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"Duality features of left Hopf algebroids"

INTRODUCTION

A characteristic feature in standard Hopf algebra theory is its self-duality, that is, the dual of a (finite-dimensional) Hopf algebra (over a field) is a Hopf algebra again. In particular, the antipode of this dual is nothing but the transpose of the original antipode; see, for example, [Sw]. In the broader setup of (*left* or *full*) Hopf algebroids over possibly noncommutative rings, this peculiar property is generally lost; see [B] or §2 for the precise definitions of these objects, we only mention here that, in contrast to full Hopf algebroids, there is no notion of antipode for left Hopf algebroids: one rather considers the inverse of a certain Hopf-Galois map and its associated *translation map*. Nevertheless, left Hopf algebroids appear as the correct generalisation of Hopf algebras over noncommutative rings, whereas full Hopf algebroids generalise Hopf algebras twisted by a character, see, for example, [Ko, §4.1.2].

Recently (after the first posting of this article), Schauenburg [Sch2] showed that the (skew) dual of a left Hopf algebroid (under a suitable finiteness assumption) carries some Hopf structure as well without giving an explicit expression for the inverse of the respective Hopf-Galois map or the associated translation map.

However, instead of one dual, a left bialgebroid U rather possesses *two*, the *right dual* U^* and the *left dual* U_*, which, on top, live in a different category compared to U as they are both right bialgebroids [KadSz]. There is no reason why one should prefer one of the duals to the other. Hence, any question concerning "the dual of U" should be converted into a question about the pair $\big(U^*, U_*\big)$.

Dealing with full Hopf algebroids (see §5.2.1) does notably worsen the situation as there are actually *four* duals to be taken into account, two of which are left and two of which are right bialgebroids. In this case, an answer to the question of the nature of the Hopf structure on the dual(s) has only been given in certain cases, more precisely, in the presence of integrals [BSz, §5].

1.1. **Aims and objectives.** As mentioned a moment ago, the object one should investigate to discover the limits of self-duality in (left) Hopf algebroid theory is a pair of duals. In short, our question reads as follows: if a left bialgebroid U is, in particular, a left (or right) Hopf algebroid, what extra structure can be found on the pair $\big(U^*, U_*\big)$ of duals?

1.2. **Main results.** After highlighting in §3 a multitude of module structures that exist on Hom-spaces and tensor products in presence of a left or right Hopf algebroid structure and that will be used in the sequel, in §4 we review (and extend) Phùngs equivalence (cf. [Phù]) of comodule categories (see the main text for all definitions and conventions used hereafter):

Theorem A. *Let* (U, A) *be a left bialgebroid.*

(1) *Let* (U, A) *be additionally a left Hopf algebroid such that* $U_\triangleleft$ *is projective. Then there exists a (strict) monoidal functor* $\mathbf{Comod}\text{-}U \to U\text{-}\mathbf{Comod}$ *: if* M *is a right* U*–comodule with coaction* $m \mapsto m_{(0)} \otimes_A m_{(1)}$ *, then*

$$M \to U_\triangleleft \otimes_A M \ , \quad m \mapsto m_{(1)-} \otimes_A m_{(0)} \epsilon(m_{(1)+})$$

defines a left comodule structure on M *over* U *.*

(2) *Let (U, A) be a right Hopf algebroid such that ${}_{\triangleright}U$ is projective. Then there exists a (strict) monoidal functor U-$\mathbf{Comod} \to \mathbf{Comod}$-$U$: if N is a left U–comodule with coaction $n \mapsto n_{(-1)} \otimes_A n_{(0)}$, then defines a right comodule structure on N over U.*

(3) *If U is both a left and right Hopf algebroid and if both $U_{\triangleleft}$ and ${}_{\triangleright}U$ are A–projective, then the functors mentioned in (i) and (ii) are quasi-inverse to each other and we have an equivalence*

$$U\text{-}\mathbf{Comod} \simeq \mathbf{Comod}\text{-}U$$

of monoidal categories.

Note that this equivalence works without the help of an antipode as there are objects that are both left and right Hopf algebroids but not full Hopf algebroids (cocommutative left Hopf algebroids, for example).

Starting from this result, under suitable finiteness hypotheses on U, one can construct functors $\mathbf{Mod}$-$U_* \to \mathbf{Mod}$-U^* resp. $\mathbf{Mod}$-$U^* \to \mathbf{Mod}$-U_*, and from this we isolate maps $U^* \to U_*$ resp. $U_* \to U^*$, which even make sense without any finiteness assumptions as proven in §5, and which are our main object of interest.

In §5.1 we can then give the following answer to the problem mentioned in §1.1, that is, elucidate the relation between the left and the right dual:

Theorem B.

(1) *If (U, A) is moreover a left Hopf algebroid, there is a morphism $S^* : U^* \to U_*$ of A^e–rings with augmentation; if, in addition, both ${}_{\triangleright}U$ and $U_{\triangleleft}$ are finitely generated A–projective, then $\big(S^*, \mathrm{id}_A\big)$ is a morphism of right bialgebroids.*

(2) *If (U, A) is a right Hopf algebroid instead, there is a morphism $S_* : U_* \to U^*$ of A^e–rings with augmentation; if, in addition, both ${}_{\triangleright}U$ and $U_{\triangleleft}$ are finitely generated A–projective, then $\big(S_*, \mathrm{id}_A\big)$ is a morphism of right bialgebroids.*

(3) *If (U, A) is simultaneously both a left and a right Hopf algebroid, then the two morphisms are inverse to each other; hence, if both ${}_{\triangleright}U$ and $U_{\triangleleft}$ are finitely generated A–projective, then $U^* \simeq U_*$ as right bialgebroids.*

Now, as said before, for a left Hopf algebroid (which is finitely generated projective with respect to both source and target map) there is no canonical choice for which dual to consider but in view of Theorem B, in case the left Hopf algebroid is simultaneously a right Hopf algebroid, both duals are isomorphic and hence can be seen as *its* dual, which carries a Hopf structure by Schauenburg's recent result [Sch2]. This seems to be as close as one can get to self-duality.

Theorem B is a straight analogue of the construction on the dual for a (finite-dimensional) Hopf algebra H (over a field) with antipode S in the following sense: here, one has $H^* = (H_*)^{\mathrm{op}}_{\mathrm{coop}}$ and S^* is exactly the transpose of S and therefore the antipode for the dual Hopf algebra.

Observe that this last case in Theorem B, *i.e.*, the presence of both a left and right Hopf structure is given, for example, when U is a full Hopf algebroid with bijective antipode but also in weaker cases such as for the universal enveloping algebra of a Lie-Rinehart algebra. In the situation of a full Hopf algebroid, U^* and U_* are additionally linked (in both directions) by the transposition ${}^tS$ of the antipode $S : U \to U^{\mathrm{op}}_{\mathrm{coop}}$. However, in Theorem 5.2.4 we show that the map ${}^tS$ in general does not coincide with S^* or S_*, in contrast to the Hopf algebra case mentioned above. Moreover, if a left Hopf algebroid U is cocommutative with both ${}_{\triangleright}U$ and $U_{\triangleleft}$ finitely generated A–projective, then $U^* = \big(U_*\big)_{\mathrm{coop}}$ is a full Hopf algebroid (with antipode precisely given by S^*), though U might be not.

We shall also see in §6 that Theorem B actually extends to a larger setup, in particular, it applies to some interesting cases (coming from geometry), where neither ${}_{\triangleright}U$ nor $U_{\triangleleft}$

are finitely generated projective but U^* and U_* are still right bialgebroids in a suitable (topological) sense, such as when U is the universal enveloping of a Lie-Rinehart algebra, or a quantisation of it.

In §6, we illustrate these results by considering some examples related to Lie-Rinehart algebras (or Lie algebroids) and their jet spaces, as well as their quantised versions. Moreover, in §6.4 we consider further duality phenomena related to dualising modules, which appear in Poincaré duality, along with their quantisations.

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