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"Algebraic supergroups of Cartan type"

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INTRODUCTION

A real milestone in classical Lie theory is the celebrated classification theorem for complex finite dimensional simple Lie algebras. A similar key result is the classification of all complex finite dimensional simple Lie superalgebras (cf. [14]); in particular, this ensures that these objects form two disjoint families: those of *classical* type, and those of *Cartan* type. The "classical" ones are strict super-analogue of simple, f.d. complex Lie algebras; the "Cartan" ones instead are a super-analogue of complex Lie algebras of Cartan type, which are simple but *infinite* dimensional.

As in the standard Lie context, one can base upon this classification result to tackle the classification problem of existence, construction and uniqueness of simple Lie supergroups, or even simple algebraic supergroups. A super-analogue of Lie's Third Theorem solves it for Lie supergroups: but the question remains for construction and for the whole algebraic point of view.

In the standard context, a constructive procedure providing all (f.d., connected) simple algebraic groups was provided by Chevalley, over fields; one starts with a (complex) f.d. simple Lie algebra $\mathfrak{g}$ a faithful $\mathfrak{g}$–module, and eventually realizes a group of requested type as a subgroup of $\mathrm{GL}(V)$. In particular, this yields all connected algebraic groups whose tangent Lie algebra is a (f.d.) simple one; this method (and result) also extends to the framework of reductive $\mathbb{Z}$–group schemes. By analogy, one might try to adapt Chevalley's method to the f.d. simple Lie superalgebras of classical type, so to provide connected algebraic supergroup-schemes (over $\mathbb{Z}$) which "integrate" any such Lie superalgebra. This is done in [8] — see also [9] and [11]. In this paper instead I implement Chevalley's idea to simple Lie superalgebras of Cartan type, with full success: the main result is an existence result, via a constructive procedure, for connected, algebraic supergroup-schemes (over any ring, e.g. $\mathbb{Z}$) whose tangent Lie superalgebra be simple of Cartan type. As a second result, I prove also a uniqueness theorem for algebraic supergroups of the above mentioned type.

Hereafter I shortly sketch how the present work is organized.

The initial datum is a f.d. simple Lie superalgebra of Cartan type, say $\mathfrak{g}$. Basing upon a detailed description of the root spaces (with respect to a fixed Cartan subalgebra), I introduce the key notion of *Chevalley basis*. Then I prove two basic results: the existence of Chevalley bases, and a PBW-like theorem for the "Kostant $\mathbb{Z}$–form" of the universal enveloping superalgebra of $\mathfrak{g}$.

Next I take a faithful $\mathfrak{g}$–module V, and I show that there exists a lattice M in V fixed by the Kostant superalgebra and also by a certain integral form $\mathfrak{g}_V$ of $\mathfrak{g}$. I define a functor G_V from the category $(\mathrm{salg})_{\Bbbk}$ of commutative $\Bbbk$–superalgebras to the category (groups) of groups as follows: for $A \in (\mathrm{salg})_{\Bbbk}$, I let $G_V(A)$ be the subgroup of $\mathrm{GL}\big(A \otimes_{\mathbb{Z}} M\big)$ generated by "homogeneous one-parameter subgroups" associated with the root vectors and with the toral elements in a Chevalley basis. Then I pick the sheafification $\mathbf{G}_V$ (in the sense of category theory) of the functor G_V.

Using commutation relations among generators, I find a factorization of $\mathbf{G}_V$ into direct product of representable (algebraic) superschemes: thus $\mathbf{G}_V$ itself is representable, hence it is an "affine (algebraic) supergroup". Some extra work shows how $\mathbf{G}_V$ depends on V, that it is independent of the choice M and that its tangent Lie superalgebra is $\mathfrak{g}_V$. So the construction of $\mathbf{G}_V$ yields an *existence theorem* of a supergroup having $\mathfrak{g}_V$ as tangent Lie superalgebra. Right after, I prove the converse, i.e. a *uniqueness theorem* showing that any such supergroup is isomorphic to some $\mathbf{G}_V$.

Finally, I illustrate the example of $\mathbf{G}_V$ for $\mathfrak{g}$ of type $W(n)$ and V its defining representation — i.e. the Grassmann algebra in n odd indeterminates, $W(n)$ being the algebra of its superderivations.

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References

[1] J. Brundan, A. Kleshchev, *Modular representations of the supergroup $Q(n)$, I*, J. Algebra **260** (2003), 64–98.

[2] Y. A. Bahturin, A. A. Mikhalev, V. M. Petrogradsky, M. V. Zaicev, *Infinite-dimensional Lie superalgebras,* de Gruyter Expositions in Mathematics, vol. 7, Walter de Gruyter & Co., Berlin, 1992.

[3] C. Carmeli, L. Caston, R. Fioresi, *Mathematical Foundation of Supersymmetry,* EMS Series of Lectures in Mathematics, vol. 15, European Mathematical Society, Zurich, 2011.

[4] M. Demazure, *Groupes réductifs: déploiements, sous-groupes, groupes-quotients*, in: M. Demazure, A. Grothendieck (eds.), *"Structure des Schémas en Groupes Réductifs"*, Schémas en Groupes (SGA 3), Lecture Notes in Mathematics **153**, Springer-Verlag, Berlin-New York, 1962/64, Exposé XXII, pp. 156–262.

[5] M. Demazure, *Groupes réductifs: unicité des groupes épinglés*, in: M. Demazure, A. Grothendieck (eds.), *"Structure des Schémas en Groupes Réductifs"*, Schémas en Groupes (SGA 3), Exposé XXIII — Lecture Notes in Mathematics **153**, Springer-Verlag, Berlin-New York, 1962/64, Exposé XXIII, pp. 263–409.

[6] P. Deligne, J. Morgan, *Notes on supersymmetry (following J. Bernstein)*, in: *Quantum fields and strings. A course for mathematicians*, Vol. 1, 2 (Princeton, NJ, 1996/1997), 41–97, American Mathematical Society, Providence, RI, 1999.

[7] M. Duflo, private communication.

[8] R. Fioresi, F. Gavarini, *Chevalley Supergroups*, Memoirs AMS **215**, no. 1014 (2011).

[9] R. Fioresi, F. Gavarini, *On the construction of Chevalley Supergroups,* in: S. Ferrara, R. Fioresi,

V. S. Varadarajan (eds.), *"Supersymmetry in Mathematics & Physics"*, UCLA Los Angeles, U.S.A., 2010, Lecture Notes in Mathematics **2027**, Springer-Verlag, Berlin-Heidelberg, 2011, pp. 101–123.

[10] R. Fioresi, F. Gavarini, *Algebraic supergroups with Lie superalgebras of classical type*, preprint `arXiv:1106.4168v3` [math.RA] (2011).

[11] F. Gavarini, *Chevalley Supergroups of type* $D(2,1;a)$, preprint `arXiv:1006.0464v1` [math.RA] (2010).

[12] D. Grantcharov, *On the structure and characters of weight modules*, Forum Mathematicum **18** (2006), 933–950.

[13] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics **9**, Springer-Verlag, New York-Heidelberg, 1972.

[14] V. G. Kac, *Lie superalgebras*, Advances in Mathematics **26** (1977), 8–26.

[15] Y. Manin, *Gauge field theory and complex geometry,* Grundlehren der Mathematischen Wissenschaften **289**, Springer-Verlag, Berlin-Heidelberg-New York, 1988.

[16] V. Serganova, *On Representations of Cartan Type Lie Superalgebras*, American Mathematical Society Translations (2) **213** (2005), 223–239.

[17] R. Steinberg, *Lectures on Chevalley groups*, Yale University, New Haven, Conn., 1968.

[18] V. S. Varadarajan, *Supersymmetry for mathematicians: an introduction,* Courant Lecture Notes **1**, American Mathematical Society, Providence, RI, 2004.

[19] A. Vistoli, *Grothendieck topologies, fibered categories and descent theory*, in: *Fundamental algebraic geometry*, 1–104, Math. Surveys Monogr. **123**, Amer. Math. Soc., Providence, RI, 2005.