j(H))B\big([C^\beta,\, K^\alpha],\, [I_0A_j,\, H]\big) =$$

$$= -\tfrac{1}{2\sin\alpha(H)\sin\beta(H)}\textstyle\sum_j \widetilde{f}(\lambda_j(H))B\big(K^\alpha,\, [[I_0A_j,\, H],\, C^\beta]\big)\,.$$

By writing

$$B\big(K^\alpha,\, [[I_0A_j,\, H],\, C^\beta]\big) = -B\big(K^\alpha,\, [[C^\beta,\, I_0A_j],\, H] + [[H,\, C^\beta],\, I_0A_j]\big) =$$

$$-\big(\alpha(H)\beta(A_j) + \beta(H)\alpha(A_j)\big)B\big(I_0P^\alpha,\, Q^\beta\big)$$

one obtains

$$-dd^c_I\rho_I(a_*iP^\alpha,\, a_*iQ^\beta) =$$

$$\tfrac{1}{2\sin\alpha(H)\sin\beta(H)}\textstyle\sum_j \widetilde{f}(\lambda_j(H))\big(\alpha(H)\beta(A_j) + \beta(H)\alpha(A_j)\big)B(I_0P^\alpha,\, Q^\beta).$$

In view of relations (1), (6) and (7), we are left to check the following cases.

Case $\boldsymbol{\alpha = e_k + e_l}$ and $\boldsymbol{\beta = e_k - e_l}$. Since $\alpha + \beta = \lambda_k$ and $-\alpha + \beta = -\lambda_l$, one has

$$-dd^c_I\rho_I(a_*iP^\alpha,\, a_*iQ^\beta) = \tfrac{1}{2\sin\alpha(H)\sin\beta(H)}\big(\widetilde{f}(\lambda_k(H))\lambda_k(H) - \widetilde{f}(\lambda_l(H))\lambda_l(H)\big)B(I_0P^\alpha,\, Q^\beta)$$

$$\tfrac{\cos\lambda_k(H) - \cos\lambda_l(H)}{2\sin\alpha(H)\sin\beta(H)}B(I_0P^\beta,\, Q^\beta) = -B(I_0P^\beta,\, Q^\beta) = \omega_I(a_*iP^\alpha,\, a_*iQ^\beta),$$

due to the trigonometric identity $\cos 2t_k - \cos 2t_l = -2\sin(t_k + t_l)\sin(t_k - t_l)$.

Case $\boldsymbol{\alpha = e_k = \beta}$. One has

$$-dd^c_I\rho_I(a_*iP^\alpha,\, a_*iQ^\alpha) = \tfrac{1}{2\sin^2\alpha(H)}\widetilde{f}(\lambda_k(H))\lambda_k(H)B(I_0P^\alpha,\, Q^\alpha)$$

$$= \tfrac{1}{2\sin^2\alpha(H)}(\cos 2\alpha(H) - 1)B(I_0P^\alpha, Q^\alpha)$$

$$= -B(I_0 P^\alpha,\, Q^\alpha) = \omega_I(a_* iP^\alpha,\, a_* iQ^\alpha)\,.$$

□

9. Appendix A: a proof of uniqueness of the adapted hyper-Kähler structure for $G = SL_2(\mathbb{R})$.

Here we carry out a proof of uniqueness of the adapted hyper-Kähler structure in the case of $G = SL_2(\mathbb{R})$, as announced in Section 7.

Consider the map $\mathfrak{p}\times\mathfrak{k}\times\Omega^+ \to \Xi$, given by $(U, C, H) \to \exp U \exp C \exp iHK^{\mathbb{C}}$, which is an analytic diffeomorphism of a neighborhood of $\{0\} \times \{0\} \times \Omega^+$ onto its image Ω'' (cf. [KrSt05], Cor. 4.2). On Ω'' we consider the vector fields

$$\breve{A}_{\exp U \exp C \exp iHK^{\mathbb{C}}} := \frac{d}{ds}\Big|_{s=0} \exp(U + sA) \exp C \exp iHK^{\mathbb{C}}\,,$$

$$\breve{P}_{\exp U \exp C \exp iHK^{\mathbb{C}}} := \frac{d}{ds}\Big|_{s=0} \exp(U + sP) \exp C \exp iHK^{\mathbb{C}}\,,$$

$$\breve{K}_{\exp U \exp C \exp iHK^{\mathbb{C}}} := \frac{d}{ds}\Big|_{s=0} \exp U \exp(C + sK) \exp iHK^{\mathbb{C}}\,,$$

$$\breve{iA}_{\exp U \exp C \exp iHK^{\mathbb{C}}} := \frac{d}{ds}\Big|_{s=0} \exp(U) \exp C \exp i(H + sA)K^{\mathbb{C}}\,,$$

where $A := [\theta E,\, E]$, $P = E - \theta E$ and $K = E + \theta E$. In particular $I_0 A = -P$, $I_0 P = A$. Moreover $[A,\, K] = \alpha(A)P = 2P$ and $[A,\, P] = \alpha(A)K = 2K$ (see (2), (4) and (6)). All above vector fields commute, since they are push-forward of coordinate vector fields in the product $\mathfrak{p} \times \mathfrak{k} \times \Omega^+$.

Proof of uniqueness of the adapted hyper-Kähler structure (case of $G = SL_2(\mathbb{R})$). Let

$$(\mathcal{I},\, \mathcal{J},\, \mathcal{K},\, \omega_{\mathcal{I}},\, \omega_{\mathcal{J}},\, \omega_{\mathcal{K}})$$

be an arbitrary G-invariant hyper-Kähler structure with the property that $\mathcal{J} = J_{ad}$ and the restriction of the Kähler structure $(\mathcal{I},\, \omega_{\mathcal{I}})$ to $\mathfrak{p}$ coincides with the standard Kähler structure $(I_0,\, \omega_0)$ of G/K. Consider the map $\overline{L} : \Omega \to GL_{\mathbb{R}}(\mathfrak{p}^{\mathbb{C}})$ which describes $\mathcal{I}$ along the slice by

$$\mathcal{I}a_* Z = a_* \overline{L}_H Z\,,$$

where $H \in \Omega$ and $a = \exp iH$. As observed in the proof of the main Theorem in Section 7, we need to show that for every H in Ω and $Z \in \mathfrak{p}^{\mathbb{C}}$ one has $\overline{L}_H Z = F_a I_0 F_a^{-1} \overline{Z}$. Note that

$$\omega_{\mathcal{J}}(a_* Z\,,\, a_* W) = \omega_{\mathcal{I}}(\mathcal{J} a_* Z\,,\, \mathcal{I} a_* W) = -\mathrm{Im} B(I_0 Z,\, \overline{L}_H W)\,. \tag{20}$$

Claim. With respect to the basis

$$\{A,\, P,\, iA,\, iP\}$$

of $\mathfrak{p}^{\mathbb{C}}$, the anti-linear anti-involution $\overline{L}_H$ of $\mathfrak{p}$ is represented by the matrix

$$\begin{pmatrix} a_1 & a_2 & b_1 & 0 \\ a_3 & -a_1 & 0 & b_1 \\ b_1 & 0 & -a_1 & -a_2 \\ 0 & b_1 & -a_3 & a_1 \end{pmatrix},$$

where a_1, a_2, a_3 and b_1 are real-analytic functions of H and $b_1^2+a_1^2+a_2a_3=-1$.

Proof of the claim. Let

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be the representative matrix of L_H with respect to the above basis, which is compatible with the decomposition $\mathfrak{p} \oplus i\mathfrak{p}$. Since $\mathcal{I}\mathcal{J} = -\mathcal{J}\mathcal{I}$, it follows that $\overline{L}_H \mathcal{J} Z = -\mathcal{J}\overline{L}_H Z$ for every Z in $\mathfrak{p}$, i.e. $\overline{L}_H$ is anti-linear. This implies that $C = B$ and $D = -A$, where

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

Since $\omega_{\mathcal{J}}(\,\cdot\,,\,\cdot\,)$ is skew-symmetric, (20) implies that

$$\mathrm{Im}B(I_0 Z,\, \overline{L}_H W) = -\mathrm{Im}B(I_0 W,\, \overline{L}_H Z) = -\mathrm{Im}B(\overline{L}_H Z,\, I_0 W)$$

for every $Z, W \in \mathfrak{p}^{\mathbb{C}}$. As ${}^tI_0 = -I_0$, one obtains

$$-\begin{pmatrix} I_0 & 0 \\ 0 & I_0 \end{pmatrix}\begin{pmatrix} 0 & Id \\ Id & 0 \end{pmatrix}\begin{pmatrix} A & B \\ B & -A \end{pmatrix} = -\begin{pmatrix} {}^tA & {}^tB \\ {}^tB & -{}^tA \end{pmatrix}\begin{pmatrix} 0 & Id \\ Id & 0 \end{pmatrix}\begin{pmatrix} I_0 & 0 \\ 0 & I_0 \end{pmatrix},$$

which implies $I_0 A = -{}^tA I_0$ and $I_0 B = {}^tB I_0$. Thus the matrix realization of $\overline{L}_H$ is as claimed and the relation $b_1^2 + a_1^2 + a_2 a_3 = -1$ follows from the fact that $(\overline{L}_H)^2 = -Id$. This concludes the proof of the claim.

Then in order to conclude the proof, we need to show that the functions a_1 and b_1 identically vanish and $a_3(H) = -\cos\alpha(H)$ (recall that $I_0 A = -P$ and $I_0 P = A$). This will be done by showing that such functions are solutions of a system of differential equations with initial conditions $a_1(0) = 0 = b_1(0)$, $a_3(0) = -1$. Without loss of generality, in the sequel we assume that the Killing form B is normalized by $B(A, A) = B(P, P) = 1$.

• $\boldsymbol{b_1} \equiv \mathbf{0}.$ Let $z = aK^{\mathbb{C}} \in \Xi''$, with $a = \exp iH$. Since the vector fields $\breve{A}$, $\breve{P}$, $\breve{iA}$ commute and $\omega_{\mathcal{J}}$ is closed, the classical Cartan's formula gives

$$d\omega_{\mathcal{J}}(\breve{A}_z,\, \breve{P}_z,\, \breve{iA}_z) = \breve{A}_z\omega_{\mathcal{J}}(\breve{P},\, \breve{iA}) - \breve{P}_z\omega_{\mathcal{J}}(\breve{A},\, \breve{iA}) + \breve{iA}_z\omega_{\mathcal{J}}(\breve{A},\, \breve{P}) = 0\,.$$

One has

$$d\omega_{\mathcal{J}}(\breve{A}_z,\, \breve{P}_z,\, \breve{iA}_z) = \tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\breve{P}_{\exp tAaK^{\mathbb{C}}},\, \breve{iA}_{\exp tAaK^{\mathbb{C}}}) +$$
$$-\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\breve{A}_{\exp tPaK^{\mathbb{C}}},\, \breve{iA}_{\exp tPaK^{\mathbb{C}}}) + \tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\breve{A}_{\exp i(H+tA)K^{\mathbb{C}}},\, \breve{P}_{\exp i(H+tA)K^{\mathbb{C}}}) =$$

$$= \tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\tfrac{d}{ds}\big|_{s=0}\exp(tA+sP)aK^{\mathbb{C}},\ \tfrac{d}{ds}\big|_{s=0}\exp tA\exp i(H+sA)K^{\mathbb{C}})+$$
$$-\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\tfrac{d}{ds}\big|_{s=0}\exp(tP+sA)aK^{\mathbb{C}},\ (\exp tPa)_*iA)+$$
$$+\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\exp i(H+tA)_*A,\ \tfrac{d}{ds}\big|_{s=0}\exp sP\exp i(H+tA)K^{\mathbb{C}}) =$$

$$= \tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\tfrac{d}{ds}\big|_{s=0}\exp tA\exp s(P-\tfrac{t}{2}[A,P]+O(t^2))aK^{\mathbb{C}},\ (\exp tAa)_*iA)+$$
$$-\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\tfrac{d}{ds}\big|_{s=0}\exp tP\exp s(A-\tfrac{t}{2}[P,A]+O(t^2))aK^{\mathbb{C}},\ (\exp tPa)_*iA)+$$
$$+\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\exp i(H+tA)_*A,\ \exp i(H+tA)_*\cos\alpha(H+tA)P)$$

which, by (20), gives

$$\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\tfrac{d}{ds}\big|_{s=0}\exp s(-t\alpha(A)K)aK^{\mathbb{C}},\ a_*iA)+$$

$$-\tfrac{d}{dt}\big|_{t=0}\cos\alpha(H+tA)\mathrm{Im}B(I_0A,\ \overline{L}_{H+tA}P) =$$

$$= \omega_{\mathcal{J}}(a_*\alpha(A)\sin\alpha(H)iP,\ a_*iA) + \tfrac{d}{dt}\big|_{t=0}\cos\alpha(H+tA)b_1(H+tA) =$$

$$= -\alpha(A)\sin\alpha(H)\mathrm{Im}B(I_0iP,\ \overline{L}_HiA) + \tfrac{d}{dt}\big|_{t=0}\cos\alpha(H+tA)b_1(H+tA) =$$

$$= -\alpha(A)\sin\alpha(H)\mathrm{Im}B(A,\ \overline{L}_HA) + \tfrac{d}{dt}\big|_{t=0}\cos\alpha(H+tA)b_1(H+tA) =$$

$$= -\alpha(A)\sin\alpha(H)b_1(H) + \tfrac{d}{dt}\big|_{t=0}\cos\alpha(H+tA)b_1(H+tA) =$$

$$= -2\alpha(A)\sin\alpha(H)b_1(H) + \cos\alpha(H)\tfrac{d}{dt}\big|_{t=0}b_1(H+tA) = 0\,.$$

Equivalently

$$\frac{d}{dt}\big|_{t=0}b_1(H+tA) = 2\tfrac{\alpha(A)\sin\alpha(H)}{\cos\alpha(H)}b_1(H)\,.$$

The solution of this differential equation is $b_1(H) = ce^{-2log\cos\alpha(H)} = \frac{c}{\cos^2\alpha(H)}$, where c is a real constant. The initial condition $b_1(0) = 0$ forces $c = 0$ and consequently $b_1 \equiv 0$.

- $\boldsymbol{a_1 \equiv 0.}$ In this case we choose the vector fields $\breve{A}$, $\breve{K}$ and $\breve{iA}$. One has

$$d\omega_{\mathcal{J}}(\breve{A}_z,\ \breve{K}_z,\ \breve{iA}_z) = \tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\breve{K}_{\exp tAaK^{\mathbb{C}}},\ \breve{iA}_{\exp tAaK^{\mathbb{C}}})+$$
$$-\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\breve{A}_{\exp tKaK^{\mathbb{C}}},\ \breve{iA}_{\exp tKaK^{\mathbb{C}}}) + \tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\breve{A}_{\exp i(H+tA)K^{\mathbb{C}}},\ \breve{K}_{\exp i(H+tA)K^{\mathbb{C}}})\,.$$

The first term on the right-hand side of the equal sign vanishes by the G-invariance of $\omega_{\mathcal{J}}$. Thus one obtains

$$-\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\tfrac{d}{ds}\big|_{s=0}\exp sA\exp tKaK^{\mathbb{C}},\ (\exp tKa)_*iA)+$$
$$\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\exp i(H+tA)_*A,\ \tfrac{d}{ds}\big|_{s=0}\exp sK\exp i(H+tA)K^{\mathbb{C}}) =$$

$$= -\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\tfrac{d}{ds}\big|_{s=0}\exp tK\exp s(A-t[K,\,A]+O(t^2))aK^{\mathbb{C}},\,(\exp tKa)_*iA)+$$
$$\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\exp i(H+tA)_*A,\,-\exp i(H+tA)_*\sin\alpha(H+tA)iP)=$$

$$= -\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(a_*(A+t\alpha(A)F_aP+O(t^2)),\,a_*iA)+$$
$$\tfrac{d}{dt}\big|_{t=0}\sin\alpha(H+tA)\mathrm{Im}B(I_0A,\,\overline{L}_{H+tA}iP)=$$

$$= -\omega_{\mathcal{J}}(a_*\alpha(A)F_aP,\,a_*iA)+\tfrac{d}{dt}\big|_{t=0}\sin\alpha(H+tA)\mathrm{Re}B(P,\,\overline{L}_{H+tA}P)=$$

$$= \alpha(A)\cos\alpha(H)\mathrm{Im}B(I_0P,\,\overline{L}_HiA)-\tfrac{d}{dt}\big|_{t=0}\sin\alpha(H+tA)a_1(H+tA)=$$

$$= -\alpha(A)\cos\alpha(H)\mathrm{Re}B(A,\,\overline{L}_HA)-\tfrac{d}{dt}\big|_{t=0}\sin\alpha(H+tA)a_1(H+tA)=$$

$$= -\alpha(A)\cos\alpha(H)a_1(H)-\tfrac{d}{dt}\big|_{t=0}\sin\alpha(H+tA)a_1(H+tA)=$$

$$= -2\alpha(A)\cos\alpha(H)a_1(H)-\sin\alpha(H)\tfrac{d}{dt}\big|_{t=0}a_1(H+tA)=0\,.$$

Equivalently
$$\tfrac{d}{dt}\big|_{t=0}a_1(H+tA)=-\tfrac{2\alpha(A)\cos\alpha(H)}{\sin\alpha(H)}a_1(H)\,.$$

For $H\neq 0$ the solution of this differential equation is $a_1(H)=ce^{-2log\sin\alpha(H)}=\frac{c}{\sin^2\alpha(H)}$ and, due to the initial condition $a_1(0)=0$, one has $\lim_{H\to 0}a_1(H)=0$. Hence $c=0$ and $a_1\equiv 0$.

• $\boldsymbol{a_3(H)=-\cos\alpha(H)}$**.** For this choose the vector fields $\breve{P}$, $\breve{K}$ and $\breve{iA}$. One has

$$d\omega_{\mathcal{J}}(\breve{P}_z,\,\breve{K}_z,\,\breve{iA}_z)=\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\breve{K}_{\exp tPaK^{\mathbb{C}}},\,\breve{iA}_{\exp tPaK^{\mathbb{C}}})+$$
$$-\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\breve{P}_{\exp tKaK^{\mathbb{C}}},\,\breve{iA}_{\exp tKaK^{\mathbb{C}}})+\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\breve{P}_{\exp i(H+tA)K^{\mathbb{C}}},\,\breve{K}_{\exp i(H+tA)K^{\mathbb{C}}})\,,$$

where the first term on the right-hand side of the equal sign vanishes by the G-invariance of $\omega_{\mathcal{J}}$. Thus one obtains

$$-\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\tfrac{d}{ds}\big|_{s=0}\exp sP\exp tKaK^{\mathbb{C}},\,\tfrac{d}{ds}\big|_{s=0}\exp tK\exp i(H+sA)K^{\mathbb{C}})+$$
$$+\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\tfrac{d}{ds}\big|_{s=0}\exp sP\exp i(H+tA)K^{\mathbb{C}},\,\tfrac{d}{ds}\big|_{s=0}\exp sK\exp i(H+tA)K^{\mathbb{C}})=$$

$$= -\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\tfrac{d}{ds}\big|_{s=0}\exp tK\exp s(P-t[K,P]+O(t^2))aK^{\mathbb{C}},\,(\exp tKa)_*iA)+$$
$$+\tfrac{d}{dt}\big|_{t=0}\omega_{\mathcal{J}}(\exp i(H+tA)_*\cos\alpha(H+tA)P,\,-\exp i(H+tA)_*\sin\alpha(H+tA)iP)=$$

(recall that $[K,P]=2A$)

$$=\omega_{\mathcal{J}}(2a_*A,\,a_*iA)$$

$$-\tfrac{d}{dt}\big|_{t=0}\cos\alpha(H+tA)\sin\alpha(H+tA)\omega_{\mathcal{J}}(\exp i(H+tA)_*P,\,\exp i(H+tA)_*iP)=$$
$$=-2\mathrm{Im}B(I_0A,\overline{L}_HiA)+\tfrac{d}{dt}\big|_{t=0}\cos\alpha(H+tA)\sin\alpha(H+tA)\mathrm{Im}B(I_0P,\,\overline{L}_{H+tA}iP)=$$
$$=-2\mathrm{Re}B(P,\overline{L}_HA)-\tfrac{d}{dt}\big|_{t=0}\cos\alpha(H+tA)\sin\alpha(H+tA)\mathrm{Re}B(A,\,\overline{L}_{H+tA}P)=$$
$$=-2a_3(H)-\tfrac{d}{dt}\big|_{t=0}\cos\alpha(H+tA)\sin\alpha(H+tA)a_2(H+tA)=$$
$$=-2a_3(H)+\tfrac{d}{dt}\big|_{t=0}\tfrac{\cos\alpha(H+tA)}{a_3(H+tA)}\sin\alpha(H+tA)=0\,.$$

For the last equality we use that, since $a_1=b_1\equiv 0$, one has $a_2=-\frac{1}{a_3}$ (see claim). Due to the initial condition $a_3(0)=-1$ and the fact that $\alpha(A)=2$, it follows that $a_3(H)=-\cos\alpha(H)$. This concludes the proof. □

10. Appendix B: the canonical Kähler form and its potential

Define $\rho_{can}:\Xi\to\mathbb{R}$ by

$$\rho_{can}(gaK^{\mathbb{C}}):=\tfrac{1}{2}B(H,H)\,,$$

for $gaK^{\mathbb{C}}\in\Xi$ with $a=\exp iH$, and set $\omega_{can}=-dd^c_J\rho_{can}$, where $J=J_{ad}$. As mentioned in the introduction, Ξ can be thought as a G-invariant domain in the cotangent bundle T^*G/K. In this realization, from the results in [GuSt91] and [LeSz91] (see also [Sz91]), it follows that ω_{can} coincides with the canonical real symplectic form on T^*G/K.

An analogous computation as in Proposition 6.2 gives the following Lie group theoretic realization of ω_{can} and of the associated moment map on $\Xi\subset G^{\mathbb{C}}/K^{\mathbb{C}}$.

Proposition 10.1. *The function ρ_{can} is a G-invariant potential of the canonical symplectic form, determined by*

$$\omega_{can}(\widetilde{Z}_{aK^{\mathbb{C}}},\,\widetilde{W}_{aK^{\mathbb{C}}}):=-\mathrm{Im}B\big(Z,\,E_a^{-1}F_a\overline{W}\big)\,,$$

for $Z,\,W\in\mathfrak{p}^{\mathbb{C}}$. Equivalently,

$$\omega_{can}(a_*Z,\,a_*W)=-\mathrm{Im}B\big(F_a^{-1}Z,\,E_a^{-1}\overline{W}\big)\,.$$

The moment map $\mu_{can}:\Xi\to\mathfrak{g}^*$ associated with ρ_{can} is given by

$$\mu_{can}(gaK^{\mathbb{C}})(X)=B\big(\mathrm{Ad}_{g^{-1}}X,\,H\big)\,.$$

Remark 10.2. *By means of Lemma 5.2(i) and Proposition 10.1, one can check that the form ω_J is the pull-back of ω_{can} via the G-equivariant map ψ defined in Section 4.* (*cf. Rem. 4.5 and* [DaSz97], *Thm. 4.1*).

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Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica 1, I-00133 Roma, Italy

E-mail address: geatti@mat.uniroma2.it, iannuzzi@mat.uniroma2.it