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# *"The global quantum duality principle: theory, examples, and applications"*

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## INTRODUCTION

*"Dualitas dualitatum*
*et omnia dualitas"*
*N. Barbecue, "Scholia"*

The most general notion of "symmetry" in mathematics is encoded in the notion of Hopf algebra. Among Hopf algebras $H$ over a field, the commutative and the cocommutative ones encode "geometrical" symmetries, in that they correspond, under some technical conditions, to algebraic groups and to (restricted, if the ground field has positive characteristic) Lie algebras respectively: in the first case $H$ is the algebra $F[G]$ of regular functions over an algebraic group $G$, whereas in the second case it is the (restricted) universal enveloping algebra $U(\mathfrak{g})$ ($\mathbf{u}(\mathfrak{g})$ in the restricted case) of a (restricted) Lie algebra $\mathfrak{g}$. A popular generalization of these two types of "geometrical symmetry" is given by quantum groups: roughly, these are Hopf algebras $H$ depending on a parameter $\hbar$ such that setting $\hbar = 0$ the Hopf algebra one gets is either of the type $F[G]$ — hence $H$ is a *quantized function algebra*, in short QFA — or of the type $U(\mathfrak{g})$ or $\mathbf{u}(\mathfrak{g})$ (according to the characteristic of the ground field) — hence $H$ is a *quantized (restricted) universal enveloping algebra*, in short QrUEA. When a QFA exists whose specialization (i.e. its "value" at $\hbar = 0$) is $F[G]$, the group $G$ inherits from this "quantization" a Poisson bracket, which makes it a Poisson (algebraic) group; similarly, if a QrUEA exists whose specialization is $U(\mathfrak{g})$ or $\mathbf{u}(\mathfrak{g})$, the (restricted) Lie algebra $\mathfrak{g}$ inherits a Lie cobracket which makes it a Lie bialgebra. Then by Poisson group theory one has Poisson groups $G^*$ dual to $G$ and a Lie bialgebra $\mathfrak{g}^*$ dual to $\mathfrak{g}$, so other geometrical symmetries are related to the initial ones.

The dependence of a Hopf algebra on $\hbar$ can be described as saying that it is defined over a ring $R$ and $\hbar \in R$: so one is lead to dwell upon the category $\mathcal{HA}$ of Hopf $R$–algebras (maybe with some further conditions), and then raises three basic questions:

— **(1)** *How can we produce quantum groups?*

— **(2)** *How can we characterize quantum groups (of either kind) within $\mathcal{HA}$?*

— **(3)** *What kind of relationship, if any, does exist between quantum groups over mutually dual Poisson groups, or mutually dual Lie bialgebras?*

A first answer to question **(1)** and **(3)** together is given, in characteristic zero, by the so-called "quantum duality principle", known in literature with at least two formulations. One claims that quantum function algebras associated to dual Poisson groups can be taken to be dual — in the Hopf sense — to each other; and similarly for quantum enveloping algebras (cf. [FRT1] and [Se]). The second one, formulated by Drinfeld in local terms (i.e., using formal groups, rather than algebraic groups, and complete topological Hopf algebras; cf. [Dr], §7, and see [Ga4] for a proof), gives a recipe to get, out of a QFA over $G$, a QrUEA over $\mathfrak{g}^*$, and, conversely, to get a QFA over $G^*$ out of a QrUEA over $\mathfrak{g}$. More precisely, Drinfeld defines two functors, inverse to each other, from the category of quantized universal enveloping algebras (in his sense) to the category of quantum formal series Hopf algebras (his analogue of QFAs) and viceversa, such that $U_\hbar(\mathfrak{g}) \mapsto F_\hbar[[G^*]]$ and $F_\hbar[[G]] \mapsto U_\hbar(\mathfrak{g}^*)$ (in his notation, where the subscript $\hbar$ stands as a reminder for "quantized" and the double square brackets stand for "formal series Hopf algebra").

In this paper we establish a *global* version of the quantum duality principle which gives a complete answer to questions **(1)** through **(3)**. The idea is to push as far as possible Drinfeld's original method so to apply it to the category $\mathcal{HA}$ of all Hopf algebras which are torsion-free — or flat, if one prefers this narrower setup — modules over some (integral) domain, say $R$, and to do it for each non-zero element $\hbar$ in $R$ such that $R/\hbar R$ be a field.

To be precise, we extend Drinfeld's recipe so to define functors from $\mathcal{HA}$ to itself. We show that the image of these "generalized" Drinfeld's functors is contained in a category of quantum groups — one gives QFAs, the other QrUEAs — so we answer question **(1)**. Then, in the characteristic zero case, we prove that when restricted to quantum groups these functors yield equivalences inverse to each other. Moreover, we show that these equivalences exchange the types of quantum group (switching QFA with QrUEA) and the underlying Poisson symmetries (interchanging $G$ or $\mathfrak{g}$ with $G^*$ or $\mathfrak{g}^*$), thus solving **(3)**. Other details enter the picture to show that these functors endow $\mathcal{HA}$ with sort of a (inner) "Galois correspondence", in which QFAs on one side and QrUEAs on the other side are the subcategories (in $\mathcal{HA}$) of "fixed points" for the composition of both Drinfeld's functors (in the suitable order): in particular, this answers question **(2)**. It is worth stressing that, since our "Drinfeld's functors" are defined for each non-trivial point $(\hbar)$ of $Spec_{max}(R)$, for any such $(\hbar)$ and for any $H$ in $\mathcal{HA}$ they yield two quantum groups, namely a QFA and a QrUEA, w.r.t. $\hbar$ itself. Thus we have a method to get, out of any single $H \in \mathcal{HA}$, several quantum groups.

Therefore the "global" in the title is meant in several respects: geometrically, we consider global objects (Poisson groups rather than Poisson *formal* groups, as in Drinfeld's approach); algebraically we consider quantum groups over any domain $R$, so there may be several different "semiclassical limits" (=specializations) to consider, one for each non-trivial point of type $(\hbar)$ in the maximal spectrum of $R$ (while Drinfeld has $R = \Bbbk[[\hbar]]$ so one can specialise only at $\hbar = 0$). More generally, our recipe applies to *any* Hopf algebra, i.e. not only to quantum groups. Finally, most of our results are characteristic-free, i.e. they hold not only in characteristic zero (as in Drinfeld's case) but also in positive characteristic. Furthermore, this "global version" of the quantum duality principle opens the way to formulate a "quantum duality principle for subgroups and homogeneous spaces", see [CG].

A key, long-ranging application of our *global quantum duality principle* (GQDP) is the following. Take as $R$ the polynomial ring $R = \Bbbk[\hbar]$, where $\Bbbk$ is a field: then for any Hopf algebra over $\Bbbk$ we have that $H[\hbar] := R \otimes_\Bbbk H$ is a torsion-free Hopf $R$–algebra, hence we can apply Drinfeld's functors to it. The outcome of this procedure is the *crystal duality principle* (CDP), whose statement strictly resembles that of the GQDP: now Hopf $\Bbbk$–algebras are looked at instead of torsionless Hopf $R$–algebras, and quantum groups are replaced by Hopf algebras with canonical filtrations such that the associated graded Hopf algebra is either commutative or cocommutative. Correspondingly, we have a method to associate to $H$ a Poisson group $G$ and a Lie bialgebra $\mathfrak{k}$ such that $G$ is an affine space (as an algebraic variety) and $\mathfrak{k}$ is graded (as a Lie algebra); in both cases, the "geometrical" Hopf algebra can be attained — roughly — through a continuous 1-parameter deformation process. This result can also be formulated in purely classical — i.e. "non-quantum" — terms and proved by purely classical means. However, the approach via the GQDP also yields further possibilities to deform $H$ into other Hopf algebras of geometrical type, which is out of reach of any classical approach.

The paper is organized as follows. In §1 we fix notation and terminology, while §2 is devoted to define Drinfeld's functors and state our main result, the GQDP (Theorem 2.2). In §3 we extend Drinfeld's functors to a broader framework, that of *(co)augmented (co)algebras*, and study their properties in general. §4 instead is devoted to the analysis of the effect of such functors on quantum groups, and prove Theorem 2.2, i.e. the GQDP. In §5 we explain the CDP, which is deduced as an application of the CDP to trivial deformations of Hopf $\Bbbk$–algebras: in particular, we study in detail the case of group algebras. In the last part of the paper we illustrate our results by studying in full detail several relevant examples. First we dwell upon some well-known quantum groups: the standard quantization of the Kostant-Kirillov structure on a Lie algebra (§6), the standard Drinfeld-Jimbo quantization of semisimple groups (§7), the quantization of the Euclidean group (§8) and the quantization of the Heisenberg group (§9). Then we study a key example of non-commutative, non-cocommutative Hopf algebra — a non-commutative version of the Hopf algebra of formal diffeomorphisms — as a nice application of the CDP (§10).

*Warning: this paper is not meant for publication! The results presented here will be published in separate articles; therefore, any reader willing to quote anything from the present preprint is kindly invited to ask the author for the precise reference(s).*

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