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"Real forms of complex Lie superalgebras and supergroups"

INTRODUCTION

The study of real forms of complex contragredient Lie superalgebras was initiated by V. G. Kac in his foundational work [13] and then carried out by M. Parker in [18] and V. Serganova in [20], where also symmetric superspaces were accounted for. Later on, Chuah in [6] gave another thorough classification of such real forms using Vogan diagrams and Cartan automorphisms. In fact, as it happens for the ordinary setting, we have a one to one correspondence between real structures on a contragredient Lie superalgebra $\mathfrak{g}$, and its Cartan automorphisms $\mathrm{aut}_{2,4}(\mathfrak{g})$, that is automorphisms that are involutions on the even part and whose square is the identity on the odd part of $\mathfrak{g}$. This translates to a bijection between the antilinear involutions $\overline{\mathrm{aut}}_{2,2}(\mathfrak{g})$ of $\mathfrak{g}$ and the linear automorphisms $\overline{\mathrm{aut}}_{2,4}(\mathfrak{g})$. In the ordinary setting, that is for $\mathfrak{g} = \mathfrak{g}_{\overline{0}}$, this correspondence is explicitly obtained through the Cartan antiinvolution $\omega_{\overline{0}}$, whose fixed points give the compact form of $\mathfrak{g}_{\overline{0}}$. In the supersetting, as we shall see, such antiinvolution is replaced by an antilinear automorphism $\omega \in \overline{\mathrm{aut}}_{2,4}(\mathfrak{g})$. This prompts for a more general treatment of real structures and real forms of superspaces and superalgebras, together with their global versions, where we consider both cases $\overline{\mathrm{aut}}_{2,s}(\mathfrak{g})$ and $\mathrm{aut}_{2,s}(\mathfrak{g})$, for $s = 2, 4$. We shall refer to such real structures and real forms as *standard* and *graded*; they were introduced in [19], [20].

The paper is organized as follows. Sec. 2 contains preliminaries that help to establish our notation. In Sec. 3, we begin by defining the notion of standard and graded real structure on a superspace V as a pair (V, ϕ) with $\phi \in \overline{\mathrm{aut}}_{2,2}(\mathfrak{g})$ or $\overline{\mathrm{aut}}_{2,4}(\mathfrak{g})$, respectively. We obtain two categories, $(\mathrm{smod})_{\mathbb{C}}^{\mathrm{st}}$ and $(\mathrm{smod})_{\mathbb{C}}^{\mathrm{gr}}$, that we compactly denote $(\mathrm{smod})_{\mathbb{C}}^{\bullet}$ whenever there is no need to remark the difference; similarly, we define the corresponding categories of superalgebras $(\mathrm{salg})_{\mathbb{C}}^{\bullet}$. As expected, given a real structure, the associated real form is given by the fixed points of the antiautomorphism, however in the graded case, the functorial point of view is most fruitful, because such points cannot be seen over the complex field. After establishing the terminology and definitions, we can then give naturally the notion of real structures and real forms of Lie superalgebras, following and extending the work [19]. These real structures and real forms do integrate: thus, in Sec. 4, we obtain the category of complex supergroups with standard or graded real structures, that we denote with $(\mathrm{sgrps})_{\mathbb{C}}^{\mathrm{st}}$ and $(\mathrm{sgrps})_{\mathbb{C}}^{\mathrm{gr}}$, or more compactly $(\mathrm{sgrps})_{\mathbb{C}}^{\bullet}$. We also briefly discuss the super Harish-Chandra pairs (sHCp) approach in this context (see also [3, 11, 16]). Our main result for this part is the following (see Theorem 3.12):

Theorem A. *If $\big(\mathbf{G}, \Phi\big) \in (\mathrm{sgrps})_{\mathbb{C}}^{\bullet}$, the real form $\mathbf{G}^{\Phi}$ of $\mathbf{G}$, given by the fixed points of Φ, is*

$$\mathbf{G}^{\Phi}(A) \;=\; \Big(G_{+}^{\Phi_{+}} \times \mathbb{A}_{\bullet,\mathbb{C}}^{0\,|d_1}\Big)(A) \quad , \qquad \forall\;\, A \in (\mathrm{salg})_{\mathbb{C}}^{\bullet}$$

where $G_{+}^{\Phi_{+}}$ is the ordinary underlying real form of G_{+} and $\mathbb{A}_{\bullet,\mathbb{C}}^{0\,|d_1}$ is a real form of a purely odd affine superspace. In particular, the supergroup functor $\mathbf{G}^{\Phi}$ is representable.

In the remaining part of the paper, we discuss compact real forms of contragredient complex Lie superalgebras and the corresponding supergroups, using the results detailed above.

In the ordinary setting, a real Lie algebra is compact if it is embedded into some orthogonal or equivalently unitary Lie algebra. For a Lie superalgebra $\mathfrak{g}$, many authors (see [6], [6], [2]) replace this notion with the requirement that $\mathfrak{g} = \mathfrak{g}_{\overline{0}}$ and the latter compact. We take a more general approach, allowing $\mathfrak{g}$ to have odd elements. For this reason, in Sec. 5, we need to examine super Hermitian forms, in the standard and graded context, and the corresponding unitary Lie superalgebras. In our Subsec. 5.4, we retrieve in our language the physicists' definition of unitary Lie superalgebra (see [21] and references therein), but also a graded version of it, obtained as fixed points of the *superadjoint* — that is, the supertranspose complex conjugate. We regard this example very significant and natural, since it is obtained via an antilinear morphism in $\overline{\mathrm{aut}}_{2,4}(\mathfrak{gl}(m|n))$, which has a categorical motivation (see [10], Ch. 1, and also [20, 19]).

In Sec. 6, we formulate our notion of *compact* Lie superalgebra as one admitting an embedding into a unitary Lie superalgebra for a suitable positive definite super Hermitian form. We shall call this *super-compact*. Then, we are finally able to introduce $\omega \in \overline{\mathrm{aut}}_{2,4}(\mathfrak{g})$, generalizing the Cartan antiinvolution $\omega_{\overline{0}}$ mentioned above, and to prove the correspondence between $\overline{\mathrm{aut}}_{2,4}(\mathfrak{g})$ and $\mathrm{aut}_{2,2}(\mathfrak{g})$ and between $\overline{\mathrm{aut}}_{2,2}(\mathfrak{g})$ and $\mathrm{aut}_{2,4}(\mathfrak{g})$. Our main result for this part is the following (see Theorems 5.10 and 5.11):

Theorem B. *Let $\mathfrak{g}$ be a simple complex contragredient Lie superalgebra. Then:*

(a) $\mathfrak{g}$ admits a graded, super-compact real form, given via $\omega \in \overline{aut}_{2,4}(\mathfrak{g})$;

(b) if $\mathfrak{g}$ is of type 1, then $\mathfrak{g}$ admits a standard, compact real form;

(c) if $\mathfrak{g}$ is of type 2, then $\mathfrak{g}$ has no standard, compact real form.

In all cases, such super-compact or compact forms are unique up to inner automorphisms.

We end our treatment giving a global version of the previous results (see Theorems 6.4, 6.5).

Theorem C. *Let $\mathbf{G}$ be a complex supergroup with $\mathfrak{g} = Lie(G)$ being simple contragredient. Then $\mathbf{G}$ admits a graded, super-compact real form, which is unique up to inner automorphisms.*

If $\mathfrak{g}$ is of type 1, then $\mathbf{G}$ admits a standard, compact real form, unique up to inner automorphisms. If $\mathfrak{g}$ is of type 2, then $\mathbf{G}$ has no standard, compact real form.

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