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"Presentation by Borel subalgebras and Chevalley generators for quantum enveloping algebras"

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INTRODUCTION

Let $\mathfrak{g}$ be a semisimple Lie algebra over a field $\Bbbk$. Classically, it has two standard presentations: Serre's one, which uses a minimal set of generators, and Chevalley's one, using a linear basis as generating set. If $\mathfrak{g}$ instead is reductive a presentation is obtained by that of its semisimple quotient by adding the center. When $\mathfrak{g} = \mathfrak{gl}_n$, Chevalley's generators are the elementary matrices, and Serre's ones form a distinguished subset of them; the general case of any classical matrix Lie algebra $\mathfrak{g}$ is a slight variation on this theme. Finally, both presentations yield also presentations of $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$.

At the quantum level, one has correspondingly a Serre-like and a Chevalley-like presentation of $U_q(\mathfrak{g})$, the quantized universal enveloping algebra associated to $\mathfrak{g}$ after Jimbo and Lusztig (i.e. defined over the field $\Bbbk(q)$, where q is an indeterminate). The first presentation is used by Jimbo (cf. [Ji1]) and Lusztig (see [Lu2]) and, *mutatis mutandis*, by Drinfeld too; in this case the generators are q–analogues of the Serre's generators, and starting from them one builds quantum root vectors via two different methods: iterated quantum brackets, as in [Ji2] — and maybe others — or braid group action, like in [Lu2]; see [Ga2] for a comparison between these two methods. The second presentation was introduced by Faddeev, Reshetikhin and Takhtajan (in [FRT]): the generators in this case, called L–operators, are q–analogues of the classical Chevalley generators; in particular, they are quantum root vectors themselves. An explicit comparison between quantum Serre-like generators and L–operators appears in [FRT], §2, for the cases of *classical* $\mathfrak{g}$; on the other hand, in [No], §1.2, a similar comparison is made for $\mathfrak{g} = \mathfrak{gl}_n$ between L–operators and quantum root vectors (for *any* root) built out of Serre's generators.

The first purpose of this note is to provide an alternative approach to the FRT presentation of $U_q(\mathfrak{g})$: it amounts to a series of elementary steps, yet the final outcome seems noteworthy. As a second, deeper result, we give an explicit presentation of the $\Bbbk\big[q,q^{-1}\big]$–subalgebra of $U_q(\mathfrak{g})$ generated by L–operators, call it $\widetilde{U}_q(\mathfrak{g})$. By construction, this is nothing but the *unrestricted* $\Bbbk\big[q,q^{-1}\big]$–integer form of $U_q(\mathfrak{g})$, defined by De Concini and Procesi (see [DP]), whose semiclassical limit is $\widetilde{U}_q(\mathfrak{g})\Big/(q-1)\,\widetilde{U}_q(\mathfrak{g}) \cong F\big[G^*\big]$, where G^* is a connected Poisson algebraic group dual to $\mathfrak{g}$ (cf. [DP], [Ga1] and [Ga3], §7.3 and §7.9): our explicit presentation of $\widetilde{U}_q(\mathfrak{g})$ yields another, independent (and much easier) proof of this fact. Third, by [DP] we know that *quantum Frobenius morphisms* exist, which embed $F\big[G^*\big]$ into the specializations of $\widetilde{U}_q(\mathfrak{g})$ at roots of 1: then our presentation of $\widetilde{U}_q(\mathfrak{g})$ provides an explicit description of them.

This analysis shows that the two presentations of $U_q(\mathfrak{g})$ correspond to different behaviors w.r.t. to specializations. Indeed, let $\widehat{U}_q(\mathfrak{g})$ be the $\Bbbk\big[q,q^{-1}\big]$–algebra given by Jimbo-Lusztig presentation *over* $\Bbbk\big[q,q^{-1}\big]$. Its specialization at $q=1$ is $\widehat{U}_q(\mathfrak{g}) \Big/ (q-1)\,\widehat{U}_q(\mathfrak{g}) \cong U(\mathfrak{g})$ (up to technicalities), with $\mathfrak{g}$ inheriting a Lie bialgebra structure (see [Ji1], [Lu2], [DL]). On the other hand, the integer form $\widetilde{U}_q(\mathfrak{g})$ mentioned above specializes to $F\big[G^*\big]$, the Poisson structure on G^* being exactly the one dual to the Lie bialgebra structure on $\mathfrak{g}$. So the existence of two different presentations of $U_q(\mathfrak{g})$ reflects the deep fact that $U_q(\mathfrak{g})$ provides, taking suitable integer forms, quantizations of two different semiclassical objects (this is a general fact, see [Ga3–4]). To the author's knowledge, this was not known so far, as the FRT presentation of $U_q(\mathfrak{g})$ was never used to study the integer form $\widetilde{U}_q(\mathfrak{g})$.

Let's sketch in short the path we follow. First, we note that $U_q(\mathfrak{g})$ is generated by the quantum Borel subgroups $U_q(\mathfrak{b}_-)$ and $U_q(\mathfrak{b}_+)$ (where $\mathfrak{b}_-$ and $\mathfrak{b}_+$ are opposite Borel subalgebras of $\mathfrak{g}$), which share a common copy of the quantum Cartan subgroup $U_q(\mathfrak{t})$. Second, there exist Hopf algebra isomorphisms $U_q(\mathfrak{b}_-) \cong F_q\big[B_-\big]$ and $U_q(\mathfrak{b}_+) \cong F_q\big[B_+\big]$, where $F_q\big[B_-\big]$ and $F_q\big[B_+\big]$ are the quantum function algebras associated to $\mathfrak{b}_-$ and $\mathfrak{b}_+$ respectively. Third, when $\mathfrak{g}$ is classical we resume the explicit presentation by generators and relations of $F_q\big[B_-\big]$ and $F_q\big[B_+\big]$, as given in [FRT], §1. Fourth, from the above we argue a presentation of $U_q(\mathfrak{g})$ where the generators are those of $F_q\big[B_-\big]$ and $F_q\big[B_+\big]$, the toral ones being taken only once, and relations are those of these quantum function algebras plus some additional relations between generators of opposite quantum Borel subgroups. We perform this last step in full detail for $\mathfrak{g}=\mathfrak{gl}_n$ and, with slight changes, for $\mathfrak{g}=\mathfrak{sl}_n$ as well. Fifth, we refine the last step to provide a presentation of $\widetilde{U}_q(\mathfrak{g})$.

As an application, our results apply also (with few changes) to the Drinfeld-like quantum groups $U_\hbar(\mathfrak{g})$: in particular we get a presentation of an $\hbar$–deformation of $F[G^*]$, say $\widetilde{U}_\hbar(\mathfrak{g}) =: F_\hbar[G^*]$. An explicit gauge equivalence between this $F_\hbar[G^*]$ and the $\hbar$–deformation provided by Kontsevitch recipe is given in [FG].

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References

[APW] H. H. Andersen, P. Polo, W. Kexin, *Representations of quantum algebras*, Invent. Math. **104** (1991), 1–59.

[DL] C. De Concini, V. Lyubashenko, *Quantum Function Algebras at Roots of 1*, Adv. Math. **108** (1994), 205–262.

[DP] C. De Concini, C. Procesi, *Quantum groups*, Lecture Notes in Mathematics **1565** (1993).

[FG] D. Fiorenza, F. Gavarini, *Deformation-quantizations for the Poisson groups GL_n^* and SL_n^**, preprint.

[FRT] L. D. Faddeev, N. Yu. Reshetikhin, L. A. Takhtajan, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. **1** (1990), 193–225.

[Ga1] F. Gavarini, *Quantization of Poisson groups*, Pacific Journal of Mathematics **186** (1998), 217–266.

[Ga2] ———, *Quantum function algebras as quantum enveloping algebras*, Communications in Algebra **26** (1998), 1795–1818.

[Ga3] ———, *The global quantum duality principle: theory, examples, and applications*, preprint math.QA/0303019 (2003).

[Ga4] ———, *The global quantum duality principle*, preprint.

[GL] I. Grojnowski, G. Lusztig, *On bases of irreducible representation of quantum group GL_n* , in: V. Deodhar (ed.), *Kazhdan-Lusztig theory and related topics*, Cont. Math. **139** (1992), 167–174.

[Ji1] M. Jimbo, *A q–difference analog of $U(\mathfrak{g})$ and the Yang-Baxter equation*, Lett. in Math. Phys. **10** (1985), 63–69.

[Ji2] ———, *A q–analog of $U\big(\mathfrak{gl}(N+1)\big)$, Hecke algebra, and the Yang-Baxter equation*, Lett. in Math. Phys. **11** (1986), 247–252.

[Lu1] G. Lustzig, *Quantum deformations of certains simple modules over enveloping algebras*, Adv. Math. **70** (1988), 237–249.

[Lu2] ———, *Quantum groups at roots of 1*, Geom. Dedicata **35** (1990), 89–113.

[Mo] P. Möseneder Frajria, *A guide to L–operators*, Rendiconti di Matematica, Serie VII **18** (1998), 65–85.

[No] M. Noumi, *Macdonald's Symmetric Polynomials as Zonal Spherical Functions on Some Quantum Homogeneous Spaces*, Adv. Math. **123** (1996), 16–77.

[PW] B. Parshall, J.P. Wang, *Quantum linear groups*, Mem. Amer. Math. Soc. **89** (1991).

[Ta] T. Tanisaki, *Killing forms, Harish-Chandra Isomorphisms, and Universal R-Matrices for Quantum Algebras*, Internat. J. Modern Phys. A **7**, Suppl. 1B (1992), 941–961.