F. Gavarini, Z. Rakić

“$F_q[Mat_2]$, $F_q[GL_2]$ and $F_q[SL_2]$ as quantized hyperalgebras”

to appear in *Communications in Algebra* (2008)

INTRODUCTION

Let G be a semisimple, connected, simply connected affine algebraic group over $\mathbb{C}$, and $\mathfrak{g}$ its tangent Lie algebra. Let $U_q(\mathfrak{g})$ be the Drinfeld-Jimbo quantum group over $\mathfrak{g}$, defined over the field $\mathbb{Q}(q)$, where q is an indeterminate. There exist two integral forms of $U_q(\mathfrak{g})$ over $\mathbb{Z}\big[q,q^{-1}\big]$, the restricted one, say $\mathfrak{U}_q(\mathfrak{g})$, and the unrestricted one, say $\mathcal{U}_q(\mathfrak{g})$ — see [CP] and references therein. Both of them bear so called “quantum Frobenius morphisms”, namely Hopf algebra morphisms linking their specialisations at 1 with their specialisations at roots of 1. In particular, $\mathfrak{U}_q(\mathfrak{g})$ for $q \to 1$ specializes to $U_{\mathbb{Z}}(\mathfrak{g})$, the Kostant $\mathbb{Z}$–form of $U(\mathfrak{g})$; so $\mathfrak{g}$ becomes a Lie bialgebra, and G a Poisson group. Also, $\mathcal{U}_q(\mathfrak{g})$ for $q \to 1$ specializes to $F_{\mathbb{Z}}[G^*]$, a $\mathbb{Z}$–form of the function algebra on a Poisson group G^* dual to G.

Dually, one constructs a Hopf algebra $F_q[G]$ of matrix coefficients of $U_q(\mathfrak{g})$. It has two $\mathbb{Z}\big[q,q^{-1}\big]$–forms, say $\mathfrak{F}_q[G]$ and $\mathcal{F}_q[G]$, defined to be the subset of $F_q[G]$ of all $\mathbb{Z}\big[q,q^{-1}\big]$–valued functions on $\mathfrak{U}_q(\mathfrak{g})$, respectively on $\mathcal{U}_q(\mathfrak{g})$. At $q=1$, $\mathfrak{F}_q[G]$ specializes to $F_{\mathbb{Z}}[G]$, while $\mathcal{F}_q[G]$ specializes to $U_{\mathbb{Z}}\big(\mathfrak{g}^*\big)$, a Kostant-like $\mathbb{Z}$–form of $U\big(\mathfrak{g}^*\big)$ — cf. [Ga1] for details. Moreover, both $\mathfrak{F}_q[G]$ and $\mathcal{F}_q[G]$ bear quantum Frobenius morphisms (relating their specialisations at 1 with those at roots of 1), which are dual to those of $\mathfrak{U}_q(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{g})$.

The aim of this paper is to describe $\mathcal{F}_q[G]$, its specializations at roots of 1 and its quantum Frobenius morphisms when $G = SL_2$. Moreover, as the construction of these objects makes sense for $G = GL_2$ and $G = M_2 := Mat_2$ as well, we find similar results for them.

By [Ga1], $\mathcal{F}_q[M_2]$ should resemble $\mathfrak{U}_q(\mathfrak{gl}_2)$. Indeed, this is the case: $\mathcal{F}_q[M_2]$ is generated by quantum divided powers and quantum binomial coefficients, a PBW-like theorem hold for $\mathcal{F}_q[M_2]$, and the quantum Frobenius morphisms are given by an “ℓ–th root operation”, if ℓ is the order of the root of unity. Similar (weaker) results hold for $\mathcal{F}_q[GL_2]$ and $\mathcal{F}_q[SL_2]$.

The general case of $M_n := Mat_n$, GL_n and SL_n is studied in [GR2], exploiting the same key ideas already developed here and the present results for $n=2$.

Warning: an expanded, more detailed version of this paper is available on line, cf. [GR1]; the quotations in [GR2] about the present work refer in fact to [GR1].

DEDICATORY

This work grew out of a cooperation supported by an official agreement between the Department of Mathematics of the University of Rome “Tor Vergata” and the Faculty of Mathematics of the University of Belgrade in the period 2000–2003. Such agreement was the outcome of a common wish of peaceful, fruitful partnership, as an answer to the military aggression of NATO countries to the Federal Republic of Yugoslavia, which started in spring of 1999.

This paper is dedicated to the memory of all victims of that war.

— — — — —

References

[CP] V. Chari, A. Pressley, *A guide to Quantum Groups*, Cambridge Univ. Press, Cambridge, 1994.

[DL] C. De Concini, V. Lyubashenko, *Quantum function algebra at roots of 1*, Adv. Math. **108** (1994), 205–262.

[DP] C. De Concini, C. Procesi, *Quantum groups*, D-modules, Representation Theory, and Quantum Groups (L. Boutet de Monvel, C. De Concini, C. Procesi, P. Schapira, M. Vergne, eds.), Lecture Notes in Mathematics **1565**, Springer & Verlag, Berlin, 1993, pp. 798–820.

[Dr] V. G. Drinfel'd, *Quantum groups*, Proc. Intern. Congress of Math. (Berkeley, 1986), 1987, pp. 798–820.

[Ga1] F. Gavarini, *Quantization of Poisson groups*, Pacific Journal of Mathematics **186** (1998), 217–266.

[Ga2] ———, *Quantum function algebras as quantum enveloping algebras*, Communications in Algebra **26** (1998), 1795–1818.

[Ga3] ———, *The global quantum duality principle: theory, examples, and applications*, preprint `math.QA/0303019` (2003).

[Ga4] ———, *The global quantum duality principle*, J. Reine Angew. Math. (to appear).

[GR1] F. Gavarini, Z. Rakić, *$F_q[M_2]$, $F_q[GL_2]$ and $F_q[SL_2]$ as quantized hyperalgebras*, preprint `math.QA/0411440` (2004).

[GR2] F. Gavarini, Z. Rakić, *$F_q[M_n]$, $F_q[GL_n]$ and $F_q[SL_n]$ as quantized hyperalgebras*, J. Algebra **315** (2007), 761–800.

[Hu] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics **9**, Springer & Verlag, Berlin–Heidelberg–New York, 1972.

[No] M. Noumi, *Macdonald's Symmetric Polynomials as Zonal Spherical Functions on Some Quantum Homogeneous Spaces*, Adv. Math. **123** (1996), 16–77.