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"The global quantum duality principle"

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INTRODUCTION

"Dualitas dualitatum
et omnia dualitas"
N. Barbecue, "Scholia"

Generalized "symmetries" in mathematics are described by Hopf algebras. Among these, the "geometrical" ones are of type $H = F[G]$, the algebra of regular functions over an algebraic group G, and $H = U(\mathfrak{g})\,\big(= \mathbf{u}(\mathfrak{g})\big)$, the (restricted, if the ground field $\Bbbk$ has positive characteristic) universal enveloping algebra of a (restricted) Lie algebra $\mathfrak{g}$. These notions of "geometrical symmetries" are generalized by quantum groups: roughly, these are Hopf algebras H depending on a parameter $\hbar$ such that, setting $\hbar = 0$, the Hopf algebra one gets is either of the type $F[G]$ — hence H is a *quantized function algebra*, in short QFA — or of the type $U(\mathfrak{g})$ or $\mathbf{u}(\mathfrak{g})$ (according to the characteristic of $\Bbbk$) — hence H is a *quantized restricted universal enveloping algebra*, in short QrUEA. When a QFA exists whose specialization at $\hbar = 0$ is $F[G]$, the algebraic group G inherits a structure of Poisson (algebraic) group. Similarly, if a QrUEA exists whose specialization is $U(\mathfrak{g})$ or $\mathbf{u}(\mathfrak{g})$, the (restricted) Lie algebra $\mathfrak{g}$ inherits a structure of Lie bialgebra. Then, by general Poisson group theory, Poisson groups G^* dual to G and a Lie bialgebra $\mathfrak{g}^*$ dual to $\mathfrak{g}$ exist.

In this setting, three basic questions rise at once:

— **(1)** *How can we produce quantum groups?*

— **(2)** *How can we characterize quantum groups (of either kind) among Hopf algebras?*

— **(3)** *What kind of relationship, if any, does exist between quantum groups over mutually dual Poisson groups, or mutually dual Lie bialgebras?*

A first answer to **(1)** and **(3)** is given, for $Char(\Bbbk) = 0$, by the "quantum duality principle", formulated by Drinfeld in terms of *formal* quantum groups (cf. [Dr], §7, and [Ga1]): it is a functorial recipe to get, out of a QFA over G, a QrUEA over $\mathfrak{g}^*$, and a QFA over G^* out of a QrUEA over $\mathfrak{g}$.

In this paper I provide a *global* version of this principle, which answers questions **(1)** through **(3)**. Indeed, I push Drinfeld's original method as far as possible, so to apply it to the category $\mathcal{HA}$ of Hopf algebras which are torsion-free (or flat) over some integral domain, say R, and to do it for each $\hbar \in R \setminus \{0\}$ such that $\Bbbk := R\big/\hbar R$ is a field. In fact, I extend Drinfeld's recipe so to define endofunctors of $\mathcal{HA}$. The image of either functor is contained in a category of quantum groups (one gives QFAs, the other QrUEAs) so we answer question **(1)**. If $\Bbbk$ has zero characteristic, when restricted to quantum groups these functors yield equivalences inverse to each other. Moreover, these equivalences exchange the types of quantum group (switching QFA with QrUEA) and the underlying Poisson symmetries (interchanging G or $\mathfrak{g}$ with G^* or $\mathfrak{g}^*$), thus solving **(3)**. Other details show that these functors endow $\mathcal{HA}$ with a (inner) Galois' correspondence, in which QFAs on one side and QrUEAs on the other side are the subcategories (in $\mathcal{HA}$) of "fixed points" for the composition of both Drinfeld's functors (in suitable order): in particular, this answers question **(2)**. Let me point out that, as my "Drinfeld's functors" are defined for each element $\hbar \in R$ as above, for any such $\hbar$ and for any H in $\mathcal{HA}$ they yield two quantum groups, a QFA and a QrUEA, w.r.t. $\hbar$ itself. Thus we have a method to get, out of any single $H \in \mathcal{HA}$, several quantum groups.

Further aspects, examples and applications of the main result are presented in [Ga2–4].

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