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"Chevalley Supergroups of type $D(2,1;a)$"

INTRODUCTION

In his work of 1955, Chevalley provided a combinatorial construction of all simple affine algebraic groups over any field. In particular, his method led to an existence theorem for simple affine algebraic groups: one starts with a simple (complex, finite-dimensional) Lie algebra and a simple module V for it, and realizes the required group as a closed subgroup of $\mathrm{GL}(V)$. This can also be recast as to provide a description of all simple affine groups as group schemes over $\mathbb{Z}$.

In [6] the philosophy of Chevalley was revisited in the context of supergeometry. The outcome is a construction of affine supergroups whose tangent Lie superalgebra is of *classical type*. However, some exceptions were left out, namely the cases when the Lie superalgebra is of type $D(2,1;a)$ and the parameter a is not an integer number; the present work fills in this gap. As the case of simple Lie superalgebras of Cartan type is solved in [9], this paper completes the program of constructing connected affine supergroups associated with any simple Lie superalgebra.

By "affine supergroup" here I mean a representable functor from the category (salg) of commutative superalgebras — over some fixed ground ring — to the category (groups) of groups: in other words, an affine supergroup-scheme, identified with its functor of points. In [6], one first constructs a functor from (salg) to (groups), recovering Chevalley's ideas to define the values of such a group functor on each superalgebra A — i.e., to define its A–points; then one proves that the sheafification of this functor is representable — hence it is an affine supergroup-scheme.

For the case $D(2,1;a)$ — with $a \notin \mathbb{Z}$ — one needs a careful modification of the general procedure of [6]; thus the presentation hereafter will detail those steps which need changes, and will simply refer to [6] for those where the original arguments still work unchanged.

The initial datum is a simple Lie superalgebra $\mathfrak{g} = D(2,1;a)$.

We start with basic results on $\mathfrak{g}$: the existence of *Chevalley bases* (with nice integrality properties) and a PBW theorem for the Kostant $\mathbb{Z}$–form of the universal enveloping superalgebra $U(\mathfrak{g})$.

Next we take a faithful, finite-dimensional $\mathfrak{g}$–module V, and we show it has suitable lattices M invariant by the Kostant superalgebra. This allows to define — functorially — additive and multiplicative one-parameter (super)subgroups of operators acting on scalar extensions of M. The additive subgroups are just like in the general case: there exists one of them for every root of $\mathfrak{g}$. The multiplicative ones instead are associated to elements of the fixed Cartan subalgebra of $\mathfrak{g}$, and are of two types: those of *classical* type, modeled on the group functor $A \mapsto U(A_0)$ — the *group of units* of A_0 — and those of *a–type*, modeled on the group functor $A \mapsto P_a(A)$ — the group of elements of A_0 "which may be raised to the a^k–th power, for all k". The second type of multiplicative one-parameter subgroups, not used in [fg1], is now needed because one has to consider the "operation" $t \mapsto t^a$, defined just for $t \in P_a(A)$; this marks a difference with the case $a \in \mathbb{Z}$.

Then we consider the functor $G : (\text{salg}) \longrightarrow (\text{groups})$ whose value $G(A)$ on $A \in (\text{salg})$ is the subgroup of $\mathrm{GL}(V(A))$ — with $V(A) := A \otimes M$ — generated by all the homogeneous one-parameter supersubgroups mentioned above. This functor is a presheaf, hence we can take its sheafification $\mathbf{G}_V = \mathbf{G} : (\text{salg}) \longrightarrow (\text{groups})$. These $\mathbf{G}_V$ are, by definition, our "Chevalley supergroups".

Acting just like in [6], one defines a "classical affine subgroup" $\mathbf{G}_0$ of $\mathbf{G}_V$, corresponding to the even part $\mathfrak{g}_0$ of $\mathfrak{g}$ (and to V), and then finds a factorization $\mathbf{G}_V = \mathbf{G}_0 \, \mathbf{G}_1 \cong \mathbf{G}_0 \times \mathbf{G}_1$, where $\mathbf{G}_1$ corresponds instead to the odd part $\mathfrak{g}_1$ of $\mathfrak{g}$. Actually, one has even a finer factorization $\mathbf{G}_V = \mathbf{G}_0 \times \mathbf{G}_1^{-,<} \times \mathbf{G}_1^{+,<}$ with $\mathbf{G}_1^{\pm,<}$ being totally odd superspaces associated to the positive or negative odd roots of $\mathfrak{g}$. Thus $\mathbf{G}_1 = \mathbf{G}_1^{-,<} \times \mathbf{G}_1^{+,<}$ is representable, and $\mathbf{G}_0$ is representable too, hence the above factorization implies that $\mathbf{G}_V$ is representable too, so it is an affine supergroup. The outcome then is that our Chevalley supergroups are affine supergroups.

Finally, one also proves that our construction is functorial in V and that $\mathrm{Lie}(\mathbf{G}_V)$ is just $\mathfrak{g}$ as one expects, like in [6] (no special changes are needed).

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