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"Twisted deformations vs. cocycle deformations for quantum groups"

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INTRODUCTION

Roughly speaking, quantum groups — in the form of quantized universal enveloping algebras — are Hopf algebra deformations of the universal enveloping algebra $U(\mathfrak{g})$ of some Lie algebra $\mathfrak{g}$. From this deformation, $\mathfrak{g}$ itself inherits (as "semiclassical limit" of the deformed coproduct) a Lie cobracket that makes it into a Lie bialgebra — the infinitesimal counterpart of a Poisson group whose tangent Lie algebra is $\mathfrak{g}$.

When $\mathfrak{g}$ is a complex simple Lie algebra, a quantum group in this sense, depending on a single parameter, was introduced by Drinfeld [Dr] as a formal series deformation $U_\hbar(\mathfrak{g})$ defined over a ring of formal power series (in the formal parameter $\hbar$) and by Jimbo and Lusztig (see [Ji], [Lu]) as a deformation $U_q(\mathfrak{g})$ defined over a ring of rational series (in the formal parameter q). Indeed, Jimbo's $U_q(\mathfrak{g})$ is actually a "polynomial version" of Drinfeld's $U_\hbar(\mathfrak{g})$.

Later on, several authors (cf. [BGH], [BW1,BW2], [CM], [CV1], [Hay], [HLT], [HPR], [Ko], [KT], [Ma], [OY], [Re], [Su], [Ta], to name a few) introduced many types of deformations of $U(\mathfrak{g})$ depending on several parameters, usually referred to as "multiparameter quantum groups". In turn, these richer deformations induce as semiclassical limits corresponding "multiparameter" bialgebra structures on $\mathfrak{g}$. The construction of these multiparameter deformations applies a general procedure, always available for Hopf algebras, following two patterns that we recall hereafter.

Let H be any Hopf algebra (in some braided tensor category). Among all possible deformations of the Hopf structure of H, we look at those in which only one of either the product or the coproduct is actually modified, while the other one is kept fixed. The general deformation will then be, somehow, an intermediate case between two such extremes. On the one hand, a *twist deformation* of H is a (new) Hopf algebra structure on H where the multiplicative structure is unchanged, whereas a new coproduct is defined by $\Delta^{\mathcal{F}}(x) := \mathcal{F}\,\Delta(x)\mathcal{F}^{-1}$ for $x \in H$: here $\mathcal{F}$ is an invertible element in $H^{\otimes 2}$ satisfying suitable axioms, called a "twist" for H. On the other hand, a 2–*cocycle deformation* of

H is one where the coproduct is unchanged, while a new product is defined via a formula which only depends on the old product and on a 2–cocycle σ of H (as an algebra): again, this procedure can be read as a suitable "conjugation" of the old product map by the 2–cocycle.

Inasmuch as a meaningful notion of "duality" applies to the Hopf algebras one is dealing with, these two constructions of deformations are *dual to each other*, directly by definition. In detail, if H^* is a Hopf algebra dual to H with respect to a non-degenerate (skew) Hopf pairing, e.g. H and $H^* := H^\circ$ (i.e., Sweedler's *restricted* dual), then the dual of the deformation by twist, resp. by 2–cocycle, of H is a deformation by 2–cocycle, resp. by twist, of H^*; moreover, the 2–cocycle, resp. the twist, on H^* is uniquely determined by the twist, resp. the 2–cocycle, on H. In order to stress this duality between the two types of deformation procedures that we are dealing with, as well as the fact that both are in fact "conjugations" of some sort, we adopt the terminology *"comultiplication twisting"* and *"multiplication twisting"*, instead of *"deformation by twist"* and of *"deformation by 2–cocycle"*, respectively.

It so happens that the large majority of multiparameter quantizations of $U(\mathfrak{g})$ considered in literature actually occur as either comultiplication twistings or multiplication twistings of a one-parameter quantization of Drinfeld's type or Jimbo-Lusztig's type. Indeed, in both cases the twists and the 2–cocycles taken into account are of special type, namely "toral" ones, in that (roughly speaking) they are defined only in terms of the (quantum) toral part of the one-parameter deformation of $U(\mathfrak{g})$.

Technically speaking, Drinfeld's $U_\hbar(\mathfrak{g})$ is better suited for comultiplication twistings, while Jimbo-Lusztig's $U_q(\mathfrak{g})$ is typically used for multiplication twistings (see [Re], [Ma], [Su], [HPR], [HLT], [CV1], [Ta]). As we aim to compare both kinds of twistings, we focus on polynomial one-parameter quantum groups $U_q(\mathfrak{g})$, and we adapt the very notion of "twist deformation", or "comultiplication twisting", to them. Then we consider both comultiplication twistings and multiplication twistings (of "toral type", in both cases) of $U_q(\mathfrak{g})$ — thus getting "twisted quantized universal enveloping algebras (=TwQUEA's)" and "multiparameter quantized universal enveloping algebras (=MpQUEA's)", respectively — and compare them. Moreover, by natural reasons we restrict ourselves to twists and cocycles that are defined by a *rational* datum, i.e., a matrix with rational entries.

As a first result, we describe the link *twist* $\longleftrightarrow$ *2–cocycle* under duality. Namely, quantum Borel (sub)groups $U_q(\mathfrak{b}_\pm)$ of opposite signs are in Hopf duality (in a proper sense): then we prove that any twisting on one side — of either comultiplication or multiplication — and the *dual* one on the other side — of either multiplication or comultiplication, respectively — are described by *the same rational datum*. Indeed, we provide an explicit bijection between the sets of toral twists and toral 2–cocycles.

As a second, more striking result (the core of our paper, indeed), we find that, in short, *twisted quantum groups and multiparameter quantum groups coincide*: namely, any TwQUEA can be realized as a MpQUEA, and viceversa. Even more precisely, the twist and the 2–cocycle involved in either realization are described by *the same (rational) datum*. This result is, in a sense, a side effect of the "autoduality" of quantum groups (in particular Borel ones). The proof is constructive, and quite explicit: indeed, switching from the realization as TwQUEA to that as MpQUEA and viceversa is a sheer *change of* *presentation*. We can shortly sketch the underlying motivation: any "standard" (=undeformed) quantum group is pointed (as a Hopf algebra); then any TwQUEA of "toral type" is pointed as well, and it is generated by the quantum torus and $(1, g)$–skew primitive elements: these new "homogeneous" generators yield a new presentation, which realizes the TwQUEA as a MpQUEA.

The direct consequence of this result is that (roughly speaking, and within the borders of our restrictions) there exists only *one* type of multiparameter quantization of $U(\mathfrak{g})$, and consequently only one type of corresponding multiparameter Lie bialgebra structure on $\mathfrak{g}$ arising as semiclassical limits, as in [GG1].

All the elements that lead us to the above mentioned results for TwQUEA's and MpQUEA's are also available for Hopf algebras that (like Borel quantum subgroups) are bosonizations of Nichols algebras of diagonal type; thus, we can replicate our work in that context too. In another direction, we extend further on this analysis in the framework of multiparametric *formal* QUEA's, à la Drinfeld — cf. [GG2].

We finish with a few words on the structure of the paper.

In Section 2 we collect the material on Hopf algebras and their deformations that will be later applied to quantum groups. Section 3 is devoted to introduce quantum groups (both in Drinfeld's version and in Jimbo-Lusztig's one) and their comultiplication twistings (of rational, toral type), i.e., the TwQUEA's: the part on *Drinfeld's* quantum groups could be dropped, yet we present it to explain the deep-rooting motivations of our work. In Section 4, instead, we present the multiplication twistings (of rational, toral type) of Jimbo-Lusztig's quantum groups, hence the MpQUEA's. Finally, in Section 5 we compare TwQUEA's and MpQUEA's, proving that — in a proper sense, under some finiteness assumption — they actually coincide.

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