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“Duality functors for quantum groupoids”

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INTRODUCTION

The classical theory of Lie groups, or of algebraic groups, has a quantum counterpart in the theory of “quantum groups”. In Drinfeld’s language, quantum groups are suitable topological Hopf algebras which are formal deformations either of the algebra of functions on a formal group, or of the universal enveloping algebras of a Lie algebra. These deformations add further structure on the classical object: the formal group inherits a structure of *Poisson* formal group, and the Lie algebra a structure of *Lie bialgebra*. Linear duality for topological Hopf algebras reasonably adapts to quantum groups, lifting the analogous duality for their semiclassical limits. On the other hand, Drinfeld revealed a more surprising feature of quantum groups, later named “quantum duality”, which somehow lifts the Poisson duality among Poisson (formal) groups. Namely, there exists an equivalence of categories between quantized enveloping algebras and quantized formal groups, which shifts from a quantization of a given Lie bialgebra L to one of the *dual* Lie bialgebra L^*.

Another extension of Lie group theory is that of Lie-Rinehart algebras (sometimes loosely called “Lie algebroids”), developed by Rinehart, Huebschmann and others. The notion of Lie-Rinehart algebra $\big(L\,,[\ \ ,\ \ ]\,,\omega\big)$ over a commutative ring A lies inbetween A–Lie algebras and k–Lie algebras of derivations of the form $Der\,(A)$. Well-known examples come from geometry, such as the global sections of a Lie algebroid, for example the 1–forms over a Poisson manifold (cf. [7], [15], [10]).

The natural algebraic gadgets attached with a Lie-Rinehart algebra are its universal enveloping algebra $V^\ell(L)$ and its algebra of jets $J^r(L)$, which are in linear duality with each other. Any Lie-Rinehart algebra L can also be seen as a *right* Lie-Rinehart algebra: thus one can also consider its right enveloping algebra, call it $V^r(L)$, anti-isomorphic to $V^\ell(L)$, and its dual $J^\ell(L)$.

All these algebraic objects — $V^\ell(L)$, $V^r(L)$, $J^r(L)$ and $J^\ell(L)$ — are *(topological) bialgebroid* — *left* ones when a superscript “ ℓ ” occurs, and *right* when “ r ” does. Indeed, they also have an additional property, about their Hopf-Galois map, such that these left/right bialgebroids are actually *left* or *right Hopf left/right bialgebroids* — an important generalization of Hopf algebras.

Linear duality for (left/right) bialgebroids is twofold: any (left/right) bialgebroid U is naturally a left A–module and a right A–module, thus one may consider its *left dual* U_*

as well as its *right dual* U^*. Under mild conditions, U^* and U_* are naturally (right/left) bialgebroids (see [Kadison and Szlachanyi]). The $\big(V^\ell(L)\,,J^r(L)\big)$ is tied together by such a linear duality, and similarly for $\big(V^r(L)\,,J^\ell(L)\big)$.

When looking for quantizations of Lie-Rinehart algebras, one should consider formal deformations of either $V^{\ell/r}(L)$ or $J^{r/\ell}(L)$, among left/right (topological) bialgebroids: these deformations automatically inherit from their semiclassical limits the additional property of being *left/right Hopf left/right bialgebroids*. We shall loosely call such deformations "quantum groupoids".

The first step in this direction was made by Ping Xu (cf. [34]): he introduced a notion of quantization of $V^\ell(L)$, called quantum universal enveloping algebroid (LQUEAd in short). Then he noticed that any such quantization endows L with a richer structure of *Lie-Rinehart bialgebra*. This is a direct extension of the notion of Lie bialgebra, in particular, it is a self-dual notion, so if L is a Lie-Rinehart bialgebra then its dual L^* is a Lie-Rinehart bialgebra as well (see [Kosmann]). Finally, Xu also provided an example of construction of a non-trivial LQUEAd $\mathcal{D}_X[[h]]^{\mathcal{F}}$, by "twisting" the trivial deformation $\mathcal{D}_X[[h]]$ of $\mathcal{D}_X := V^\ell\big(\Gamma(TX)\big)$, where X is a Poisson manifold.

The purpose of this paper is to move some further steps in the theory of "quantum groupoids".

After recalling some basics of the theory of Lie-Rinehart algebras and bialgebras (Sec. 2), we introduce also some basics of the theory of bialgebroids (Sec. 3): in particular, we dwell on the relevant examples, i.e. universal enveloping algebras and jet spaces for Lie-Rinehart algebras.

Then we introduce "quantum groupoids" (Sec. 4). Besides Xu's original notion of LQUEAd, we introduce its right counterpart (in short RQUEAd): a topological right bialgebroid which is a formal deformation of some $V^r(L)$. Similarly, we introduce quantizations of jet spaces; a topological right bialgebroid which is a formal deformation of some $J^r(L)$ will be called a *right quantum formal series algebroid* (RQFSAd in short); similarly, the left-handed version of this notion gives rise to the definition of *left quantum formal series algebroid* (LQFSAd in short). Altogether, this gives us four kinds of quantum groupoids; each one of these induces a Lie-Rinehart *bialgebra* structures on the original Lie-Rinehart algebra one deals with, extending what happens with LQUEAd's.

As a next step, we discuss linear duality for quantum groupoids (Sec. 5). The natural language is that of linear duality for bialgebroids, with some precisions. First, by infinite rank reasons we are lead to consider *topological* duals. Second, both left and right duals are available, thus taking duals might cause a proliferation of objects. Nevertheless, we can keep this phenomenon under control, so eventually we can bound ourselves to deal with only a handful of duality functors.

In the end, our main result on the subject claims the following: our duality functors provide (well-defined) anti-equivalences between the category of all LQUEAd's and the category of all RQFSAd's (on a same, fixed ring A_h), and similarly also anti-equivalences between the category of all RQUEAd's and the category of all LQFSAd's (on A_h again). In addition, if one starts with a given quantum groupoid, which induces a specific (Lie-Rinehart) bialgebra structure on the underlying Lie-Rinehart algebra, then the dual quantization yields the same or the coopposite Lie-Rinehart bialgebra structure — see Theorems 5.1.5 and 5.2.2 for further precisions.

Finally (Sec. 6), we develop a suitable "Quantum Duality Principle" for quantum groupoids. Indeed, we introduce functors "à la Drinfeld", denoted by $(\ )^\vee$ and $(\ )'$, which turns (L/R)QFSAd's into (L/R)QUEAd's and viceversa, so to provide an equivalence between the category of LQFSAd's and that of LQUEAd's, and a similar equivalence between RQFSAd's and RQUEAd's. In addition, if one starts with a quantization of some Lie-Rinehart bialgebra L, then the (appropriate) Drinfeld's functor gets out of it a quantization of the *dual* Lie-Rinehart bialgebra L^*.

For the functor $(\ )^\vee$, Drinfeld's original definition for quantum groups can be easily extended to quantum groupoids. Instead, this is not the case for the functor $(\ )'$: therefore we have resort to a different characterization (for quantum groups) of it, and adopt that as a definition (for quantum groupoids): this requires linear duality, which sets a strong link with the first part of the paper.

It is worth remarking that linear duality for quantum groupoids interchanges "left" and "right"; instead, quantum duality takes either one to itself: at the end of the day, this means that if one aims to have both linear duality and quantum duality then he/she is forced to deal with *all four types of quantum groupoids* that we introduced — none of them can be left apart.

At the end (Sec. 7) we present an example, just to illustrate some of our main results on a single — and simple, yet significant enough — toy model.

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