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"The crystal duality principle: from general symmetries to geometrical symmetries"

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INTRODUCTION

Among all Hopf algebras over a field $\Bbbk$, there are two special families which are of relevant interest for their geometrical meaning. The function algebras $F[G]$ of algebraic groups G and the universal enveloping algebras $U(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$, if $\mathit{Char}(\Bbbk) = 0$, or the restricted universal enveloping algebras $\mathbf{u}(\mathfrak{g})$ of restricted Lie algebras $\mathfrak{g}$, if $\mathit{Char}(\Bbbk) > 0$; to be short, we call both the latters "enveloping algebras" and denote them by $\mathcal{U}(\mathfrak{g})$, and similarly by "restricted Lie algebra" when $\mathit{Char}(\Bbbk) = 0$ we shall simply mean "Lie algebra". Function algebras are exactly those Hopf algebras which are commutative, and enveloping algebras those which are connected, cocommutative and generated by their primitives.

In this paper we give functorial recipes to get, out of any Hopf algebra, two pairs of Hopf algebras of geometrical type, namely one pair $\big(F[G_+], \mathcal{U}(\mathfrak{g}_-)\big)$ and a second pair $\big(F[K_+], \mathcal{U}(\mathfrak{k}_-)\big)$. In addition, the algebraic groups thus obtained are *Poisson groups*, and the (restricted) Lie algebras are (restricted) *Lie bialgebras*. Therefore, to each Hopf algebra, encoding a general notion of "symmetry", we can associate in a functorial way some symmetries — "global" ones when taking an algebraic group, "infinitesimal" when considering a Lie algebra — of geometrical type, where the geometry involved is in fact Poisson geometry. Moreover, the groups concerned are always connected, and if $\mathit{Char}(\Bbbk) > 0$ they have dimension 0 and height 1, which makes them pretty interesting from the point of view of arithmetic geometry (hence in number theory).

The construction of the pair $(G_+, \mathfrak{g}_-)$ uses pretty classical (as opposite to "quantum") methods: in fact, it might part of be the content of any basic textbook on Hopf algebras (and, surprisingly enough, it is not!). Instead, to make out the pair $(K_+, \mathfrak{k}_-)$ one relies on the construction of the first pair, and make use of the theory of quantum groups.

Let's describe our results in some detail. Let $J := \mathit{Ker}(\epsilon_H)$ be the augmentation ideal of H (where ϵ_H is the counit of H), and let $\underline{J} := \big\{J^n\big\}_{n \in \mathbb{N}}$ be the associated J–adic filtration, $\widehat{H} := G_{\underline{J}}(H)$ the associated graded vector space and $H^\vee := H \Big/ \bigcap_{n \in \mathbb{N}} J^n$. One proves that $\underline{J}$ is a Hopf algebra filtration, hence $\widehat{H}$ is a graded Hopf algebra: the latter happens to be connected, cocommutative and generated by its primitives, so $\widehat{H} \cong \mathcal{U}(\mathfrak{g}_-)$ for some (restricted) Lie algebra $\mathfrak{g}_-$; in addition, since $\widehat{H}$ is graded also $\mathfrak{g}_-$ itself is graded (as a restricted Lie algebra). The fact that $\widehat{H}$ be cocommutative allows to define on it a Poisson cobracket (from the natural Poisson cobracket $\nabla := \Delta - \Delta^{\text{op}}$ on H) which makes $\widehat{H}$ into a graded *co-Poisson* Hopf algebra, and eventually this implies that $\mathfrak{g}_-$ is a *Lie bialgebra*. So the right-hand side half of the first pair of "Poisson geometrical" Hopf algebras is just $\widehat{H}$.

On the other hand, one consider a second filtration — increasing, whereas $\underline{J}$ is decreasing — namely $\underline{D}$ which is defined in a dual manner to $\underline{J}$: for each $n \in \mathbb{N}$, let δ_n the composition of the n–fold iterated coproduct followed by the projection onto $J^{\otimes n}$ (note that $H = \Bbbk \cdot 1_H \oplus J$); then $\underline{D} := \big\{ D_n := \mathit{Ker}\,(\delta_{n+1}) \big\}_{n \in \mathbb{N}}$. Let now $\widetilde{H} := G_{\underline{D}}(H)$ be the associated graded vector space and $H' := \bigcup_{n \in \mathbb{N}} D_n$. Again, one shows that $\underline{D}$ is a Hopf algebra filtration, hence $\widetilde{H}$ is a graded Hopf algebra: moreover, the latter is commutative, so $\widetilde{H} = F[G_+]$ for some algebraic group G_+. One proves also that $\widetilde{H} = F[G_+]$ has no non-trivial idempotents, thus G_+ is connected; a deeper analysis shows that in the positive characteristic case G_+ has dimension 0 and height 1; in addition, since $\widetilde{H}$ is graded, G_+ as a variety is just an affine space. The fact that $\widetilde{H}$ be commutative allows to define on it a Poisson bracket (from the natural Poisson bracket on H given by the commutator) which makes $\widetilde{H}$ into a graded *Poisson* Hopf algebra: this means G_+ is an algebraic *Poisson group*. So the left-hand side half of the first pair of "Poisson geometrical" Hopf algebras is just $\widetilde{H}$.

The relationship among H and the "geometrical" Hopf algebras $\widehat{H}$ and $\widetilde{H}$ can be expressed in terms of "reduction steps" and regular 1-parameter deformations, namely

$$\widetilde{H} \xleftrightarrow[\mathcal{R}^t_{\underline{D}}(H)]{0 \leftarrow t \rightarrow 1} H' \hookrightarrow H \twoheadrightarrow H^\vee \xleftrightarrow[\mathcal{R}^t_{\underline{J}}(H^\vee)]{1 \leftarrow t \rightarrow 0} \widehat{H} \qquad (\bigstar)$$

where the "one-way" arrows are Hopf algebra morphisms and the "two-ways" arrows are 1-parameter regular deformations of Hopf algebras, realized through the *Rees* Hopf algebras $\mathcal{R}^t_{\underline{D}}(H)$ and $\mathcal{R}^t_{\underline{J}}(H^\vee)$ associated to the filtration $\underline{D}$ of H and to the filtration $\underline{J}$ of $H^\vee$.

The construction of the pair $(K_+, \mathfrak{k}_-)$ uses quantum group theory, the basic ingredients being $\mathcal{R}^t_{\underline{D}}(H)$ and $\mathcal{R}^t_{\underline{J}}(H^\vee)$. In the present context, by quantum group we mean, loosely speaking, a Hopf $\Bbbk[t]$–algebra (t an indeterminate) H_t such that either *(a)* $H_t \Big/ t\,H_t \cong F[G]$ for some connected Poisson group G — then we say H_t is a QFA — or *(b)* $H_t \Big/ t\,H_t \cong \mathcal{U}(\mathfrak{g})$, for some restricted Lie bialgebra $\mathfrak{g}$ — then we say H_t is a QrUEA. Formula $(\bigstar)$ says that $H'_t := \mathcal{R}^t_{\underline{D}}(H)$ is a QFA, with $H'_t \Big/ t\,H'_t \cong \widetilde{H} = F[G_+]$, and also that $H^\vee_t := \mathcal{R}^t_{\underline{J}}(H)$ is a QrUEA, with $H^\vee_t \Big/ t\,H^\vee_t \cong \widehat{H} \cong \mathcal{U}(\mathfrak{g}_-)$. Now, a general result — the "Global Quantum Duality Principle", in short GQDP — teaches us how to construct from the QFA H'_t a QrUEA, call it $\big(H'_t\big)^\vee$, and how to build out of the QrUEA $H^\vee_t$ a QFA, say $\big(H^\vee_t\big)'$; then $\big(H'_t\big)^\vee \Big/ t\,\big(H'_t\big)^\vee \cong \mathcal{U}(\mathfrak{k}_-)$ for some (restricted) Lie bialgebra $\mathfrak{k}_-$, and $\big(H^\vee_t\big)' \Big/ t\,\big(H^\vee_t\big)' \cong F[K_+]$ for some connected Poisson group K_+. This provides the pair $(K_+, \mathfrak{k}_-)$. The very construction implies that $\big(H'_t\big)^\vee$ and $\big(H^\vee_t\big)'$ yield another frame of regular 1-parameter deformations for H' and $H^\vee$, namely

$$\mathcal{U}(\mathfrak{k}_-) \xleftrightarrow[(H'_t)^\vee]{0 \leftarrow t \rightarrow 1} H' \hookrightarrow H \twoheadrightarrow H^\vee \xleftrightarrow[(H^\vee_t)']{1 \leftarrow t \rightarrow 0} F[K_+] \qquad (\maltese)$$

which is the analogue of (★). In addition, when $Char(\Bbbk) = 0$ the GQDP also claims that the two pairs $(G_+, \mathfrak{g}_-)$ and $(K_+, \mathfrak{k}_-)$ are related by Poisson duality: namely, $\mathfrak{k}_-$ *is the cotangent Lie bialgebra to* G_+, and $\mathfrak{g}_-$ *is the cotangent Lie bialgebra* of K_+ (in short, we write $\mathfrak{k}_- = \mathfrak{g}^\times$ and $K_+ = G_-^\star$). Therefore the four "Poisson symmetries" G_+, $\mathfrak{g}_-$, K_+ and $\mathfrak{k}_-$, attached to H are actually encoded simply by the pair (G_+, K_+).

In particular, when $H' = H = H^\vee$ from (★) and (✠) together we find

$$\begin{array}{ccccc} F[G_+] & \xleftarrow[H'_t]{0 \leftarrow t \rightarrow 1} \!\!\!\!\xrightarrow{} & H' & \xleftarrow[(H'_t)^\vee]{1 \leftarrow t \rightarrow 0} \!\!\!\!\xrightarrow{} & \mathcal{U}(\mathfrak{k}_-) \quad \Big(= U\big(\mathfrak{g}_+^\times\big) \text{ if } Char(\Bbbk) = 0 \Big) \\ & & \| & & \\ & & H & & \\ & & \| & & \\ \mathcal{U}(\mathfrak{g}_-) & \xleftarrow[H_t^\vee]{0 \leftarrow t \rightarrow 1} \!\!\!\!\xrightarrow{} & H^\vee & \xleftarrow[(H_t^\vee)']{1 \leftarrow t \rightarrow 0} \!\!\!\!\xrightarrow{} & F[K_+] \quad \Big(= F\big[G_-^\star\big] \text{ if } Char(\Bbbk) = 0 \Big) \end{array}$$

which gives *four* different regular 1-parameter deformations from H to Hopf algebras encoding geometrical objects of Poisson type (i.e. Lie bialgebras or Poisson algebraic groups).

When the Hopf algebra H we start from is already of geometric type, the result involves Poisson duality. Namely, if $Char(\Bbbk) = 0$ and $H = F[G]$ then $\mathfrak{g}_- = \mathfrak{g}^*$ (where $\mathfrak{g} := Lie(G)$), and if $H = \mathcal{U}(\mathfrak{g}) = U(\mathfrak{g})$ then $Lie(G_+) = \mathfrak{g}^*$, i.e. G_+ has $\mathfrak{g}$ as cotangent Lie bialgebra. If instead $Char(\Bbbk) > 0$ we have only a slight variation on this result.

The construction of $\widehat{H}$ and $\widetilde{H}$ needs only "half the notion" of a Hopf bialgebra: in fact, we construct $\widehat{A}$ for any "augmented algebra" A (i.e., roughly, an algebra with an augmentation, or counit, that is a character), and we construct $\widetilde{C}$ for any "coaugmented coalgebra" C (i.e. a coalgebra with a coaugmentation, or "unit", that is a coalgebra morphism from $\Bbbk$ to C). In particular this applies to bialgebras, for which both $\widehat{B}$ and $\widetilde{B}$ are (graded) Hopf algebras; we can also perform a second construction as above using $\big(B'_t\big)^\vee$ and $\big(B_t^\vee\big)'$ (thanks to a stronger version of the GQDP), and get from these by specialization at $t = 0$ a second pair of bialgebras $\Big(\big(B'_t\big)^\vee\Big|_{t=0}, \big(B_t^\vee\big)'\Big|_{t=0} \Big)$: then again $\big(B'_t\big)^\vee\Big|_{t=0} \cong \mathcal{U}(\mathfrak{k}_-)$ for some restricted Lie bialgebra $\mathfrak{k}_-$, but on the other hand $\big(B_t^\vee\big)'\Big|_{t=0}$ is commutative and with no non-trivial idempotents, but it's not, in general, a Hopf algebra! Thus the spectrum of $\big(B_t^\vee\big)'\Big|_{t=0}$ is an algebraic Poisson monoid, irreducible as an algebraic variety, but it is not necessarily a Poisson group.

It is worth stressing that everything in fact follows from the GQDP, which — in the stronger formulation — deals with augmented algebras and coaugmented coalgebras over 1-dimensional domain. All the content of this paper can in fact be obtained as a corollary of the GQDP as follows: pick any augmented algebra or coaugmented coalgebra over $\Bbbk$, and take its scalar extension from $\Bbbk$ to $\Bbbk[t]$; the latter ring is a 1-dimensional domain, hence we can apply the GQDP, and every result in the present paper will follow.

This note is the written version of the author's talk at the international workshop *"Contemporary Geometry and Related Topics"*, held in Belgrade in May 15-21, 2002. We dwell somewhat in detail upon the very constructions under study, but we skip proofs and other technicalities, which are postponed to a forthcoming article (namely [Ga3]).

Finally, a few words about the organization of the paper. In §1 we collect a bunch of definitions, and some standard, technical results. In §2 we introduce the "connecting functors" $A \mapsto A^\vee$ (on augmented algebras) and $C \mapsto C'$ (on coaugmented coalgebras), and the (associated) "crystal functors" $A \mapsto \widehat{A}$ and $C \mapsto \widetilde{C}$; we also explain the relationship between these two pairs of functors with respect to Hopf duality. §3 cope with the effect of connecting and crystal functors on bialgebras and Hopf algebras. §4 considers the deformations provided by Rees modules, while §5 treats deformations arising from the previous ones via quantum group theory, introducing the "Drinfeld-like functors". In §6 we look at function algebras and enveloping algebras, we collect all our results in the "Crystal Duality Principle", and explain how this result can be also proved via quantum group theory.

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