N. Ciccoli, F. Gavarini

“A quantum duality principle for coisotropic subgroups and Poisson quotients”

Advances in Mathematics **199** (2006), no. 1, 104–135.

INTRODUCTION

The natural semiclassical counterpart of the study of quantum groups is the theory of Poisson groups: indeed, Drinfeld himself introduced Poisson groups as the semiclassical limits of quantum groups. Therefore, it should be no surprise to anyone, anymore, that the geometry of quantum groups gain in clarity and comprehension when its connection with Poisson geometry is more transparent. The same can be observed when referring to homogeneous spaces.

In fact, in the study of Poisson homogeneous spaces, a special rôle is played by *Poisson quotients*. These are those Poisson homogeneous spaces whose symplectic foliation has at least one zero-dimensional leaf, so they can be thought of as pointed Poisson homogeneous spaces, just like Poisson groups themselves are pointed by the identity element. When looking at quantizations of a Poisson homogeneous space, one finds that the existence is guaranteed only if the space is a quotient (cf. [EK2]). Thus the notion of Poisson quotient shows up naturally also from the point of view of quantization (see [Ci]).

Poisson quotients are a natural subclass of Poisson homogeneous G–spaces (G a Poisson group), best adapted to the usual relation between homogeneous G–spaces and subgroups of G: they correspond to *coisotropic* subgroups. The quantization process for a Poisson G–quotient then corresponds to a like procedure for the attached coisotropic subgroup of G. Also, when following an infinitesimal approach one deals with Lie subalgebras of the Lie algebra $\mathfrak{g}$ of G, and the coisotropy condition has its natural counterpart in this Lie algebra setting; the quantization process then is to be carried on for the Lie subalgebra corresponding to the initial homogeneous G–space.

When quantizing Poisson groups (or Lie bialgebras), a precious tool is the quantum duality principle (QDP). Loosely speaking this guarantees that any quantized enveloping algebra can be turned (roughly speaking) into a quantum function algebra for the dual Poisson group; viceversa any quantum function algebra can be turned into a quantization of the enveloping algebra of the dual Lie bialgebra. More precisely, let $\mathcal{QUEA}$ and $\mathcal{QFSHA}$ respectively be the category of all quantized universal enveloping algebras (QUEA) and the category of all quantized formal series Hopf algebras (QFSHA), in Drinfeld's sense. After its formulation by Drinfeld (see [Dr1], §7) the QDP establishes a category equivalence between $\mathcal{QUEA}$ and $\mathcal{QFSHA}$ via two functors, $(\ )'\colon \mathcal{QUEA} \longrightarrow \mathcal{QFSHA}$ and $(\ )^\vee\colon \mathcal{QFSHA} \longrightarrow \mathcal{QUEA}$, such that, starting from a QUEA over a Lie bialgebra (resp. from a QFSHA over a Poisson group) the functor $(\ )'$ (resp. $(\ )^\vee$) gives a QFSHA (resp. a QUEA) over the dual Poisson group (resp. the dual Lie bialgebra). In a nutshell,

$U_\hbar(\mathfrak{g})' = F_\hbar[[G^*]]$ and $F_\hbar[[G]]^\vee = U_\hbar(\mathfrak{g}^*)$ for any Lie bialgebra $\mathfrak{g}$. So from a quantization of any Poisson group this principle gets out a quantization of the dual Poisson group too.

In this paper we establish a similar quantum duality principle for (closed) coisotropic subgroups of a Poisson group G, or equivalently for Poisson G–quotients, sticking to the formal approach which is best suited for dealing with quantum groups à la Drinfeld. Namely, given a Poisson group G, assume quantizations $U_\hbar(\mathfrak{g})$ and $F_\hbar[[G]]$ of it are given; then any formal coisotropic subgroup K of G has two possible algebraic descriptions via objects related to $U(\mathfrak{g})$ or $F[[G]]$, and similarly for the formal Poisson quotient $G\big/K$. Thus the datum of K or equivalently of $G\big/K$ is described algebraically in four possible ways: by quantization of such a datum we mean a quantization of any one of these four objects. Our "QDP" now is a series of functorial recipes to produce, out of a quantization of K or $G\big/K$ as before, a similar quantization of the so-called *complementary dual* of K, i.e. the coisotropic subgroup $K^\perp$ of G^* whose tangent Lie bialgebra is just $\mathfrak{k}^\perp$ inside $\mathfrak{g}^*$, or of the associated Poisson G^*–quotient, namely $G^*\big/K^\perp$.

We would better stress that, just like the QDP for quantum groups, ours is by no means an existence result: instead, it can be thought of as a *duplication result*, in that it yields a new quantization (for a complementary dual object) out of one given from scratch.

As an aside remark, let us comment on the fact that the more general problem of quantizing coisotropic manifolds of a given Poisson manifold, in the context of deformation quantization, has recently raised quite some interest (see [BGHHW,CF]).

As an example, in the last section we show how we can use this quantum duality principle to derive new quantizations from known ones. The example is given by the Poisson structure introduced on the space of Stokes matrices by Dubrovin (see [Du]) and Ugaglia (see [Ug]) in the framework of moduli spaces of semisimple Frobenius manifolds. It was Boalch (cf. [Bo]) that first gave an interpretation of Dubrovin–Ugaglia brackets in terms of Poisson–Lie groups. We will rather follow later work by Xu (see [Xu]) where it was shown how Boalch's construction may be equivalently interpreted as quotient Poisson structure of the dual Poisson-Lie group G^* of the standard $SL_n(\Bbbk)$. In more detail the Poisson space of Stokes matrices $G^*\big/H^\perp$ is the dual Poisson space to the Poisson space $SL_n(\Bbbk)\big/SO_n(\Bbbk)$. It has to be noted that the embedding of $SO_n(\Bbbk)$ in $SL_n(\Bbbk)$ is known to be coisotropic but not Poisson. Starting, then, from results obtained by Noumi in [No] related to a quantum version of the embedding $SO_n(\Bbbk) \longrightarrow SL_n(\Bbbk)$ we are able to interpret them as an explicit quantization of the Dubrovin-Ugaglia structure. We provide explicit computations for the case $n = 3$, and draw a sketch with the main guidelines for the general case.

Finally, another, stronger formulation of our QDP for subgroups and homogeneous spaces can be given in terms of quantum groups of *global* type, see [CG].

— — — — —

References

[BGHHW] M. Bordemann, G. Ginot, G. Halbout, H.-C. Herbig, S. Waldmann, *Star-répresentation sur des sous-varietés coïsotropes*, preprint (2003); electronic version on `arXiv: math.QA/0309321`.

[Bo] P. Boalch, *Stokes matrices, Poisson-Lie groups and Frobenius manifolds*, Invent. Math. **146** (2001), 479–506.

[CF] A. Cattaneo, G. Felder, *Coisotropic submanifolds in Poisson geometry, branes and Poisson σ–models*, Lett. Math. Phys. **69** (2004), 157–175.

[CG] N. Ciccoli, F. Gavarini, *A global quantum duality principle for subgroups and homogeneous spaces*, preprint.

[Ci] N. Ciccoli, *Quantization of coisotropic subgroups*, Lett. Math. Phys. **42** (1997), 123–138.

[Dr1] V. G. Drinfel'd, *Quantum groups*, Proc. Intern. Congress of Math. (Berkeley, 1986), 1987, pp. 798–820.

[Dr2] V. G. Drinfel'd, *On Poisson homogeneous spaces of Poisson-Lie groups*, Theoret. and Math. Phys. **95** (1993), 524–525.

[Du] B. Dubrovin, *Geometry of 2d topological field theories*, in: Integrable Systems and Quantum Groups (Montecatini Terme, 1993), 120–348, Lecture Notes in Mathematics **1620**, Springer, Berlin, 1996.

[EK1] P. Etingof, D. Kazhdan, *Quantization of Lie bialgebras, I*, Selecta Math. (N.S.) **2** (1996), 1–41.

[EK2] ______, *Quantization of Poisson algebraic groups and Poisson homogeneous spaces*, Symétries quantiques (Les Houches, 1995), North-Holland, Amsterdam, 1998, pp. 935–946.

[EL] S. Evens, J.-H. Lu, *On the variety of Lagrangian subalgebras. I*, Ann. Sci. École Norm. Sup. (N.S.) **34** (2001), 631–668.

[ES] P. Etingof, O. Schiffman, *Lectures on Quantum Groups*, International Press, Cambridge, 1998.

[FRT] L. D. Faddeev, N. Yu. Reshetikhin, L. A. Takhtajan, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. **1** (1990), 193–225.

[Ga1] F. Gavarini, *The quantum duality principle*, Annales de l'Institut Fourier **52** (2002), 809–834.

[Ga2] ______, *The global quantum duality principle: theory, examples, and applications*, preprint math.QA/0303019 (2003).

[GIK] A. M. Gavrilik, N. Z. Iorgov, A. U. Klimyk, *Nonstandard deformation $U_q'(so_n)$: the imbedding $U_q'(so_n) \subset U_q(sl_n)$ and representations*, Symmetries in Science X, B. Gruber, M. Ramek (eds.), Plenum Press, New York, 1998, pp. 121–133.

[HKP] M. Havlíček, A. U. Klimyk, S. Pošta, *Representations of the cyclically symmetric q–deformed algebra $so_q(3)$*, Jour. Math. Phys. **40** (1999), 2135–2161.

[KT] C. Kassel, V. Turaev, *Biquantization of Lie bialgebras*, Pac. Jour. Math. **195** (2000), 297–369.

[Lu] J. H. Lu, *Multiplicative and affine Poisson structures on Lie groups*, Ph.D. thesis University of California, Berkeley (1990).

[LW] J. H. Lu, A. Weinstein, *Poisson-Lie groups, dressing transformations and Bruhat decompositions*, J. Diff. Geom. **31** (1990), 501–526.

[Ma] A. Masuoka, *Quotient theory of Hopf algebras*, in: Advances in Hopf algebras, Lecture Notes in Pure and Appl. Math. **158** (1994), 107–133.

[No] M. Noumi, *Macdonald's Symmetric Polynomials as Zonal Spherical Functions on Some Quantum Homogeneous Spaces*, Adv. Math. **123** (1996), 16–77.

[Ug] M. Ugaglia, *On a Poisson structure on the space of Stokes matrices*, International Mathematics Research Notices **9** (1999), 473–493.

[Xu] P. Xu, *Dirac submanifolds and Poisson involutions*, Ann. Sci. Ecole Norm. Sup. (3) **36** (2003), 403–430.

[Za] S. Zakrzewski, *Poisson homogeneous spaces*, in: J. Lukierski, Z. Popowicz, J. Sobczyk (eds.), *Quantum groups (Karpacz, 1994)*, PWN, Warszaw, 1995, 629–639.