R. Fioresi, F. Gavarini

"On the construction of Chevalley Supergroups"

in: *Supersymmetry in Mathematics and Physics*, pp. 101–123,
S. Ferrara, R. Fioresi, V. S. Varadarajan (eds.), UCLA Los Angeles, U.S.A. 2010,
Lecture Notes in Mathematics **2027**, Springer-Verlag, Berlin-Heidelberg, 2011
— **DOI:** 10.1007/978-3-642-21744-9_5 —

the original publication is available at
http://www.springerlink.com/content/j378521716k7204p/

INTRODUCTION

The notion of Chevalley group, introduced by Chevalley in 1955, provided a unified combinatorial construction of all simple algebraic groups over a generic field k. The consequences of Chevalley's work were many and have had tremendous impact in the following decades. His construction was motivated by issues linked to the problem of the classification of semisimple algebraic groups: he provided an existence theorem for such groups, essentially exhibiting an example of simple group for each of the predicted possibility. In the course of this discussion, he discovered new examples of finite simple groups, which had escaped to the group theorists up to then. Later on, in the framework of a modern treatment of algebraic geometry, his work was instrumental to show that all simple algebraic groups are algebraic schemes over $\mathbb{Z}$ and to study arithmetic questions over arbitrary fields.

We may say that we have similar motivations: we want a unified approach to describe all algebraic supergroups, which have Lie superalgebras of classical type and we also want to give new examples of supergroups, over arbitrary fields. For instance, our discussion enables us to provide an explicit construction of algebraic supergroups associated with the exceptional and the strange Lie superalgebras. To our knowledge these supergroups have not been examined before, though an approach in the differential setting can be very well carried through via the language of *super Harish-Chandra pairs*. In such approach a supergroup is understood as a pair $(G_0, \mathfrak{g})$, consisting of an ordinary group G_0 and a super Lie algebra $\mathfrak{g}$, with even part $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$, together with some natural compatibility conditions involving the adjoint action of the group G_0 on $\mathfrak{g}$. It is clear that in positive characteristic this method shows severe limitations.

In the present work we outline the construction of the Chevalley supergroups associated with Lie superalgebras of classical type. We shall not present complete proofs for our statements, they will appear in [9], however we shall concentrate on the key ideas and examples that will help to understand our construction.

In our statements, we shall leave out the strange Lie superalgebra $Q(n)$ and some low dimensional cases, which can be treated very well with the same method, with minor modifications, but present extra difficulties that make our construction and notation opaque.

Essentially, we are going to follow Chevalley's recipe and push it as far as we can, before resorting to more sophisticated algebraic geometry techniques, when the supergeometric nature of our objects forces us to do so.

We start with a complex Lie superalgebra of classical type $\mathfrak{g}$, together with a fixed Cartan subalgebra $\mathfrak{h}$, and we define the *Chevalley basis* of $\mathfrak{g}$. This is an homogeneous basis of $\mathfrak{g}$, as super vector space, whose elements have the brackets expressed as a linear combination of the basis elements with just *integral* coefficients. Consequently they give us an integral form of $\mathfrak{g}$, that we call $\mathfrak{g}_{\mathbb{Z}}$ the *Chevalley Lie superalgebra* associated with $\mathfrak{g}$ and $\mathfrak{h}$. Such integral form gives raise to the Kostant integral form $K_{\mathbb{Z}}(\mathfrak{g})$ of the universal enveloping superalgebra $U(\mathfrak{g})$ of $\mathfrak{g}$. $K_{\mathbb{Z}}(\mathfrak{g})$ is free over $\mathbb{Z}$ with basis given by the ordered monomials in the divided powers of the root vectors and the binomial coefficients in the generators of $\mathfrak{h}$ in the Chevalley basis: $X^m/m!$, $\binom{H_i}{n}$, $\alpha \in \Delta$ (root system) and $m, n \in \mathbb{N}$.

Next, we look at a faithful rational representation of $\mathfrak{g}$ in a finite dimensional complex vector space V. Inside V we can find an *integral lattice* M which is invariant under the action of $K_{\mathbb{Z}}(\mathfrak{g})$ and its stabilizer $\mathfrak{g}_V$ in $\mathfrak{g}$ defines an integral form of $\mathfrak{g}$. In complete analogy with Chevalley, for an arbitrary field k, we can give the following key definitions:

$$V_k := k \otimes_{\mathbb{Z}} M, \qquad \mathfrak{g}_k := k \otimes_{\mathbb{Z}} \mathfrak{g}_V, \qquad U_k := k \otimes_{\mathbb{Z}} K_{\mathbb{Z}}(\mathfrak{g}).$$

We could even take k to be a commutative ring, however for the scope of the present work and to stress the analogy with Chevalley's construction, we prefer the restrictive hypothesis of k to be a field.

This is the point where our construction departs dramatically from Chevalley's one. In fact, starting from the faithful representation V_k of $\mathfrak{g}_k$, Chevalley defines the Chevalley group G_V as generated by the exponentials $\exp(tX_\alpha) := 1 + tX_\alpha + (t^2/2)X_\alpha^2 + \dots$, for $t \in k$ and X_α the root vector corresponding to the root α in the Chevalley basis. Such an expression makes sense since the X_α's act as nilpotent elements. If we were to repeat without changes this construction in the super setting, we shall find only ordinary groups over k associated with the Lie algebra $\mathfrak{g}_0$, the even part of $\mathfrak{g}$. This is because over a field, we cannot see any supergeometric behaviour; the only thing we can recapture is the underlying classical object. For this reason, we need to go beyond Chevalley's construction and build our supergroups as *functors*.

We define $\mathbf{G}$ the *Chevalley supergroup* associated with $\mathfrak{g}$ and the faithful representation V, as the functor $\mathbf{G} : (\text{salg}) \longrightarrow (\text{sets})$, with $\mathbf{G}(A)$ the subgroup of $\mathbf{GL}(A \otimes V_k)$ generated by $\mathbf{G}_0(A)$ and the elements $1 + \theta_\beta X_\beta$, for $\beta \in \Delta_1$. In other words we have:

$$\mathbf{G}_V(A) = \langle \mathbf{G}_0(A), 1 + \theta_\beta X_\beta \rangle \subset \mathbf{GL}(A \otimes V_k), \qquad A \in (\text{salg}), \qquad \theta_\beta \in A_1$$

where (salg) and (sets) are the categories of commutative superalgebras and sets respectively and (as always) we use X_β to denote also the image of the root vector X_β in the chosen faithful representation V_k. $\mathbf{G}_0$ is the functor of points of the (reductive) algebraic supergroup associated to $\mathfrak{g}_0$ and the representation V_k.

This is a somehow natural generalization of what Chevalley does in his original construction: he provides the k-points of the algebraic group scheme constructed starting from a complex semisimple Lie algebra and a faithful representation, for all the fields k, while we give the A-points of the supergroup scheme for any commutative k-superalgebra A.

Once this definition is properly established, we need to show that $\mathbf{G}$ is the functor of points of an algebraic supergroup, in other words, that it is representable. This is the price to pay when we employ the language of the functor of points: it is much easier to define geometric objects, however we need to prove representability in order to speak properly of supergroup schemes. As customary, we use the same letter to denote both the superscheme and its functor of points.

We shall obtain the representability of $\mathbf{G}$ by showing that

$$\mathbf{G} \cong \mathbf{G}_0 \times \mathbf{A}^{0|N}$$

where $\mathbf{A}^{0|N}$ is the functor of points of an affine superspace of dimension $0|N$. Once this isomorphism is established the representability follows at once, since both $\mathbf{G}_0$ and $\mathbf{A}^{0|N}$ are representable, i.e. they are the functors of points of superschemes, hence their product is.

The next question we examine is how much our construction depends on the chosen representation. In complete analogy to Chevalley approach, we show that if we have two representations V and V', with weight lattices $L_V \subset L_{V'}$, then there is a surjective morphism $\mathbf{G}_{V'} \longrightarrow \mathbf{G}_V$, with kernel in the center of $\mathbf{G}_{V'}$. This implies right away that our construction depends only on the weight lattice of the chosen representation V and in particular it shows that it is independent from the choice of the lattice M inside V.

This paper is organized as follows.

In section 2 we review quickly some facts of algebraic supergeometry and the theory of Lie superalgebras.

In sections 3 and 4 we go to the heart of the construction of Chevalley's supergroups going through all the steps detailed above.

Finally in section 5 we provide some insight into our construction with some examples and observations.

We wish to thank the UCLA Mathematics Department, for hosting the conference. We thank I. Dimitrov, V. Serganova and A. Schwarz for helpful comments. We finally thank our referee for helping us to improve the readibility of our manuscript.

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