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" $F_q[M_n]$, $F_q[GL_n]$ and $F_q[SL_n]$ as quantized hyperalgebras"

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INTRODUCTION

Let G be a semisimple, connected, simply connected affine algebraic group over \mathbb{Q} , and \mathfrak{g} its tangent Lie algebra. Let $U_q(\mathfrak{g})$ be the Drinfeld-Jimbo quantum group over \mathfrak{g} defined (after Jimbo) as a Hopf algebra over the field $\mathbb{Q}(q)$, where q is an indeterminate. After Lusztig, one has an integral form over $\mathbb{Z}[q,q^{-1}]$, say $\mathfrak{U}_q(\mathfrak{g})$, which for $q \to 1$ specializes to $U_{\mathbb{Z}}(\mathfrak{g})$, the integral \mathbb{Z} -form of $U(\mathfrak{g})$ defined by Kostant (see [CP], §9.3, and references therein, and [DL], §§2–3). As $U_{\mathbb{Z}}(\mathfrak{g})$ is usually called "hyperalgebra", we call $\mathfrak{U}_q(\mathfrak{g})$ "quantum" (or quantized) hyperalgebra". In particular, as $U_{\mathbb{Z}}(\mathfrak{g})$ is generated by divided powers (in the simple root vectors) and binomial coefficients (in the simple coroots) so $\mathfrak{U}_q(\mathfrak{g})$ is generated by quantum analogues of divided powers and of binomial coefficients. Moreover, if ε is a root of 1 with odd order ℓ , if \mathbb{Z}_{ε} is the formal extension of \mathbb{Z} by ϵ (see §1.4) and $\mathfrak{U}_{\varepsilon}(\mathfrak{g})$ is the corresponding specialization of $\mathfrak{U}_q(\mathfrak{g})$, there is a Hopf algebra epimorphism $\mathfrak{F}_{\mathfrak{g}}^{\mathbb{Z}}: \mathfrak{U}_{\varepsilon}(\mathfrak{g}) \longrightarrow \mathbb{Z}_{\varepsilon} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{g})$, described as an " ℓ -th root operation" on generators. This is a quantum analogue of the Frobenius morphism in positive characteristic, so is called quantum Frobenius morphism for \mathfrak{g} .

In a Hopf-dual setting, one constructs ([DL], §§4–6) a Hopf algebra $F_q[G]$ of matrix coefficients of $U_q(\mathfrak{g})$, and a $\mathbb{Z}[q,q^{-1}]$ -form $\mathfrak{F}_q[G]$ of it which specializes to $F_{\mathbb{Z}}[G]$ (the algebra of regular functions over $G_{\mathbb{Z}}$, the algebraic scheme of \mathbb{Z} -points of G) as a Hopf algebra, for $q \to 1$. In particular, $\mathfrak{F}_q[G]$ is nothing but the set of "functions" in $F_q[G]$ which take values in $\mathbb{Z}[q,q^{-1}]$ when "evaluated" on $\mathfrak{U}_q(\mathfrak{g})$: in a word, the $\mathbb{Z}[q,q^{-1}]$ valued functions on $\mathfrak{U}_q(\mathfrak{g})$. When specializing at roots of 1 (with notation as above) there is a Hopf algebra monomorphism $\mathfrak{Fr}_G^{\mathbb{Z}}: F_{\mathbb{Z}}[G] \longrightarrow \mathfrak{F}_{\varepsilon}[G]$ dual to the above epimorphism and described, roughly, as an " ℓ -th power operation" on generators. This also is a quantum analogue of the classical Frobenius morphism, which is therefore called the quantum Frobenius morphism for G.

The quantization $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{g})$ of $U_{\mathbb{Z}}(\mathfrak{g})$ endows the latter with a co-Poisson (Hopf) algebra structure which makes \mathfrak{g} into a Lie bialgebra; similarly, $\mathfrak{F}_q[G]$ endows $F_{\mathbb{Z}}[G]$ with a Poisson (Hopf) algebra structure which makes G into a Poisson group. The Lie bialgebra structure on \mathfrak{g} is exactly the one induced by the Poisson structure on G. Then one can consider the dual Lie bialgebra \mathfrak{g}^* , and dual Poisson groups G^* having \mathfrak{g}^* as tangent Lie bialgebra.

Lusztig's $\mathbb{Z}[q, q^{-1}]$ -integral forms $\mathfrak{U}_q(\mathfrak{g})$ and $\mathfrak{F}_q[G]$ are said to be restricted. On the other hand, another $\mathbb{Z}[q, q^{-1}]$ -integral form of $U_q(\mathfrak{g})$, say $\mathcal{U}_q(\mathfrak{g})$, has been introduced by De Concini and Procesi (cf. [CP], §9.2, and [DP], §12.1 — the original construction is over $\mathbb{C}[q, q^{-1}]$, but it works the same over $\mathbb{Z}[q, q^{-1}]$ too), called unrestricted. It is generated by suitably rescaled quantum root vectors and by toral quantum analogues of

simple root vectors, and for $q \to 1$ specializes to $F_{\mathbb{Z}}[G^*]$ (notation as before). Moreover, at roots of 1 there is a Hopf algebra monomorphism $\mathcal{F}r_{\mathfrak{g}}^{\mathbb{Z}}: F_{\mathbb{Z}}[G^*] = \mathcal{U}_1(\mathfrak{g}) \hookrightarrow \mathcal{U}_{\varepsilon}(\mathfrak{g})$, defined on generators as an " ℓ -th power operation". This is a quantum analogue of the classical Frobenius morphisms (for G^*), strictly parallel to $\mathcal{F}r_{\mathfrak{g}}^{\mathbb{Z}}$ above, so is called the *quantum Frobenius morphism for* G^* . In the Hopf-dual setting, one can consider — [Ga1], §§4, 7 — as "dual" of $\mathcal{U}_q(\mathfrak{g})$ the subset $\mathcal{F}_q[G]$ of "functions" in $F_q[G]$ which take values in $\mathbb{Z}[q, q^{-1}]$ when "evaluated" on $\mathcal{U}_q(\mathfrak{g})$; this subset is a Hopf subalgebra, which for $q \to 1$ specializes to $U_{\mathbb{Z}}(\mathfrak{g}^*)$. When specializing at roots of 1 there is a Hopf epimorphism $\mathcal{F}r_G^{\mathbb{Z}_{\varepsilon}}: \mathcal{F}_{\varepsilon}[G] \longrightarrow \mathbb{Z}_{\varepsilon} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{g}^*)$, dual to the previous monomorphism; this again is a quantum analogue of the classical Frobenius morphism (for \mathfrak{g}^*), hence it is called the *quantum Frobenius morphism for* \mathfrak{g}^* .

In this paper we provide an explicit description of $\mathcal{F}_q[G]$, its specializations at roots of 1 and its quantum Frobenius morphisms $\mathcal{F}r_G^{\mathbb{Z}_{\varepsilon}}$ for $G = SL_n$. The whole construction makes sense for $G = GL_n$ and $G = M_n$ (the Poisson algebraic monoid of square matrices of size n) too, and we find similar results for them. In fact, we first approach the case of $\mathcal{F}_q[M_n]$, for which the strongest results are found; then from these we get those for $\mathcal{F}_q[GL_n]$ and $\mathcal{F}_q[SL_n]$.

Our starting point is the well-known description of $\mathfrak{F}_q[M_n]$ by generators and relations, as a $\mathbb{Z}[q, q^{-1}]$ -algebra generated by the entries of a quantum $(n \times n)$ -matrix (see [APW], Appendix). In particular, this is an algebra of skew-commutative polynomials, much like $\mathcal{U}_q(\mathfrak{g})$ is just an algebra of skew-commutative polynomials (which are Laurent in some variables). Dually, this leads us to expect that, like $\mathfrak{U}_q(\mathfrak{gl}_n)$, also $\mathcal{F}_q[M_n]$ be generated by quantum divided powers and quantum binomial coefficients: also, we expect that a suitable PBW-like theorem holds for $\mathcal{F}_q[M_n]$, like for $\mathcal{U}_q(\mathfrak{gl}_n)$. Similarly, as $\mathfrak{Fr}_{M_n}^{\mathbb{Z}}: F_{\mathbb{Z}}[M_n] \longrightarrow \mathfrak{F}_{\varepsilon}[M_n]$ is defined on generators as an " ℓ -th power operation", dually we expect that $\mathcal{F}r_{M_n}^{\mathbb{Z}_{\varepsilon}}: \mathcal{F}_{\varepsilon}[M_n] \longrightarrow \mathbb{Z}_{\varepsilon} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{gl}_n^*)$ be given by an " ℓ -th root operation", much like $\mathfrak{Fr}_{\mathfrak{gl}_n}^{\mathbb{Z}_{\varepsilon}}: \mathfrak{U}_{\varepsilon}(\mathfrak{gl}_n) \longrightarrow \mathbb{Z}_{\varepsilon} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{gl}_n)$.

In fact, all these conjectural expectations turn out to be true.

From this we get similar (yet slightly weaker) results for $\mathcal{F}_q[GL_n]$ and $\mathcal{F}_q[SL_n]$. On the way, we also improve the (already known) above mentioned results about specializations and quantum Frobenius epimorphisms.

The intermediate step is the quantum group $U_q(\mathfrak{g}^*)$, analogue for \mathfrak{g}^* of what $U_q(\mathfrak{g})$ is for \mathfrak{g} (see [Ga1], §6). In particular, there are integral $\mathbb{Z}[q, q^{-1}]$ -forms $\mathfrak{U}_q(\mathfrak{g}^*)$ and $\mathcal{U}_q(\mathfrak{g}^*)$ of $U_q(\mathfrak{g}^*)$ for which PBW theorems and presentations hold. Moreover, a Hopf algebra embedding $\mathcal{F}_q[M_n] \hookrightarrow \mathfrak{U}_q(\mathfrak{gl}_n^*)$ exists, via which we "pull back" a PBW-like basis and a presentation from $\mathfrak{U}_q(\mathfrak{gl}_n^*)$ to $\mathcal{F}_q[M_n]$. These arguments work, *mutatis mutandis*, for GL_n and SL_n as well. As aside results, we provide (in §3) explicit descriptions of these embeddings, and related results which turn useful in studying specializations at roots of 1.

The present work bases upon the analysis of the case n = 2, which is treated in [GR].

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