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"A new equivalence between super Harish-Chandra pairs and Lie supergroups"

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INTRODUCTION

To every Lie supergroup G one can associate the $\big(G_{\mathbf{0}}, \mathfrak{g}\big)$ formed by the classical Lie group $G_{\mathbf{0}}$ underlying G and the tangent Lie superalgebra $\mathfrak{g} = \mathit{Lie}(G)$ of G; these two objects are "compatible" in a natural sense, so that their pair is what is called a "super Harish-Chandra pair", or just "sHCp" for short. Overall, mapping $G \mapsto \big(G_{\mathbf{0}}, \mathfrak{g}\big)$ yields a functor, call it Φ, from the category of Lie supergroups — either smooth, analytic or holomorphic — to the category of super Harish-Chandra pairs — of smooth, analytic or holomorphic type respectively. Is there any functor Ψ from sHCp's to Lie supergroups which be a quasi-inverse for Φ, so that the two categories be equivalent? And how much explicit such a functor (if any) is?

A first answer to this question was given by Kostant and by Koszul in the real smooth case (see [18] and [19]), providing an explicit quasi-inverse for Φ. Later on, Vishnyakova (see [24]) fixed the complex holomorphic case, and her proof works for the real analytic case as well. More recently, this result was increasingly extended to the setup of algebraic supergeometry (see [8], [21], [22]). It is worth remarking, though, that all these subsequent results were, in the end, further improvements of the original idea by Koszul (while Kostant's method was a slight variation of that), who defined a Lie supergroup out of a sHCp $\big(K_+, \mathfrak{k}\big)$ as a super-ringed space, defining the "proper" sheaf of superalgebras onto K_+ by means of $\mathfrak{k}$.

In this paper we present a new solution, namely we provide a new functor Ψ — in two different versions — from sHCp's to Lie supergroups that is quasi-inverse to Φ. For this we follow the approach where, instead of thinking of supermanifolds as being super-ringed manifolds, one treats them as suitable functors, defined on the category of "Weil superalgebras". This point of view allows to unify several different approaches to supergeometry (see [3]) and also to treat the infinite-dimensional setup (see [2]); for a broader discussion about this, we refer to classical sources as [4], [10], [20], [25] or more recent ones like [3], [5], [7], [23].

Now, if we want a functor Ψ from sHCp's to Lie supergroups, we need a Lie supergroup $G_{\mathcal{P}}$ for each sHCp $\mathcal{P}$; to have such a $G_{\mathcal{P}}$ (as a functor) we need a Lie group $G_{\mathcal{P}}(A)$ for each Weil superalgebra A, whose definition must be natural in A: moreover, one still has to show that the resulting functor have those additional properties that make it into a Lie supergroup. Finally, all this should aim to find a Ψ that is quasi-inverse to Φ — and this fixes ultimate bounds to the construction we aim to.

Bearing all this in mind, the construction that we present goes as follows. Given a super Harish-Chandra pair $\mathcal{P} = (G_+, \mathfrak{g})$, for each Weil superalgebra A, we define a group $G_{\mathcal{P}}(A)$ abstractly, by generators and relations: this definition is natural in A, hence it yields a functor from Weil algebras to (abstract) groups, call it $G_{\mathcal{P}}$ — cf. §3.1 and §3.3. As a key step in the work, we prove that $G_{\mathcal{P}}$ admits a "global splitting", i.e. it is the direct product of G_+ times a totally odd affine superspace (isomorphic to $\mathfrak{g}_{\mathbf{1}}$, the odd part of $\mathfrak{g}$): as both these are supermanifolds, it turns out that $G_{\mathcal{P}}$ itself is a supermanifold as well, hence it is a Lie supergroup because (as a functor) it is group-valued too — cf. §3.2 and §3.4. One more step proves that the construction of $G_{\mathcal{P}}$ is natural in $\mathcal{P}$, so it yields a functor Ψ from sHCp's to Lie supergroups: this is our candidate to be a quasi-inverse to Φ — cf. Theorem 3.2.6 and Theorem 3.4.6.

It is immediate to check that $\Phi \circ \Psi$ is isomorphic to the identity functor onto sHCp's, while proving that $\Psi \circ \Phi$ is isomorphic to the identity on Lie supergroups is much more demanding. For this we need to know that every Lie supergroup G has a "global splitting" on its own: this fact is more or less known among specialists, but we need it stated in a genuine geometrical form, while it is usually given in sheaf-theoretic terms — so we work it out explicitly (cf. §2.4). In fact, we find *two* different formulations of such a result: this is why, building upon them, we can provide *two* versions, Ψ° and Ψ^{e}, of a functor Ψ as required.

Finally, the reader can find a more detailed treatment in the expanded version [17] of this paper. Moreover, specific examples of application can be realized by suitably adapting the constructions of *algebraic* supergroups presented in [11], [12], [13], [14] and [15].

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