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"Global splittings and super Harish-Chandra pairs for affine supergroups"

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INTRODUCTION

The study of "supergroups" is a chapter of "supergeometry", i.e. geometry in a $\mathbb{Z}_2$–graded sense. In particular, the relevant structure sheaves of (commutative) algebras sitting on top of the topological spaces one works with are replaced with sheaves of (commutative) *superalgebras*.

Every superalgebra A is built from (homogeneous) even and odd elements. It is then natural — especially in the commutative case, when these elements can be thought of as "functions" on some superspace — to look for some "separation of variables" result for A, in the form of a "splitting", i.e. a factorization of type $A = \overline{A} \otimes A'$ where $\overline{A}$ is a totally even subalgebra and A' is a second algebra which encodes the "odd part" of A. Actually, in the commutative case the best one can hope for is that A' be an algebra freely generated by some subsets of odd elements in A, hence A' is a Grassmann (super)algebra, i.e. the "polynomial (super)algebra" on some set of odd variables.

When coming to supergeometry, we deal with "superspaces" such as smooth or analytic supermanifolds (in the differential and complex holomorphic setup) or superschemes (in the algebro-geometric framework). Any such superspace can be considered as a *classical* (i.e. non-super) space – in the appropriate category — endowed with a suitable sheaf of commutative superalgebras.

A natural question then arises: can one parallelize this sheaf? In other words, is it globally trivial, in some "natural" sense? For superspaces (in any sense: differential, analytic, etc.) the answer in general is in the negative: indeed, counterexamples do exist. Instead, if we restrict to *supergroups* then the answer in most cases is *positive*. Indeed, this is the case for real Lie supergroups (see [1], [17], [6]) and for complex analytic supergroups (see [21] and [7]); in the algebro-geometric setting, the best result I am aware of is by Masuoka (see [18]), who proved that for all affine supergroups over fields of characteristic different from 2 the answer still is positive.

It might be worth minding the analogy with the situation of the tangent bundle on a classical space: for a generic space (manifold, complex analytic variety or scheme) in general it is *not* parallelizable; for *groups* instead (real Lie group, complex analytic Lie groups and group-schemes) it is known to be parallelizable. This might lead us to expect, from scratch, that a similar result occur with supergroups and their structure sheaf — although this is nothing but a sheer analogy.

Note that in the *affine* case having a parallelization of the structure sheaf on a superspace X amounts to having a "splitting" of its superalgebra of global sections $\mathcal{O}(X)$: this sets a link with the previously mentioned theme of splitting (commutative) superalgebras, and also leads us to saying that X has a "global splitting", or it is globally split, whenever its structure sheaf is parallelizable.

On the other hand, one can study any supergroup G, like any superspace, via its functor of points: then, for each commutative superalgebra A one has the group $G(A)$ of A–points of G. Such a group may have remarkable "splittings" (in group-theoretical sense) on its own; this kind of "pointwise splitting" is often considered in literature (e.g. in Boseck's papers [3], [4], [5]), but must not be confused with the notion of "global splitting".

Roughly speaking, a parallelized "supersheaf" $\mathcal{S}$ over a superspace X is "encoded" by a pair $\big(\mathcal{S}_{\mathbf{0}}\,, S_{x_0}\big)$ where $\mathcal{S}_{\mathbf{0}}$ is the "even part" of $\mathcal{S}$ and S_{x_0} is the fiber of $\mathcal{S}$ over some point x_0; as $\mathcal{S}_{\mathbf{0}}$ is encoded in the classical (i.e. non-super) space $X_{\mathbf{0}}$ underlying X, one can also use the pair $\big(X_{\mathbf{0}}\,, S_{x_0}\big)$ instead. When $X = G$ is a supergroup, we can take x_0 to be the identity element in the (classical) group $G_{\mathbf{0}}$ and approximate S_{x_0} with the cotangent space at $G_{\mathbf{0}}$ in that point; we can also replace this cotangent space with its dual, i.e. the *tangent Lie superalgebra* $\mathfrak{g} := \mathit{Lie}(G)$ *of* G.

This leads us to another — tightly related — way of formulating the problem, namely inquiring whether it is possible (via a "parallelization" of the structure sheaf, etc.) to describe a supergroup G in terms of the pair $\big(G_{\mathbf{0}}\,, \mathfrak{g}\big)$ which is naturally associated with it. Indeed, this is the core of the problem of studying supergroups via "super Harish-Chandra pairs", as I now explain.

The notion of "super Harish-Chandra pair" (a terminology first found in [9]), or just sHCp in the sequel, was first introduced in the real differential setup, but naturally adapts to the complex analytic or the algebro-geometric context (see, e.g., [21] and [7]). Whatever the setup, a sHCp is a pair $(G_+, \mathfrak{g})$ made of a classical group (real Lie, complex analytic, etc.) and a Lie superalgebra obeying natural compatibility constraints. Indeed, the definition itself is tailored in a such a way that there exists a natural functor Φ from the category of supergroups to the category of sHCp's which associates with each supergroup G its sHCp $\big(G_{ev}\,, \mathit{Lie}(G)\big)$ made of the "classical subgroup" and the "tangent Lie superalgebra" of G. The question is: can one recover a supergroup out of its associated sHCp? In other words, does there exist any functor Ψ from sHCp's to supergroups which be a quasi-inverse for Φ? And if the answer is positive, how much explicit such a functor is?

In the real differential framework — i.e. for real Lie supergroups and real smooth sHCp's — Kostant proved (see [15], and also [16]) that Φ is an equivalence i.e. one has a quasi-inverse for it.

Besides, Vishnyakova (see [21]) fixed both the real smooth and the complex analytic cases.

As to the *algebraic* setup, more recently Carmeli and Fioresi (see [7]) proved the same result for *algebraic affine supergroup schemes* (and the corresponding category of sHCp's) over a ground ring $\Bbbk$ that is an algebraically closed field of characteristic zero. Indeed, their method — which extends Vishnyakova's idea, so applies to the real smooth and complex analytic setup too — provides an explicit construction of a quasi-inverse functor Ψ for Φ. This was improved by Masuoka (in [19]), who only required that $\Bbbk$ be a field whose characteristic is not 2, and applied his result to a characteristic-free study of affine supergroup schemes. Later on (see [20]), Masuoka and Shibata further extended Koszul's method up to work on every commutative ring, via an algebraic version of the notion of sHCp devised to treat the matter with Hopf (super)algebra techniques.

In the second part of this paper I present a new solution to these problems, providing explicitly a new functor Ψ (different from those by other authors), which does the job; in particular, I also show that any positive answer is possible if and only if we restrict our attention to those (affine) supergroups which are globally strongly split — thus setting a link with the first part of the paper.

The above mentioned construction of the functor Ψ is made in the setup (and with the language) of algebraic supergeometry. Nevertheless, it is worth stressing that one can easily reformulate everything in the setup (and with the language) of real differential supergeometry or complex analytic supergeometry: in other words, the method presented here also applies, *mutatis mutandis*, to real or complex Lie supergroups (which, as we mentioned, are known to be all globally split).

The paper is organized as follows. First (Sec. 2) we establish the language and notations we need. Then (Sec. 3) we treat the notions of "splittings" for superalgebras, Hopf superalgebras, superschemes and supergroups; in particular, we present some results about global splittings of supergroups and about their "local" splittings, i.e. splittings on A–points. Finally (Sec. 4), we study the relation between supergroups and super Harish-Candra pairs, and the construction of a functor Ψ which is quasi-inverse to the natural one Φ associating a sHCp with any supergroup.

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