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"The global quantum duality principle: theory, examples, and applications"

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ABSTRACT

Let R be an integral domain, let $\hbar \in R \setminus \{0\}$ be such that $R/\hbar R$ is a field, and $\mathcal{HA}$ the category of torsionless (or flat) Hopf algebras over R. We call $H \in \mathcal{HA}$ a *"quantized function algebra" (=QFA)*, resp. *"quantized restricted universal enveloping algebras" (=QrUEA)*, at $\hbar$ if $H/\hbar H$ is the function algebra of a connected Poisson group, resp. the (restricted, if $R/\hbar R$ has positive characteristic) universal enveloping algebra of a (restricted) Lie bialgebra.

We establish an "inner" Galois correspondence on $\mathcal{HA}$, via the definition of two endofunctors, $(\ )^\vee$ and $(\ )'$, of $\mathcal{HA}$ such that: *(a)* the image of $(\ )^\vee$, resp. of $(\ )'$, is the full subcategory of all QrUEAs, resp. QFAs, at $\hbar$; *(b)* if $p := Char\big(R/\hbar R\big) = 0$, the restrictions $(\ )^\vee\big|_{\mathrm{QFAs}}$ and $(\ )'\big|_{\mathrm{QrUEAs}}$ yield equivalences inverse to each other; *(c)* if $p = 0$, starting from a QFA over a Poisson group G, resp. from a QrUEA over a Lie bialgebra $\mathfrak{g}$, the functor $(\ )^\vee$, resp. $(\ )'$, gives a QrUEA, resp. a QFA, over the dual Lie bialgebra, resp. a dual Poisson group. In particular, *(a)* yields a machine to produce quantum groups of both types (either QFAs or QrUEAs), *(b)* gives a characterization of them among objects of $\mathcal{HA}$, and *(c)* gives a "global" version of the so-called "quantum duality principle" (after Drinfeld's, cf. [Dr]).

We then apply our result to Hopf algebras of the form $\Bbbk[\hbar] \otimes_{\Bbbk} H$ where H is a Hopf algebra over the field $\Bbbk$: this yields quantum groups, hence "classical" geometrical symmetries of Poisson type (Poisson groups or Lie bialgebras, via specialization) associated to the "generalized" symmetry encoded by H. Both our main result and the above mentioned application are illustrated by means of several examples, which are studied in some detail.

Warning: this paper is not meant for publication! The results presented here will be published in separate articles; therefore, any reader willing to quote anything from the present preprint is kindly invited to ask the author for the precise reference(s).

"Dualitas dualitatum
et omnia dualitas"

N. Barbecue, "Scholia"

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