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"Multiparameter quantum groups at roots of unity"

INTRODUCTION

In literature, by "quantum group" one usually means some deformation of an algebraic object that in turn encodes a geometrical object describing symmetries (such as a Lie or algebraic group or a Lie algebra): we are interested now in the case when the geometrical object is a Lie bialgebra $\mathfrak{g}$, and the algebraic one its universal enveloping algebra $U(\mathfrak{g})$, with its full structure of co-Poisson Hopf algebra.

In the most studied case, such a deformation explicitly depends on one single parameter, in a "formal" version — like with Drinfeld's $U_\hbar(\mathfrak{g})$ — or in a "polynomial" one — such as with Jimbo-Lusztig's $U_q(\mathfrak{g})$. However, since the beginning of the theory, more general deformations depending on many parameters have been considered too: taking this many parameters as a single "multiparameter" one then talks of "multiparameter quantum groups" — or MpQG's in short — that again exist both in formal and in polynomial version; see for instance [BGH], [BW1, BW2], [CM2], [CV1], [Hay], [HLT], [HP1], [HPR], [Ko], [KT], [Man], [OY], [Re], [Su], [Ta] — and the list might be longer.

In the previously mentioned papers, multiparameter quantum enveloping algebras where often introduced via *ad hoc* constructions. A very general recipe, instead, was that devised by Reshetikhin (cf. [Re]), that consists in performing a so-called *deformation by twist* on a "standard" one-parameter quantum group.

Similarly, a somehow dual method was also developed, that starts again from a usual one-parameter quantum group and then performs on it a deformation by a 2–cocycle. In addition, as the usual uniparameter quantum group is a quotient of the Drinfeld's quantum double of two Borel quantum (sub)groups, one can start by deforming (e.g., by a 2–cocycle) the Borel quantum subgroups and then look at their quantum double and its quotient. This is indeed the point of view adopted, for instance, in [AA2], [AAR1, AAR2], [An1, An2, An3, An4], [AS1, AS2], [AY], [Gar], [He1, He2], [HK], [HY] and [Mas1], where in addition the Borel quantum (sub)groups are always thought of as bosonizations of Nichols algebras.

In our forthcoming paper [GaGa] we shall thoroughly compare deformations by twist or by 2–cocycles on the standard uniparameter quantum group; up to technicalities, it turns out that the two methods yield the same results. Taking this into account, we adopt the point of view of deformations by 2–cocycles, implemented on uniparameter quantum groups, that are realized as (quotients of) quantum doubles of Borel quantum (sub)groups. With this method, the multiparameter $\mathbf{q}$ codifying our MpQG is used from scratch as the core datum to construct the Borel quantum (sub)groups and eventually remains in the description of our MpQG by generators and relations. In this approach, a natural constraint arises for $\mathbf{q}$, namely that it be *of Cartan type*, to guarantee that the so-constructed MpQG have finite Gelfand-Kirillov dimension.

In order to have meaningful specializations of a MpQG, one needs to choose a suitable integral form of that MpQG, and then specialize the latter: indeed, by "specialization of a MpQG" one means in short the specialization of such an integral form of it. The outcome of the specialization process then can strongly depend on the choice of the integral form. For the usual case of uniparameter "canonical" quantum groups, one usually considers two types of integral forms, namely *restricted* ones (after Lusztig's) and *unrestricted* ones (after De Concini and Procesi), whose specialization yield entirely different outcomes — dual to each other, in a sense. There also exist *mixed* integral forms (due to Habiro and Thang Le) that are very interesting for applications in algebraic topology.

For general MpQG's, we introduce integral forms of restricted, unrestricted and mixed type, by directly extending the construction of the canonical setup: although this is quite a natural step, it seems (to the best of the authors' knowledge) that it had not yet been considered so far. Moreover, for restricted forms — for which the multiparameter has to be "integral", i.e. made of powers (with integral exponents) of just one single, "basic" parameter q — we consider two possible variants, which gives something new even in the canonical case. For these integral forms (of either type) we state and prove all those fundamental structure results (triangular decompositions, PBW Theorems, duality, etc.) that one needs to work with them.

When taking specialization at $q = 1$ (where "q" is again sort of a "basic parameter" underlying the multiparameter $\mathbf{q}$), co-Poisson and Poisson Hopf structures pop up, yielding classical objects that bear some Poisson geometrical structure. In detail, when specializing the restricted form one gets the enveloping algebra of a Lie bialgebra, and when specializing the unrestricted one the function algebra of a Poisson group is found: this shows some duality phenomenon, which is not surprising because the two integral forms are in a sense related by Hopf duality. This feature already occurs in the uniparameter, canonical case: but in the present, multiparameter setup, the additional relevant fact is that the involved (co)Poisson structures directly depend on the multiparameter $\mathbf{q}$.

Now consider instead a non-trivial root of 1, say ε. Then the specialization of a MpQG at $q = \varepsilon$ is tightly related with its specialization at $q = 1$: this link is formalized in a so-called *quantum Frobenius morphism* — a Hopf algebra morphism with several remarkable properties between these two specialized MpQG's — moving to opposite directions in the restricted and the unrestricted case. We complete these morphisms to short exact sequences, whose middle objects are our MpQG's at $q = \varepsilon$; the new Hopf algebras we add to complete the sequences are named *small MpQG's*.

Remarkably enough, we prove that the above mentioned short exact sequences have the additional property of being *cleft*; as a consequence, our specialized MpQG's at $q = \varepsilon$ are in fact *cleft extensions* of the corresponding small MpQG's and the corresponding specialized MpQG's at $q = 1$ — which are *classical* geometrical objects, see above. Furthermore, implementing this construction in both cases — with restricted and with unrestricted forms — literally yields *two* small MpQG's: nevertheless, we eventually prove that they do coincide indeed.

To some extent, these results (at roots of 1) are a direct generalisation of what happens in the uniparameter case (i.e., for the canonical multiparameter). However, some of our results seem to be entirely new even for the uniparameter context.

Finally, here is the plan of the paper.

In section 2 we set some basic facts about Hopf algebras, the bosonization process, cocycle deformations, braided spaces, etc. — along with all the related notation.

Section 3 introduces our MpQG's: we define them by generators and relations, and we recall that we can get them as 2–cocycle deformations of the canonical one.

We collect in section 4 some fundamental results on MpQG's, such as the construction of quantum root vectors and PBW-like theorems (and related facts). In addition, we compare the multiplicative structure in the canonical MpQG with that in a general MpQG, the latter being thought of as cocycle deformation of the former.

In section 5 we introduce integral forms of our MpQG's — of restricted type and of unrestricted type — providing all the basic results one needs when working with them. We also shortly discuss mixed integral forms.

Section 6 focuses on specializations at 1, and the semiclassical structures arising from MpQG's by means of this process.

At last, in section 7 we finally harvest our main results. Namely, we deal with specializations at non-trivial roots of 1, with quantum Frobenius morphisms and with small MpQG's, for both the restricted version and the unrestricted one.

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