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"On the radical of Brauer algebras"

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INTRODUCTION

The Brauer algebras first arose in Invariant Theory (cf. [Br]) in connection with the study of invariants of the action of the orthogonal or the symplectic group — call it $G(U)$ — on the tensor powers of its standard representation U. More precisely, the centralizer algebra $End_{G(U)}\big(U^{\otimes f}\big)$ of such an action can be described by generators and relations: the latter depend on the relationship among two integral parameters, f and x — the latter being related to $dim(U)$ — but when x is big enough (what is called "the stable case") the relations always remain the same. These "stable" relations then define an algebra $\mathcal{B}_f^{(x)}$ of which the centralizer one is a quotient, obtained by adding the further relations, when necessary. The abstract algebra $\mathcal{B}_f^{(x)}$ is the one which bears the name of "Brauer algebra".

The definition of $\mathcal{B}_f^{(x)}$ still makes sense with x arbitrarily chosen in a fixed ground ring. An alternative description is possible too, by displaying an explicit basis of $\mathcal{B}_f^{(x)}$ and assigning the multiplication rules for elements in this basis.

Assume the ground field $\Bbbk$ has characteristic zero. Then $G(U)$ is linearly reductive, so by Schur duality the algebra $End_{G(U)}\big(U^{\otimes f}\big)$ is semisimple: hence in the stable case, when $\mathcal{B}_f^{(x)} \cong End_{G(U)}\big(U^{\otimes f}\big)$, the Brauer algebra is semisimple too. Otherwise, $\mathcal{B}_f^{(x)}$ may fail to be semisimple, i.e. it may have a non-trivial radical.

The most general result on $Rad\Big(\mathcal{B}_f^{(x)}\Big)$, for $Char(\Bbbk) = 0$, was found in [Wz]: for "general values" of x — i.e., all those out of a finite range (depending on f, and yielding the stable case) of values in the prime subring of $\Bbbk$ — the Brauer algebra $\mathcal{B}_f^{(x)}$ is semisimple. So the problem only remained of computing $Rad\Big(\mathcal{B}_f^{(x)}\Big)$ when x is an integer and we are not in the stable case. In this framework, the first contributions came from Brown, who reduced the task to studying the radical of "generalized matrix algebras" (cf. [Bw1–2]). In particular, this radical is strictly related with the nullspace of the matrix of structure constants of such an algebra: later authors mainly followed the same strategy, see e.g. [HW1–2]. Further results were obtained using new techniques: see [GL], [DHW], [KX], [CDM], [Hu], [DH].

In the present paper we rather come back to the Invariant Theory viewpoint. The idea we start from is a very naïve one: as the algebra $End_{G(U)}\big(U^{\otimes f}\big)$ is semisimple, we have

$$Rad\Big(\mathcal{B}_f^{(x)}\Big) \;\subseteq\; Ker\Big(\pi_U : \mathcal{B}_f^{(x)} \twoheadrightarrow End_{G(U)}\big(U^{\otimes f}\big)\Big)$$

where $\pi_U : \mathcal{B}_f^{(x)} \twoheadrightarrow End_{G(U)}\big(U^{\otimes f}\big)$ is the natural epimorphism. The second step is an intermediate result, namely a description of the kernel $Ker\left(\pi_U : \mathcal{B}_f^{(x)} \twoheadrightarrow End_{G(U)}\left(U^{\otimes f}\right)\right)$. Indeed, using the Second Fundamental Theorem of classical invariant theory we find a set of linear generators for it: they are explicitly written in terms of the basis of diagrams, and called *(diagrammatic) minors* or *Pfaffians*, depending on the sign of x. As $Ker\left(\pi_U\right)$ contains $Rad\left(\mathcal{B}_f^{(x)}\right)$, every element of $Rad\left(\mathcal{B}_f^{(x)}\right)$ is a linear combination of these special elements (minors or Pfaffians). As a last step, a basic knowledge of $\mathcal{B}_f^{(x)}$–modules yields some more information on the structure of the semisimple quotient of $\mathcal{B}_f^{(x)}$. Thus we determine exactly which ones among minors, or Pfaffians, belong to $Rad\left(\mathcal{B}_f^{(x)}\right)$: so we find a great part of $Rad\left(\mathcal{B}_f^{(x)}\right)$, and we conjecture that this is *all* the part of the radical inside the proper step of the standard filtration. We then find a similar result and conjecture for the generic indecomposable $\mathcal{B}_f^{(x)}$–modules too.

Our approach applies directly only in case x is an integer which is not zero nor odd negative; but *a posteriori*, we find also similar results for $x = 0$, via an *ad hoc* approach.

Also, we discuss how much of these results can be extended to the case of $Char\left(\Bbbk\right) > 0$.

Finally, we provide some more precise results for the module of pointed chord diagrams, and the Temperley-Lieb algebra — realised as a subalgebra of $\mathcal{B}_f^{(1)}$ — acting on it.

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