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QUANTUM DUALITY PRINCIPLE FOR QUANTUM CONTINUOUS KAC–MOODY ALGEBRAS

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1. Introduction

Quantum groups, in their standard formulation as suitable topological Hopf algebras on a ring of formal power series $\Bbbk[[\hbar]]$, exist in two versions. Namely, our quantum group is called a quantized universal enveloping algebra (or QUEA, in short), when its specialization at $\hbar = 0$ is the universal enveloping algebra of some Lie algebra (actually a *Lie bialgebra*), or a quantum formal series Hopf algebra (in short, a QFSHA) when its specialization is (the algebra of functions on) a formal algebraic/Lie group — actually, a *Poisson group*. The categories of QUEA's and of QFSHA's are antiequivalent to each other via linear duality, just like it happens for their semiclassical counterparts. Surprisingly enough, they are also *equivalent*, through explicit equivalence functors, originally sketched in [Dri87, §7], and later detailed in [Gav02]: in a sloppy formulation, this phenomenon is known as *Quantum Duality Principle* — hereafter shortened as QDP.

Roughly speaking, the QDP claims that every QUEA, resp. every QFSHA, can be "renormalized" as to give rise to a QFSHA, resp. to a QUEA: in either case, the new quantum algebra — sometimes called "the Drinfeld-Gavarini dual" of the original one — is a quantization of the object (Poisson group or Lie bialgebra, respectively) which is *Poisson dual* to the object that the original quantum algebra is a quantization of. In particular, if $U_\hbar(\mathfrak{g})$ is a QUEA quantizing $U(\mathfrak{g})$ then the QDP provides an explicit, functorial construction of a suitable Hopf subalgebra $U_\hbar(\mathfrak{g})'$ of $U_\hbar(\mathfrak{g})$ which is a quantization of $F[[G^*]]$, where G^* is the formal Poisson group dual to the Lie bialgebra $\mathfrak{g}$. In fact, by construction $U_\hbar(\mathfrak{g})'$ is in fact a $\Bbbk[[\hbar]]$–integral form of $\Bbbk((\hbar)) \otimes_{\Bbbk[[\hbar]]} U_\hbar(\mathfrak{g})$, just like $U_\hbar(\mathfrak{g})$ itself is. In the other direction, if $F_\hbar[[G]]$ is any QFSHA for the formal Poisson group G then the QDP provides a different $\Bbbk[[\hbar]]$–integral form $F_\hbar[[G]]^\vee$ of $\Bbbk((\hbar)) \otimes_{\Bbbk[[\hbar]]} F_\hbar[[G]]$ that is indeed a QUEA for $\mathfrak{g}^*$.

Note that the geometrical objects $\mathfrak{g}$ and G (and their Poisson dual) considered by the QDP in its original formulation are finite dimensional, though some aspects of its functorial construction do apply to the infinite setup as well.

In a different approach, where quantum groups are defined as standard (i.e., non-topological) Hopf algebras over the field $\Bbbk(q)$ — that is, à la Jimbo-Lusztig, say — so that one deals with "polynomial QUEA" and "polynomial QFA (=Quantum Function Algebras)", a suitable polynomial version of the QDP has been developed (cf. [Gav02]). In short, in this context one considers a Hopf algebra $\mathbb{H}$ over $\Bbbk(q)$

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and an $\Bbbk\big[\,q\,,q^{-1}\big]$–integral form H of it: the latter then are called QUEA or QFA depending on whether $H\Big/(\,q-1)\,H$ has the form $U(\mathfrak{g})$ or $F[G]$, whence one writes $H = \mathcal{U}_q(\mathfrak{g})$ or $H = \mathcal{F}_q[G]$, respectively. Then the "polynomial" QDP provides functorial recipes (direct adaptation of Drinfeld's original ones) $\mathcal{U}_q(\mathfrak{g}) \mapsto \mathcal{U}_q(\mathfrak{g})'$ and $\mathcal{F}_q[G] \mapsto \mathcal{F}_q[G]^\vee$ such that $\mathcal{U}_q(\mathfrak{g})'$ is a QFA for G^* and $\mathcal{F}_q[G]^\vee$ is a QUEA for $\mathfrak{g}^*$ (actually, the complete result is much stronger, see [Gav02, Theorem 2.2]). In particular, $\mathcal{U}_q(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{g})'$ are two $\Bbbk\big[\,q\,,q^{-1}\big]$–integral forms of the same $\mathbb{H}$, and similarly for $\mathcal{F}_q[G]$ and $\mathcal{F}_q[G]^\vee$. Indeed, in concrete examples, when the $\Bbbk(q)$–algebra $\mathbb{H}$ is given by a presentation by generators and relations, the difference between the two integral forms $\mathcal{U}_q(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{g})'$ amounts to a different choice of generators (roughly, a different "normalization" of them), and similarly for $\mathcal{F}_q[G]$ and $\mathcal{F}_q[G]^\vee$ again. For instance, for the usual Jimbo-Lusztig quantum group $\mathbb{U}_q(\mathfrak{g})$ over a finite-dimensional semisimple $\mathfrak{g}$ one can realize that (up to details) $\mathcal{U}_q(\mathfrak{g})$ is nothing but Lusztig's *restricted* form, while $\mathcal{U}_q(\mathfrak{g})'$ is De Concini-Procesi's *unrestricted* one. As both can be defined even over $\mathbb{Z}\big[\,q\,,q^{-1}\big]$, thus leading to (different) theories of *quantum groups at roots of 1*, we also see how the polynomial QDP is somehow deeply intertwined with the theory of quantum groups at roots of 1 — although a formal, sound theory about that correlation has still to be unveiled.

To date, the impact of the QDP — either in formal or in polynomial version — on the development of quantum group theory has been paramount, in a pervasive manner (although not always explicitly recognized). Nevertheless, as in real life examples and constructions of QUEA's are available way more than of QFSHA's (or QFA's), people mostly applied the QDP in the direction QUEA $\mapsto$ QFSHA (or QUEA $\mapsto$ QFA, in the polynomial case).

For instance, the formal QDP was used in the very construction of (formal) QUEA's of Lie bialgebras - possibly extending its range to infinite-dimensional ones — cf. [EtK96, Enr01, Enr05, Hal06, EnH07] — even extending it to the infinite-dimensional framework — as in [ATL18, ATL19]. These results were also extended to broader contexts, such as that of quasi-Hopf QUEA (over quasi-Lie bialgebras) — cf. [EnH04] — that of super Lie bialgebras — cf. [Gee06] — that of (quantum) groupoids — cf. [ChG15] — that of (quantization of) Γ–Lie bialgebras and Poisson-Hopf stacks over groupoids — cf. [EnH08, HXT06] — and that of Yangians — cf. [KaWWY, FiT19]. In another direction, the formal QDP was also applied to study quantum R–matrices and associated structures (and variations on this theme), both from a geometrical point of view or a representation-theoretic one, as in [GaH01, GaH03, EGH03, EEM05]. Another geometrical application was to quantum homogeneous spaces, as in [CiG06], where the QDP was suitably extended to formal quantizations (both infinitesimal and global) of Poisson homogeneous spaces.

On the other hand, the polynomial version of the QDP is applied to (or is definitely underlying) the construction and study of new QFA's — in a finite dimensional setup (cf. [DPr95] for the uniparameter case, and [Gav98-2, GaGa] for the multiparameter case) or an infinite one (cf. [Bec94, Bec96, BeK96, Gav00]) — or new QUEA's — in a finite (cf. [Gav98-1, GaR07]) or infinite (cf. [Gav00]) dimensional setup. In a more geometrical perspective, it was applied — again in the "direction" QUEA $\mapsto$ QFA to the study of quantum R–matrices (and related subjects) in [Gav97, Gav01] — respectively for finite and affine type Kac–Moody Lie bialgebras, and in the study of Poisson homogeneous spaces in [CFG08] — where a suitable version of polynomial QDP is tailored ad hoc for the projective case — in [FiG11] — where quantum Grassmannians are treated — and in [CiG14] — where a general result is provided. Still on a geometrical side, in the wake of a very fruitful research line, the polynomial QDP was applied in [HaL16] to provide a new topological invariant of integral homology spheres.

Finally, despite being a phenomenon that is intrinsically "quantum" in nature, the QDP (in polynomial version) had also found a remarkable application back in "classical" Hopf algebra theory — cf. [Gav05-2] — with lot of immediate applications at hand (see [Gav05-1] for an example).

The purpose of the present work is to prove yet another instance of the QDP, both formal and polynomial, namely in the direction QUEA $\mapsto$ QF(SH)A for the quantization of the *continuous Kac–Moody algebras* by Appel, Sala and Schiffmann (see [ASS18, ApS20]. Indeed, these (topological) Lie bialgebras, hereafter denoted by $\mathfrak{g}_X$, are uncountably infinite-dimensional, hence one cannot directly apply the QDP as stated and proved in [Gav02]. Instead, starting from the formal QUEA $U_\hbar(\mathfrak{g}_X)$ we provide a direct definition of a suitable subalgebra $\widetilde{U}_\hbar(\mathfrak{g}_X)$ of $U_\hbar(\mathfrak{g}_X)$ and then we prove that it has exactly the properties predicted by the QDP, in particular $\widetilde{U}_\hbar(\mathfrak{g}_X)$ is a QFSHA whose semiclassical limit is $F\big[\big[G^*\big]\big]$. Finally, we also prove that this $\widetilde{U}_\hbar(\mathfrak{g}_X)$ actually admits also a description that coincides with the one prescribed by the usual Drinfeld's functor $U_\hbar(\mathfrak{g}) \mapsto U_\hbar(\mathfrak{g})'$.

As a second step, we introduce a suitable *polynomial* QUEA $\mathcal{U}_q(\mathfrak{g}_X)$ — easy to guess as a subalgebra of $U_\hbar(\mathfrak{g}_X)$ — and we realize for it the (polynomial) QDP by introducing by hands its appropriate Drinfeld-Gavarini dual $\mathcal{U}_q(\mathfrak{g}_X)'$. Here again, we cannot apply the general recipe given in [Gav07] (as the latter applies to the finite dimensional case only), but we give instead a direct definition of a suitable integral form $\widetilde{\mathcal{U}}_q(\mathfrak{g}_X)$, inspired by what is done for $\mathcal{U}_q(\mathfrak{g})$ when $\mathfrak{g}$ finite Kac–Moody (cf. [DPr95] and [Gav98-2]) or affine Kac–Moody (see [Bec96, BeK96] and [Gav00]). Later on, we also prove that this $\widetilde{\mathcal{U}}_q(\mathfrak{g}_X)$ does coincide with what comes out if one literally applies the recipe for Drinfeld's functor $\mathcal{U}_q(\mathfrak{g}) \mapsto \mathcal{U}_q(\mathfrak{g})'$ as given in [Gav07].

An important feature of the construction sketched above is the following. In the "indirect" construction, mentioned above, of the Drinfeld-Gavarini dual $\mathcal{U}_q(\mathfrak{g})'$ as a suitable $\Bbbk\big[q\,,q^{-1}\big]$–integral form of $\Bbbk(q)\otimes\mathcal{U}_q(\mathfrak{g})$ when $\mathfrak{g}$ is Kac–Moody finite or affine (as in the works of De Concini-Procesi, Beck and the author), a critical step is the construction of suitable "quantum root vectors" for any root, that are *not* available from scratch. However, the Lie bialgebras $\mathfrak{g}_X$ have a Kac–Moody like presentation which includes, as generators, the (analogue of) "root vectors" for any possible roots; even more, the same is true for the QUEA's $U_\hbar(\mathfrak{g}_X)$ and $\mathcal{U}_q(\mathfrak{g}_X)$ alike. Therefore, the "critical step" mentioned before is already fixed from scratch, so that performing the same construction of the $\Bbbk\big[q\,,q^{-1}\big]$–integral form $\widetilde{\mathcal{U}}_q(\mathfrak{g}_X) = \mathcal{U}_q(\mathfrak{g}_X)'$ of $\Bbbk(q)\otimes\mathcal{U}_q(\mathfrak{g}_X)$ as mentioned above becomes an easy task. Up to technicalities, the very same strategy can be followed in order to define $U_\hbar(\mathfrak{g}_X)' = \widetilde{U}_\hbar(\mathfrak{g}_X)$, again because all needed quantum root vectors are already given by definition.

As a last remark, we point out that both the QUEA $\mathcal{U}_q(\mathfrak{g}_X)$ — for the (topological) Lie bialgebra $\mathfrak{g}_X$ — and the QFA $\widetilde{\mathcal{U}}_q(\mathfrak{g}_X) = \mathcal{U}_q(\mathfrak{g}_X)'$ — for the Poisson group G_X^* — are actually defined over $\mathbb{Z}\big[q\,,q^{-1}\big]$: hence, an "arithmetic theory" for specializations at roots of 1, much like Lusztig (for the QUEA side) and De Concini-Procesi and Beck (for the QFA side) did, in principle is at hand.

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