N. Ciccoli, R. Fioresi, F. Gavarini

"Quantization of Projective Homogeneous Spaces and Duality Principle"

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INTRODUCTION

A projective variety can be described via its homogeneous graded coordinate ring. This ring is not an invariant associated to the variety, but depends on a chosen embedding of the variety into some projective space. Different embeddings will, in general, produce non isomorphic graded rings.

When a projective variety is homogeneous, i.e. endowed with a transitive action of an (affine) algebraic group on it, it can be realized as quotient of affine algebraic groups G/H. In this case a projective embedding can be obtained via sections of a line bundle on G/H, uniquely given once a character of H is specified.

If one approaches a quantization of this picture in the context of quantum groups the problem immediately arising is that standard quantum groups have a very limited set of quantum subgroups. This explains why usually the preferred approach goes through representation theoretic techniques.

An explanation of the lack of quantum subgroups, together with a way to circumvent this problem, is suggested by considering the semiclassical picture, i.e. in the context of algebraic Poisson groups. In such setting algebraic Poisson subgroups are quite rare too; however there is no need of an algebraic Poisson subgroup to cook up a Poisson quotient. The existence of a surjective Poisson map $G \to G/H$ is guaranteed simply by requiring H to be a coisotropic subgroup of G. This condition can be expressed by saying that the defining ideal of H, in the function algebra of G, is required to be a Poisson subalgebra rather than a Poisson ideal, as required for Poisson subgroups.

Let $\mathcal{O}_q(G)$ be a quantization of the affine algebraic Poisson groups G. At the quantum level, a quantization $\mathcal{O}_q(H)$ of its *coisotropic subgroup* H can be defined through conditions on the projection $\pi\colon \mathcal{O}_q(G) \longrightarrow \mathcal{O}_q(H)$. We will see this in full detail in the sections 2, 3.

Our first aim is to build a quantum deformation $\mathcal{O}_q\big(G/H\big)$ of the projective variety G/H, i.e. of its graded ring $\mathcal{O}\big(G/H\big)$, subject to the following requirements:

(1) there exists a one dimensional corepresentation of the quantum coisotropic subgroup $\mathcal{O}_q(H)$ which is a deformation of the corepresentation of $\mathcal{O}(H)$ corresponding to the character of H which defines the line bundle giving the projective embedding of G/H;

(2) a quantum analogue $\mathcal{O}_q\big(G/H\big)$ to $\mathcal{O}\big(G/H\big)$ is defined as the subset — inside $\mathcal{O}_q(G)$ — of "semi-invariant functions" with respect to the given corepresentation of $\mathcal{O}_q(H)$;

(3) the subset $\mathcal{O}_q\big(G/H\big)$ is a graded subalgebra of $\mathcal{O}_q(G)$;

(4) the graded subalgebra $\mathcal{O}_q\big(G/H\big)$ is a graded left coideal of $\mathcal{O}_q(G)$, so the coproduct in $\mathcal{O}_q(G)$ induces a (left) $\mathcal{O}_q(G)$–coaction on $\mathcal{O}_q\big(G/H\big)$, and the latter can be thought of as a *quantum homogeneous space*.

(5) the semiclassical limit of $\mathcal{O}_q\big(G/H\big)$ is $\mathcal{O}\big(G/H\big)$ — embedded into $\mathcal{O}(G)$ — as a graded subalgebra, left coideal and graded Poisson subalgebra.

In other words, a quantum deformation of a projective homogeneous space, embedded into some projective space, consists of the deformation of the graded algebra associated to the embedding, in such a way that the action of the group on the homogeneous space is also naturally quantized.

We will work out the details of the construction for the case of the Grassmannian and its Plücker embedding, that is when G is the special linear group and $H = P$ is a maximal parabolic subgroup and we will sketch it in the more general case of quantum flag varieties of simple Lie groups.

Our main motivation to develop this point of view is to adapt to projective homogeneous spaces, the correspondence introduced by Ciccoli and Gavarini [5] for coinvariant subalgebras. This recipe allows to associate functorially to a quantum quasi-affine homogeneous space another quantum homogeneous space, through a generalization of the *quantum duality principle* (QDP), defined by Drinfeld for quantum groups. A part of the arguments in [5] does not directly apply to *projective* homogeneous spaces, since it is based on the realization of the ring of the homogeneous space as the set of coinvariant functions inside the ring of the quantum group acting on it. But this is possible — as in the classical case — if and only if the homogeneous space is quasi-affine, which is not the case of projective varieties. The coordinate ring of the homogeneous space is replaced by a graded ring inside the quantum group ring, consisting of semi-coinvariants with respect to one dimensional corepresentation, which can be seen as a deformation of the line bundle that classically determines the projective embedding. The definitions introduced in section 3 will allow us to define a quantum duality functor and obtain the QDP construction in this more general setting. In the last chapter we will discuss applications to quantum flag manifolds.

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