F. Gavarini, *"Poisson geometrical symmetries associated to non-commutative formal diffeomorphisms"*

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INTRODUCTION

"A series of outlaws joined and formed the Nottingham group, whose renowned chieftain was the famous Robin Hopf"

N. Barbecue, "Robin Hopf"

The most general notion of "symmetry" in mathematics is encoded in the notion of Hopf algebra. Then, among all Hopf algebras (over a field $\Bbbk$), there are two special families which are of relevant interest for their geometrical meaning: assuming for simplicity that $\Bbbk$ have zero characteristic, these are the function algebras $F[G]$ of algebraic groups G and the universal enveloping algebras $U(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$. Function algebras are exactly those Hopf algebras which are commutative, and enveloping algebras those which are connected (in the general sense of Hopf algebra theory) and cocommutative.

Given a Hopf algebra H, encoding some generalized symmetry, one can ask whether there are any other Hopf algebras "close" to H, which are of either one of the above mentioned *geometrical types*, hence encoding geometrical symmetries associated to H. The answer is affirmative: namely (see [Ga4]), it is possible to give functorial recipes to get out of any Hopf algebra H two pairs of Hopf algebras of geometrical type, say $\big(F[G_+], U(\mathfrak{g}_-)\big)$ and $\big(F[K_+], U(\mathfrak{k}_-)\big)$. Moreover, the algebraic groups thus obtained are connected *Poisson* groups, and the Lie algebras are *Lie bialgebras*; therefore in both cases *Poisson geometry* is involved. In addition, the two pairs above are related to each other by Poisson duality (see below), thus only either one of them is truly relevant. Finally, these four "geometrical" Hopf algebras are "close" to H in that they are 1-parameter deformations (with pairwise isomorphic fibers) of a quotient or a subalgebra of H.

The method above to associate Poisson geometrical Hopf algebras to general Hopf algebras, called "Crystal Duality Principle" (CDP in short), is explained in detail in [Ga4]. It is a special instance of a more general result, the "Global Quantum Duality Principle" (GQDP in short), explained in [Ga2–3], which in turn is a generalization of the "Quantum Duality Principle" due to Drinfeld (cf. [Dr], §7, and see [Ga1] for a proof).

Drinfeld's QDP deals with quantum universal enveloping algebras (QUEAs in short) and quantum formal series Hopf algebras (QFSHAs in short) over the ring of formal power series $\Bbbk[[\hbar]]$. A QUEA is any topologically free, topological Hopf $\Bbbk[[\hbar]]$–algebra whose quotient modulo $\hbar$ is the universal enveloping algebra $U(\mathfrak{g})$ of some Lie algebra $\mathfrak{g}$; in this case we denote the QUEA by $U_\hbar(\mathfrak{g})$. Instead, a QFSHA is any topological Hopf $\Bbbk[[\hbar]]$–algebra of type $\Bbbk[[\hbar]]^S$ (as a $\Bbbk[[\hbar]]$–module, S being a set) whose quotient modulo $\hbar$ is the function algebra $F[[G]]$ of some formal algebraic group G; then we denote the QFSHA by $F_\hbar[[G]]$.

The QDP claims that the category of all QUEAs and the category of all QFSHAs are equivalent, and provides an equivalence in either direction. From QFSHAs to QUEAs it goes as follows: given a QFSHA, say $F_\hbar[[G]]$, let J be its augmentation ideal (the kernel of its counit map) and set $F_\hbar[[G]]^\vee := \sum_{n \geq 0} \hbar^{-n} J^n$. Then $F_\hbar[[G]] \mapsto F_\hbar[[G]]^\vee$ defines (on objects) a functor from QFSHAs to QUEAs. To go the other way round, i.e. from QUEAs to QFSHAs, one uses a perfectly dual recipe. Namely, given a QUEA, say $U_\hbar(\mathfrak{g})$, let again J be its augmentation ideal; for each $n \in \mathbb{N}$, let δ_n be the composition of the n–fold iterated coproduct followed by the projection onto $J^{\otimes n}$ (this makes sense since $U_\hbar(\mathfrak{g}) = \Bbbk[[\hbar]] \cdot 1_{U_\hbar(\mathfrak{g})} \oplus J$): then set $U_\hbar(\mathfrak{g})' := \bigcap_{n \geq 0} {\delta_n}^{-1}\big(\hbar^n U_\hbar(\mathfrak{g})^{\otimes n}\big)$, or more explicitly $U_\hbar(\mathfrak{g})' := \big\{ \eta \in U_\hbar(\mathfrak{g}) \,\big|\, \delta_n(\eta) \in \hbar^n U_\hbar(\mathfrak{g})^{\otimes n}, \ \forall\, n \in \mathbb{N} \big\}$. Then $U_\hbar(\mathfrak{g}) \mapsto U_\hbar(\mathfrak{g})'$ defines (on objects) a functor from QUEAs to QFSHAs. The functors $(\ )^\vee$ and $(\ )'$ are inverse to each other, hence they provide the claimed equivalence.

Note that the objects (QUEAs and QFSHAs) involved in the QDP are *quantum groups*; their semiclassical limits then are endowed with Poisson structures: namely, every $U(\mathfrak{g})$ is in fact a co-Poisson Hopf algebra and every $F[[G]]$ is a (topological) Poisson Hopf algebra. The geometrical structures they describe are then *Lie bialgebras* and *Poisson groups*. The QDP then brings further information: namely, the semiclassical limit of the image of a given quantum group is *Poisson dual* to the Poisson geometrical object we start from. In short

$$F_\hbar[[G]]^\vee \Big/ \hbar\, F_\hbar[[G]]^\vee \, = \, U(\mathfrak{g}^\times)\,, \qquad \text{i.e. (roughly)} \qquad F_\hbar[[G]]^\vee \, = \, U_\hbar(\mathfrak{g}^\times) \tag{I.1}$$

where $\mathfrak{g}^\times$ is the *cotangent Lie bialgebra* of the Poisson group G, and

$$U_\hbar(\mathfrak{g})' \Big/ \hbar\, U_\hbar(\mathfrak{g})' \, = \, F\big[[G^\star]\big]\,, \qquad \text{i.e. (roughly)} \qquad U_\hbar(\mathfrak{g})' \, = \, F_\hbar\big[[G^\star]\big] \tag{I.2}$$

where $G^\star$ is a connected Poisson group with cotangent Lie bialgebra $\mathfrak{g}$. So the QDP involves both Hopf duality (switching enveloping and function algebras) and Poisson duality.

The generalization from QDP to GQDP stems from a simple observation: the construction of Drinfeld's functors needs not to start from quantum groups! Indeed, in order to define either $H^\vee$ or H' one only needs that H be a torsion-free Hopf algebra over some 1-dimensional doamin R and $\hbar \in R$ be *any non-zero prime* (actually, even less is truly necessary, see [Ga2–3]). On the other hand, the outcome still is, in both cases, a "quantum group", now meant in a new sense. Namely, a QUEA now will be any torsion-free Hopf algebra H over R such that $H \big/ \hbar\, H \cong U(\mathfrak{g})$, for some Lie (bi)algebra $\mathfrak{g}$. Also, instead of QFSHAs we consider "quantum function algebras", QFAs in short: here a QFA will be any torsion-free Hopf algebra H over R such that $H \big/ \hbar\, H \cong F[G]$ (plus one additional technical condition) for some connected (Poisson) group G. In this new framework Drinfeld's recipes give that $H^\vee$ is a QUEA and H' is a QFA, whatever is the torsion-free Hopf R–algebra H one starts from. Moreover, when restricted to quantum groups Drinfeld's functors $(\ )^\vee$ and $(\ )'$ again provide equivalences of quantum group categories, respectively from QFAs to QUEAs and viceversa; then Poisson duality is involved once more, like in (I.1–2).

Therefore, the generalization process from the QDP to the GQDP spreads over several concerns. Arithmetically, one can take as $(\hbar)$ any non-generic point of the spectrum of

R, and define Drinfeld's functors and specializations accordingly; in particular, the corresponding quotient field $\Bbbk_\hbar := R\big/\hbar\,R$ might have *positive* characteristic. Geometrically, one considers algebraic groups rather than formal groups, i.e. *global* vs. *local* objects. Algebraically, one drops any topological worry ($\hbar$–adic completeness, etc.), and deals with *general* Hopf algebras rather than with quantum groups. This last point is the one of most concern to us now, in that it means that we have (functorial) recipes to get several quantum groups, hence — taking semiclassical limits — Poisson geometrical symmetries, springing out of the "generalized symmetry" encoded by a torsion-free Hopf algebra H over R: namely, for each non-trivial point of the spectrum of R, the quantum groups $H^\vee$ and H' given by the corresponding Drinfeld's functors. Note, however, that *a priori* nothing prevents any of these $H^\vee$ or H' or their semiclassical limits to be (essentially) trivial.

The CDP comes out when looking at Hopf algebras over a field $\Bbbk$, and then applying the GQDP to their scalar extensions $H[\hbar] := \Bbbk[\hbar] \otimes_\Bbbk H$ with $R := \Bbbk[\hbar]$ (and $\hbar := \hbar$ itself). A first application of Drinfeld's functors to $H_\hbar := H[\hbar]$ followed by specialization at $\hbar = 0$ provides the pair $\big(F[G_+]\,, U(\mathfrak{g}_-)\big)$ mentioned above: in a nutshell, $\big(F[G_+]\,, U(\mathfrak{g}_-)\big) = \Big({H_\hbar}'\big|_{\hbar=0}\,, {H_\hbar}^\vee\big|_{\hbar=0}\Big)$, where hereafter $X\big|_{\hbar=0} := X\Big/\hbar\,X$. Then applying once more Drinfeld's functors to ${H_\hbar}^\vee$ and to ${H_\hbar}'$ and specializing at $\hbar = 0$ yields the pair $\big(F[K_+]\,, U(\mathfrak{k}_-)\big)$, namely $\big(F[K_+]\,, U(\mathfrak{k}_-)\big) = \Big(\big({H_\hbar}^\vee\big)'\Big|_{\hbar=0}\,, \big({H_\hbar}'\big)^\vee\Big|_{\hbar=0}\Big)$. Finally, the very last part of the GQDP explained before implies that $K_+ = G_-^\star$ and $\mathfrak{k}_- = \mathfrak{g}_+^\times$.

While in the second step above one really needs the full strength of the GQDP, for the first step instead it turns out that the construction of Drinfeld's functors on $H[\hbar]$, can be fully "tracked through" and described at the "classical level", i.e. in terms of H alone. In addition, the exact relationship among H and the pair $\big(F[G_+]\,, U(\mathfrak{g}_-)\big)$ can be made quite clear, and more information is available about this pair. We now sketch it in some detail.

Let J be the augmentation ideal of H, let $\underline{J} := \big\{J^n\big\}_{n \in \mathbb{N}}$ be the associated (decreasing) J–adic filtration, $\widehat{H} := G_{\underline{J}}(H)$ the associated graded vector space and $H^\vee := H\Big/\bigcap_{n\in\mathbb{N}} J^n$. One can prove that $\underline{J}$ is a Hopf algebra filtration, hence $\widehat{H}$ is a graded Hopf algebra. The latter happens to be connected and cocommutative, so $\widehat{H} \cong U(\mathfrak{g}_-)$ for some Lie algebra $\mathfrak{g}_-$; in addition, since $\widehat{H}$ is graded also $\mathfrak{g}_-$ itself is graded as a Lie algebra. The fact that $\widehat{H}$ be cocommutative allows to define on it a Poisson cobracket which makes $\widehat{H}$ into a graded *co-Poisson* Hopf algebra; eventually, this implies that $\mathfrak{g}_-$ is a *Lie bialgebra*. The outcome is that our $U(\mathfrak{g}_-)$ is just $\widehat{H}$.

On the other hand, one considers a second (increasing) filtration defined in a dual manner to $\underline{J}$, namely $\underline{D} := \big\{D_n := Ker\,(\delta_{n+1})\big\}_{n\in\mathbb{N}}$. Let now $\widetilde{H} := G_{\underline{D}}(H)$ be the associated graded vector space and $H' := \bigcup_{n\in\mathbb{N}} D_n$. Again, one shows that $\underline{D}$ is a Hopf algebra filtration, hence $\widetilde{H}$ is a graded Hopf algebra. Moreover, the latter is commutative, so $\widetilde{H} = F[G_+]$ for some algebraic group G_+. One proves also that $\widetilde{H} = F[G_+]$ has no non-trivial idempotents, thus G_+ is connected; in addition, since $\widehat{H}$ is graded, G_+ as a variety is just an affine space. The fact that $\widetilde{H}$ be commutative allows to define on it a Poisson bracket which makes $\widetilde{H}$ into a graded *Poisson* Hopf algebra: this means that G_+ is an algebraic *Poisson group*. Thus eventually $F[G_+]$ is just $\widetilde{H}$.

The relationship among H and the "geometrical" Hopf algebras $\widehat{H}$ and $\widetilde{H}$ can be expressed in terms of "reduction steps" and regular 1-parameter deformations, namely

$$\widetilde{H} \xleftarrow[\mathcal{R}_{\underline{D}}^{\hbar}(H)]{0 \leftarrow \hbar \rightarrow 1} H' \hookrightarrow H \twoheadrightarrow H^{\vee} \xleftarrow[\mathcal{R}_{\underline{J}}^{\hbar}(H^{\vee})]{1 \leftarrow \hbar \rightarrow 0} \widehat{H} \tag{I.3}$$

where one-way arrows are Hopf algebra morphisms and two-ways arrows are regular 1-parameter deformations of Hopf algebras, realized through the *Rees* Hopf algebras $\mathcal{R}_{\underline{D}}^{\hbar}(H)$ and $\mathcal{R}_{\underline{J}}^{\hbar}(H^{\vee})$ associated to the filtration $\underline{D}$ of H and to the filtration $\underline{J}$ of $H^{\vee}$. Hereafter "regular" for a deformation means that all its fibers are pairwise isomorphic as vector spaces. In classical terms, (I.3) comes directly from the construction above; on the other hand, in terms of the GQDP it comes from the fact that $\mathcal{R}_{\underline{D}}^{\hbar}(H) = H_{\hbar}{}'$ and $\mathcal{R}_{\underline{J}}^{\hbar}(H^{\vee}) = H_{\hbar}{}^{\vee}$.

As we mentioned above, next step is the "application" of (suitable) Drinfeld's functors to the Rees algebras $\mathcal{R}_{\underline{D}}^{\hbar}(H) = H_{\hbar}{}'$ and $\mathcal{R}_{\underline{J}}^{\hbar}(H^{\vee}) = H_{\hbar}{}^{\vee}$ occurring in (I.3). The outcome is a second frame of regular 1-parameter deformations for H' and $H^{\vee}$, namely

$$U\big(\mathfrak{g}_{+}^{\times}\big) = U(\mathfrak{k}_{-}) \xleftarrow[(H_{\hbar}')^{\vee}]{0 \leftarrow \hbar \rightarrow 1} H' \hookrightarrow H \twoheadrightarrow H^{\vee} \xleftarrow[(H_{\hbar}^{\vee})']{1 \leftarrow \hbar \rightarrow 0} F[K_{+}] = F\big[G_{-}^{\star}\big] \tag{I.4}$$

which is the analogue of (I.3). In particular, when $H^{\vee} = H = H'$ from (I.3) and (I.4) together we find H as the mid-point of four deformation families, whose "external points" are Hopf algebras of "Poisson geometrical" type, namely

$$\begin{array}{ccccc} U(\mathfrak{g}_{-}) & \xleftarrow[H_{\hbar}^{\vee}]{0 \leftarrow \hbar \rightarrow 1} & H & \xleftarrow[(H_{\hbar}^{\vee})']{1 \leftarrow \hbar \rightarrow 0} & F\big[G_{-}^{\star}\big] \\ & & \| & & \\ F[G_{+}] & \xleftarrow[H_{\hbar}']{0 \leftarrow \hbar \rightarrow 1} & H & \xleftarrow[(H_{\hbar}')^{\vee}]{1 \leftarrow \hbar \rightarrow 0} & U\big(\mathfrak{g}_{+}^{\times}\big) \end{array} \tag{✠}$$

which gives *four* different regular 1-parameter deformations from H to Hopf algebras encoding Poisson geometrical objects. Then each of these four Hopf algebras may be thought of as a semiclassical geometrical counterpart of the "generalized symmetry" encoded by H.

The purpose of the present paper is to show the effectiveness of the CDP, applying it to a key example, the Hopf algebra of non-commutative formal diffeomorphisms of the line. Indeed, the interest of the latter, besides its own reasons, grows bigger as we can see it as a toy model for a broad family of Hopf algebras of great concern in mathematical physics, non-commutative geometry and beyond. Now I go and present the results of this paper.

Let $\mathcal{G}^{\mathrm{dif}}$ be the set of all formal power series starting with x with coefficients in a field $\Bbbk$ of zero characteristic. Endowed with the composition product, this is an infinite dimensional prounipotent proalgebraic group — known as the "(normalised) Nottingham group" among group-theorists and the "(normalised) group of formal diffeomorphisms of the line" among mathematical physicists — whose tangent Lie algebra is a special subalgebra of the one-sided Witt algebra. The function algebra $F\big[\mathcal{G}^{\mathrm{dif}}\big]$ is a graded, commutative Hopf algebra with countably many generators, which admits a neat combinatorial description.

In [BF] a non-commutative version of $F\big[\mathcal{G}^{\mathrm{dif}}\big]$ is introduced: this is a non-commutative non-cocommutative Hopf algebra $\mathcal{H}^{\mathrm{dif}}$ which is presented exactly like $F\big[\mathcal{G}^{\mathrm{dif}}\big]$ but dropping commutativity, i.e. taking the presentation as one of a unital *associative* — and *not* commutative — algebra; in other words, $\mathcal{H}^{\mathrm{dif}}$ is the outcome of applying to $F\big[\mathcal{G}^{\mathrm{dif}}\big]$ a raw "disabelianization" process. In particular, $H = \mathcal{H}^{\mathrm{dif}}$ is *graded* and verifies $H^\vee = H = H'$, hence the scheme (✠) makes sense and yields four Poisson symmetries associated to $\mathcal{H}^{\mathrm{dif}}$.

Note that in each line in (✠) there is essentially only *one* Poisson geometry involved, since Poisson duality relates mutually opposite sides; thus any classical symmetry on the same line carries as much information as the other one (but for global-to-local differences). Nevertheless, in the case of $H = \mathcal{H}^{\mathrm{dif}}$ we shall prove that the pieces of information from either line in (✠) are complementary, because G_+ and $G_-^\star$ happen to be isomorphic as proalgebraic Poisson varieties but *not* as groups. In particular, we find that the Lie bialgebras $\mathfrak{g}_-$ and $\mathfrak{g}_+^\times$ are both isomorphic as Lie algebras to the free Lie algebra $\mathcal{L}(\mathbb{N}_+)$ over a countable set, but they have different, non-isomorphic Lie coalgebra structures. Moreover, $G_-^\star \cong \mathcal{G}^{\mathrm{dif}} \times \mathcal{N} \cong G_+$ as Poisson varieties, where $\mathcal{N}$ is a proaffine Poisson variety whose coordinate functions are in bijection with a basis of the derived subalgebra $\mathcal{L}(\mathbb{N}_+)$; indeed, the latter are obtained by iterated Poisson brackets of coordinate functions on $\mathcal{G}^{\mathrm{dif}}$, in short because both $F\big[G_-^\star\big]$ and $F\big[G_+\big]$ are freely generated as Poisson algebras by a copy of $F\big[\mathcal{G}^{\mathrm{dif}}\big]$. For $G_-^\star$ we have a more precise result, namely $G_-^\star \cong \mathcal{G}^{\mathrm{dif}} \ltimes \mathcal{N}$ (a semidirect product) *as proalgebraic groups*: thus in a sense $G_-^\star$ is the *free Poisson group over* $\mathcal{G}^{\mathrm{dif}}$, which geometrically speaking is obtained by "pasting" to $\mathcal{G}^{\mathrm{dif}}$ all 1-parameter subgroups freely obtained via iterated Poisson brackets of those of $\mathcal{G}^{\mathrm{dif}}$; in particular, these Poisson brackets iteratively yield 1-parameter subgroups which generate $\mathcal{N}$.

We perform the same analysis simultaneously for $\mathcal{G}^{\mathrm{dif}}$, for its subgroup of *odd* formal diffeomorphisms and for all the groups $\mathcal{G}_\nu$ of *truncated* (at order $\nu \in \mathbb{N}_+$) formal diffeomorphisms, whose projective limit is $\mathcal{G}^{\mathrm{dif}}$ itself; *mutatis mutandis*, the results are the like.

The case of $\mathcal{H}^{\mathrm{dif}}$ is just one of many samples of the same type: indeed, several cases of Hopf algebras built out of combinatorial data — graphs, trees, Feynman diagrams, etc. — have been introduced in (co)homological theories (see e.g. [LR] and [Fo1–2], and references therein) and in renormalization studies (see [CK1–3]); in most cases these algebras — or their (graded) duals — are commutative polynomial, like $F\big[\mathcal{G}^{\mathrm{dif}}\big]$, and admit non-commutative analogues (thanks to [Fo1–2]), so our discussion apply almost *verbatim* to them too, with like results. Thus the given analysis of the "toy model" Hopf algebra $\mathcal{H}^{\mathrm{dif}}$ can be taken as a general pattern for all those cases.

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References

[BF] C. Brouder, A. Frabetti, *Noncommutative renormalization for massless QED*, preprint hep-th/0011161 (2000).

[Ca] R. Carmina, *The Nottingham Group*, in: M. Du Sautoy, D. Segal, A. Shalev (eds.), *New Horizons in pro-p Groups*, Progress in Math. **184** (2000), 205–221.

[CK1] A. Connes, D. Kreimer, *Hopf algebras, Renormalization and Noncommutative Geometry*, Comm. Math. Phys. **199** (1998), 203–242.

[CK2] ———, *Renormalization in quantum field theory and the Riemann-Hilbert problem I: the Hopf algebra structure of graphs and the main theorem*, Comm. Math. Phys. **210** (2000), 249–273.

[CK3] ———, *Renormalization in quantum field theory and the Riemann-Hilbert problem II: the β function, diffeomorphisms and the renormalization group*, Comm. Math. Phys. **216** (2001), 215–241.

[Dr] V. G. Drinfeld, *Quantum groups*, Proc. Intern. Congress of Math. (Berkeley, 1986), 1987, pp. 798–820.

[Fo1] L. Foissy, *Les algèbres de Hopf des arbres enracinés décorés, I*, Bull. Sci. Math. **126** (2002), 193–239.

[Fo2] ———, *Les algèbres de Hopf des arbres enracinés décorés, II*, Bull. Sci. Math. **126** (2002), 249–288.

[Fo3] ———, *Finite dimensional comodules over the Hopf algebra of rooted trees*, J. Algebra **255** (2002), 89–120.

[Ga1] F. Gavarini, *The quantum duality principle*, Annales de l'Institut Fourier **52** (2002), 809–834.

[Ga2] ———, *The global quantum duality principle: theory, examples, and applications*, preprint math.QA/0303019 (2003).

[Ga3] ———, *The global quantum duality principle*, to appear (2004).

[Ga4] ———, *The Crystal Duality Principle: from Hopf Algebras to Geometrical Symmetries*, J. of Algebra **285** (2005), 399–437.

[Je] S. Jennings, *Substitution groups of formal power series*, Canadian J. Math. **6** (1954), 325–340.

[LR] J.-L. Loday, M. O. Ronco, *Hopf algebra of the planar binary trees*, Adv. Math. **139** (1998), 293–309.

[Re] C. Reutenauer, *Free Lie Algebras*, London Mathematical Society Monographs, New Series **7**, Oxford Science Publications, New York, 1993.