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"A global quantum duality principle for subgroups and homogeneous spaces"

Documenta Mathematica **19** (2014), 333-380.

INTRODUCTION

In this paper we work with quantizations of (algebraic) complex and real groups, their subgroups and homogeneous spaces, and a special symmetry among such quantum objects which we refer to as the "Global Quantum Duality Principle". This is just a last step in a process, which is worth recalling in short.

In any possible sense, quantum groups are suitable deformations of some algebraic objects attached with algebraic groups, or Lie groups. Once and for all, we adopt the point of view of algebraic groups: nevertheless, all our analysis and results can be easily converted in the language of Lie groups.

The first step to deal with is describing an algebraic group G via suitable algebraic object(s). This can be done following two main approaches, a *global* one or a *local* one.

In the *global geometry* approach, one considers $U(\mathfrak{g})$ — the universal enveloping algebra of the tangent Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ — and $F[G]$ — the algebra of regular functions on G. Both these are Hopf algebras, and there exists a non-degenerate pairing among them so that they are dual to each other. Clearly, $U(\mathfrak{g})$ only accounts for the local data of G encoded in $\mathfrak{g}$, whereas $F[G]$ instead totally describes G: thus $F[G]$ yields a global description of G, which is why we speak of "global geometry" approach.

In this context, one describes (globally) a subgroup K of G — always assumed to be Zariski closed — via the ideal in $F[G]$ of functions vanishing on it; alternatively, an infinitesimal description is given taking in $U(\mathfrak{g})$ the subalgebra $U(\mathfrak{k})$, where $\mathfrak{g} := \mathrm{Lie}(K)$.

For a homogeneous G–space, say M, one describes it in the form $M \cong G/K$ — which amounts to fixing some point in M and its stabilizer subgroup K in G. After this, a local description of $M \cong G/K$ is given by representing its left-invariant differential operators as $U(\mathfrak{g})/U(\mathfrak{g})\,\mathfrak{k}$: therefore, we can select $U(\mathfrak{g})\,\mathfrak{k}$ — a left ideal, left coideal in $U(\mathfrak{g})$ — as algebraic object to encode $M \cong G/K$, at least infinitesimally. For a global description instead, obstructions might occur. Indeed, we would like to describe $M \cong G/K$ via some algebra $F[M] \cong F\big[G/K\big]$ strictly related with $F[G]$. This varies after the nature of $M \cong G/K$ — hence of K — and in general might be problematic. Indeed, there exists a most natural candidate for this job, namely the set $F[G]^K$ of K–invariants of $F[G]$, which is a subalgebra and left coideal. The problem is that $F[G]^K$ permits to recover exactly G/K if and only if $M \cong G/K$ is a quasi-affine variety (which is not always the case). This yields a genuine obstruction, in the sense that this way of (globally) encoding the space $M \cong G/K$ only works with quasi-affine G–spaces; for the other cases, we just drop this approach — however, for a complete treatment of the case of *projective* G–spaces see [6].

In contrast, the approach of *formal geometry* is a looser one: one replaces $F[G]$ with a topological algebra $F[[G]] = F\big[\big[G_f\big]\big]$ — the algebra of "regular functions on the formal group G_f" associated with G — which can be realized either as the suitable completion of the local ring of G at its identity or as the (full) linear dual of $U(\mathfrak{g})$. In any case, both algebraic objects taken into account now only encode the local information of G.

In this formal geometry context, the description of (formal) subgroups and (formal) homogeneous spaces goes essentially the same. However, in this case no problem occurs with (formal) homogeneous space, as any one of them can be described via a suitably defined subalgebra of invariants $F\big[\big[G_f\big]\big]^{K_f}$: in a sense, "all formal homogeneous spaces are quasi-affine". As a consequence, the overall description one eventually achieves is entirely symmetric.

When dealing with quantizations, Poisson structures arise (as semiclassical limits) on groups and Lie algebras, so that we have to do with Poisson groups and Lie bialgebras. In turn, there exist distinguished subgroups and homogeneous spaces — and their infinitesimal counterparts — which are "well-behaving" with respect to these extra structures: these are *coisotropic subgroups* and *Poisson quotients*. Moreover, the well-known Poisson duality — among Poisson groups G and G^* and among Lie bialgebras $\mathfrak{g}$ and $\mathfrak{g}^*$ — extends to similar dualities among coisotropic subgroups (of G and G^*) and among Poisson quotients (of G and G^* again). It is also useful to notice that each subgroup contains a maximal coisotropic subgroup (its "coisotropic interior"), and accordingly each homogeneous space has a naturally associated Poisson quotient.

As to the algebraic description, all properties concerning Poisson (or Lie bialgebra) structures on groups, Lie algebras, subgroups and homogeneous spaces have unique characterizations in terms of the algebraic codification one adopts for these geometrical objects. Details change a bit according to whether one deals with global or formal geometry, but everything goes in parallel in either context.

By (complex) "quantum group" of *formal type* we mean any topological Hopf algebra $H_\hbar$ over the ring $\mathbb{C}[[\hbar]]$ whose semiclassical limit at $\hbar = 0$ — i.e., $H_\hbar \big/ \hbar\, H_\hbar$ — is of the form $F\big[\big[G_f\big]\big]$ or $U(\mathfrak{g})$ for some formal group G_f or Lie algebra $\mathfrak{g}$. Accordingly, one writes $H_\hbar := F_\hbar\big[\big[G_f\big]\big]$ or $H_\hbar := U_\hbar(\mathfrak{g})$, calling the former a QFSHA and the latter a QUEA. If such a quantization (of either type) exists, the formal group G_f is Poisson and $\mathfrak{g}$ is a Lie bialgebra; accordingly, a dual formal Poisson group G_f^* and a dual Lie bialgebra $\mathfrak{g}^*$ exist too.

In this context, as formal quantizations of subgroups or homogeneous spaces one typically considers suitable subobjects of either $F_\hbar\big[\big[G_f\big]\big]$ or $U_\hbar(\mathfrak{g})$ such that: *(1)* with respect to the containing formal Hopf algebra, they have the same relation as a in the "classical" setting — such as being a one-sided ideal, a subcoalgebra, etc.; *(2)* taking their specialization at $\hbar = 0$ is the same as restricting to them the specialization of the containing algebra (this is typically mentioned as a "flatness" property). This second requirement has a key consequence, i.e. the semiclassical limit object is necessarily "good" w.r. to the Poisson structure: namely, if we are quantizing a subgroup, then the latter is necessarily coisotropic, while if we are quantizing a homogeneous space then it is indeed a Poisson quotient.

In the spirit of global geometry, by (complex) "quantum group" of *global type* we mean any Hopf algebra H_q over the ring $\mathbb{C}\big[q,q^{-1}\big]$ whose semiclassical limit at $q=1$ — i.e., $H_q\big/(q-1)\,H_q$ — is of the form $F[G]$ or $U(\mathfrak{g})$ for some algebraic group G or Lie algebra $\mathfrak{g}$. Then one writes $H_q := F_q[G]$ or $H_q := U_\hbar(\mathfrak{g})$, calling the former a QFA and the latter a QUEA. Again, if such a quantization (of either type) exists the group G is Poisson and $\mathfrak{g}$ is a Lie bialgebra, so that dual formal Poisson groups G^* and a dual Lie bialgebra $\mathfrak{g}^*$ exist too.

As to subgroups and homogeneous spaces, global quantizations can be defined via a sheer reformulation of the same notions in the formal context: we refer to such quantizations as *strict*. In this paper, we introduce two more versions of quantizations, namely *proper* and *weak* ones, ordered by increasing generality, namely $\{\textit{strict}\} \subsetneq \{\textit{proper}\} \subsetneq \{\textit{weak}\}$. This is achieved by suitably weakening the condition *(2)* above which characterizes a quantum subgroup or quantum homogeneous space. Remarkably enough, one finds that now the existence of a *proper* quantization is already enough to force a subgroup to be coisotropic, or a homogeneous space to be a Poisson quotient.

The *Quantum Duality Principle* (=QDP) was first developed by Drinfeld (cf. [7], §7) for formal quantum groups (see [10] for details). It provides two functorial recipes, inverse to each other, acting as follows: one takes as input a QFSHA for G_f and yields as output a QUEA for $\mathfrak{g}^*$; the other one as input a QUEA for $\mathfrak{g}$ and yields as output a QFSHA for G_f^*.

The *Global Quantum Duality Principle* (=GQDP) is a version of the QDP tailored for global quantum groups (see [11,12]): now one functorial recipe takes as input a QFA for G and yields a QUEA for $\mathfrak{g}^*$, while the other takes a QUEA for $\mathfrak{g}$ and provides a QFA for G^*.

An appropriate version of the QDP for formal subgroups and formal homogeneous spaces was devised in [5]. Quite in short, the outcome there was an explicit recipe which taking as input a formal quantum subgroup, or a formal quantum homogeneous space, respectively, of G_f provides as output a quantum formal homogeneous space, or a formal quantum subgroup, respectively, of G_f^*. In short, these recipes come out as direct "restriction" (to formal quantum subgroups or formal quantum homogeneous spaces) of those in the QDP for formal quantum groups. This four-fold construction is fully symmetric, in particular all duality or orthogonality relations possibly holding among different quantum objects are preserved. Finally, Poisson duality is still involved, in that the semiclassical limit of the output quantum object is always the coisotropic dual of the semiclassical limit of the input quantum object.

The main purpose of the present work is to provide a suitable version of the GQDP for global quantum subgroups and global quantum homogeneous spaces — extending the GQDP for global quantum groups — as much general as possible. The inspiring idea, again, is to "adapt" (by restriction, in a sense) to these more general quantum objects the functorial recipes available from the GQDP for global quantum groups. Remarkably enough, this approach is fully successful: indeed, it does work properly not only with *strict* quantizations (which should sound natural) but also for *proper* and for *weak* ones. Even more, the output objects always are global quantizations (of subgroups or homogeneous spaces) of *proper* type — which gives an independent motivation to introduce the notion of proper quantization.

Also in this setup, Poisson duality, in a generalized sense, shows up again as the link between the input and the output of the GQDP recipes: namely, the semiclassical limit of the output quantum object is always the coisotropic dual of the coisotropic interior of the semiclassical limit of the input quantum object.

Besides the wider generality this GQDP applies to (in particular, involving also non-coisotropic subgroups, or homogeneous spaces which are not Poisson quotients), we pay a drawback in some lack of symmetry for the final result — compared to what one has in the formal quantization context. Nevertheless, such a symmetry is almost entirely recovered if one restricts to dealing with *strict* quantizations, or to dealing with "double quantizations" — involving simultaneously a QFA and a QUEA in perfect (i.e. non-degenerate) pairing.

At the end of the paper (Section 6) we present some applications of our GQDP: this is to show how it effectively works, and in particular that it does provide explicit examples of global quantum subgroups and global quantum homogeneous spaces. Among these, we also provide an example of a quantization which is *proper* but is *not strict* — which shows that the former notion is a non-trivial generalization of the latter.

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