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"Algebraic supergroups with Lie superalgebras of classical type"

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INTRODUCTION

In [7] we have given the supergeometric analogue of the classical Chevalley's construction (see [16]), which enabled us to build a supergroup out of data involving only a complex Lie superalgebra $\mathfrak{g}$ of classical type and a suitable complex faithful representation. Such a supergroup is (affine) connected, with associated classical subgroup being k–split reductive and with tangent Lie superalgebra isomorphic to $\mathfrak{g}$: thus we obtained an *existence result* for such supergroups. In particular, this provided the first unified construction of affine algebraic supergroups with Lie superalgebras of classical type; it was also the very first explicit construction of algebraic supergroups corresponding to the basic exceptional and to the strange Lie superalgebras.

In this paper we discuss the *uniqueness* problem, cast in the following form: "is any such supergroup isomorphic to a supergroup obtained via the Chevalley's construction"?

Our answer is positive and we get an analogue of the classical result — mainly due to Chevalley and Demazure — which classifies $\mathbb{Z}$–split reductive connected affine group-schemes via pairs of the form $(\mathfrak{g}\,, V)$ where $\mathfrak{g}$ is a finite dimensional complex reductive Lie algebra and V is a faithful finite-dimensional simple $\mathfrak{g}$–module.

We also prove a side result, which we believe has an interest on its own: every Chevalley supergroup, realized through our recipe as a supersubgroup of some linear supergroup $\mathrm{GL}(V)$, is actually *closed* inside $\mathrm{GL}(V)$ itself. This implies that such a group is *globally split* and *smooth*.

Let us briefly describe how we obtain our main result. We start with an affine algebraic supergroup G, defined over a field k with associated classical subgroup G_0 k–split, reductive, and with Lie superalgebra a k–form of a complex Lie superalgebra of classical type. Moreover, we assume G to be linearizable, a fact that is automatically true when mild conditions are satisfied (e.g., when k is a field). Note that all of these conditions appear to be necessary, since they do hold for Chevalley supergroups.

Since G_0 is k–split and reductive, by Chevalley-Demazure theory it can be realized as a closed subgroup of some $\mathrm{GL}\big(V'\big)$, where V' is a suitable G_0–module. The fact G is linearizable, that is $G \subset \mathrm{GL}_{m|n}$ (for suitable m and n), allows us to build the induced $\big(\mathrm{GL}_{m|n}\big)_0$–module $W' := \mathit{Ind}_{\mathbf{G}_0}^{(\mathrm{GL}_{m|n})_0}\big(V'\big)$, which is also naturally a $\big(\mathrm{gl}_{m|n}\big)_0$–module. Inducing again we obtain the $\mathrm{gl}_{m|n}$–module $W := \mathit{Ind}_{(\mathrm{gl}_{m|n})_0}^{\mathrm{gl}_{m|n}}\big(W'\big) = \mathcal{U}\big(\mathrm{gl}_{m|n}\big) \otimes_{\mathcal{U}((\mathrm{gl}_{m|n})_0)} W'$. W is also a $\mathrm{GL}_{m|n}$-module and (by restriction) a G-module. Furthermore it contains a G–submodule $V := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} V'$.

The very construction of V allows us to build the Chevalley supergroup G_V associated with the $\mathfrak{g}$–representation V. This allows us to view both our given supergroup G and the Chevalley supergroup G_V as closed subgroups of the *same* $\mathrm{GL}(V)$. The last step is to note that both G and G_V are globally split — as any affine supergroup over a field. Since the ordinary algebraic groups are the same, $G_0 = (G_V)_0$, we have that both supergroups are smooth as well. We conclude then $G = G_V$ by infinitesimal considerations, since both supergroups are globally split and smooth and they have the same Lie superalgebra.

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