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"Braiding structures on formal Poisson groups and classical solutions of the QYBE"

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INTRODUCTION

In the study of classical Hamiltonian systems, one is naturally interested in those which are completely integrable. A natural condition to achieve complete integrability for the system is that it admit a so called "Lax pair", thus one typical goal is to find Hamiltonian systems admitting such a pair; a standard recipe to obtain this has been provided by Semenov-Tian-Shansky (see [Se]), which explain how to get such a system proceeding from a pair $(\mathfrak{g}, \boldsymbol{r})$ where $\mathfrak{g}$ is a Lie quasitriangular Lie bialgebra and $\boldsymbol{r}$ is its $\boldsymbol{r}$–matrix, a classical solution of the classical Yang-Baxter equation (CYBE): the system is built up on $\mathfrak{g}^*$, the Lie bialgebra dual to $\mathfrak{g}$, as phase space, and the $\boldsymbol{r}$–matrix $\boldsymbol{r}$ provides (a recipe for) the Poisson bracket on $C^\infty(\mathfrak{g}^*)$. This raises the question of studying quasitriangular bialgebras, as objects of special interest within the category of Lie bialgebras: in particular, since we think at $\mathfrak{g}^*$ as a phase space, so that $\mathfrak{g}$ is its cotangent space, one's desire is to understand the geometrical meaning of the classical $\boldsymbol{r}$–matrix.

A second motivation for studying the geometrical meaning of the classical $\boldsymbol{r}$–matrix arises from conformal, quantum and topological quantum field theories. Indeed, all these are concerned with the notion of "fusion rules" which, roughly, rule the tensor product in a quasitensor category (see e.g. [FK]): as an application — among others — one has a recipe which provides tangle and link invariants as well as invariants of 3-manifolds (cf. [Tu]). In this setting, the common notion one start with is that of a quasitensor (or "braided monoidal") category; such an object can be built up as category of representations of a quasitriangular Hopf algebra (QTHA): indeed, by Tannaka-Krein reconstruction theorems the two notions — quasitensor categories and quasitriangular Hopf algebras — are essentially equivalent, so one may switch to the study of QTHAs. A key example of QTHA is given by a quantum group, in the shape of a quantum universal enveloping algebra (QUEA) together with its (universal) R–matrix. Now, the semiclassical counterpart of a QUEA is a Lie bialgebra $\mathfrak{g}$ (i.e., the given QUEA is the quantization of $U(\mathfrak{g})$): if the QUEA is also quasitriangular, then the semiclassical counterpart of its R–matrix is a classical $\boldsymbol{r}$–matrix $\boldsymbol{r}$ on $\mathfrak{g}$, the pair $(\mathfrak{g}, \boldsymbol{r})$ being a quasitriangular Lie bialgebra. The question then rises of whether — or at least how far — one can perform the constructions which are usually made via the QUEA and its R–matrix (such as that of link invariants) using instead only the "semiclassical" datum of $(\mathfrak{g}, \boldsymbol{r})$: then again the key point will be to understand the geometrical meaning of the classical $\boldsymbol{r}$–matrix.

With this kind of motivations, we go and study the following problem. It is known that if $\mathfrak{g}$ is a Lie bialgebra (over a field $\Bbbk$ of zero characteristic), then its dual space $\mathfrak{g}^*$ is a Lie bialgebra as well. Also, let G be an algebraic Poisson group — or Poisson-Lie group, say, when $\Bbbk \in \{\mathbb{R}, \mathbb{C}\}$ — whose tangent Lie bialgebra is $\mathfrak{g}$. Now assume $\mathfrak{g}$ is quasitriangular, with $\boldsymbol{r}$–matrix $\boldsymbol{r}$: this gives to $\mathfrak{g}$ some additional properties; two questions then rise:

(∗) What an additional structure one obtains on the dual Lie bialgebra $\mathfrak{g}^*$?

(•) What is the geometrical global datum on G which is the result of "integrating" $\boldsymbol{r}$?

Of course, the two questions and their answers are necessarily tightly related.

First, an answer to question (∗) was given by the authors in [GH] (cf. also [Re], [Ga1], [Ga2]): the topological Poisson Hopf algebra $F[[\mathfrak{g}^*]]$ (the function algebra of the formal Poisson group associated to $\mathfrak{g}^*$) is *braided* (see the definition later on).

The result in [GH] was proved using the theory of quantum groups. Indeed, after Etingof-Kazhdan (cf. [EK]) every Lie bialgebra admits a quantization $U_\hbar(\mathfrak{g})$, namely a (topological) Hopf algebra over $\Bbbk[[\hbar]]$ whose specialisation at $\hbar = 0$ is isomorphic to $U(\mathfrak{g})$ as a co-Poisson Hopf algebra; in addition, if $\mathfrak{g}$ is quasitriangular and $\boldsymbol{r}$ is its $\boldsymbol{r}$–matrix, then such a $U_\hbar(\mathfrak{g})$ exists which is quasitriangular too, as a Hopf algebra, with an R–matrix $R_\hbar\,(\in U_\hbar(\mathfrak{g}) \otimes U_\hbar(\mathfrak{g}))$ such that $R_\hbar \equiv 1 + \boldsymbol{r}\,\hbar \mod \hbar^2$ (here one identifies, as $\Bbbk[[\hbar]]$–modules, $U_\hbar(\mathfrak{g}) \cong U(\mathfrak{g})[[\hbar]]$). Using Drinfeld's *Quantum Duality Principle* ([Dr1]; cf. [Ga5] for a proof), from any QUEA $U_\hbar(\mathfrak{g})$ with semiclassical limit $U(\mathfrak{g})$ one can extract a certain quantum formal series Hopf algebra (QFSHA) $U_\hbar(\mathfrak{g})'$ such that the semiclassical limit of $U_\hbar(\mathfrak{g})'$ is $F[[\mathfrak{g}^*]]$. In [GH], starting from a quasitriangular QUEA $\big(U_\hbar(\mathfrak{g}), R\big)$, we showed that, although *a priori* $R \notin U_\hbar(\mathfrak{g})' \otimes U_\hbar(\mathfrak{g})'$ (so that the pair $\big(U_\hbar(\mathfrak{g})', R\big)$ is *not* in general a quasitriangular Hopf algebra), nevertheless its adjoint action $\mathfrak{R}_\hbar := \mathrm{Ad}(R_\hbar) : U_\hbar(\mathfrak{g}) \otimes U_\hbar(\mathfrak{g}) \longrightarrow U_\hbar(\mathfrak{g}) \otimes U_\hbar(\mathfrak{g})\,,\ x \otimes y \mapsto R_\hbar \cdot (x \otimes y) \cdot R_\hbar^{-1}$ stabilises the subalgebra $U_\hbar(\mathfrak{g})' \otimes U_\hbar(\mathfrak{g})'$, hence induces by specialisation an operator $\mathfrak{R}_0$ over $F[[\mathfrak{g}^*]] \otimes F[[\mathfrak{g}^*]]$: moreover, the properties which make $R_\hbar$ an R–matrix imply that $\mathfrak{R}_\hbar$ is a braiding operator, hence the same holds for $\mathfrak{R}_0$: thus, the pair $\big(F[[\mathfrak{g}^*]], \mathfrak{R}_0\big)$ is a braided Hopf algebra. In particular, this gives us a new method to produce set-theoretical solutions of the QYBE, thus giving a positive answer to a question set in [Dr2] (also tackled, for instance, in [ESS]). Note also that for igniting our construction we only need a quantisation functor $(\mathfrak{g}, \boldsymbol{r}) \mapsto \big(U_\hbar(\mathfrak{g}), R\big)$, and several of them exist (see [En]).

Second, an answer to question (•) was given by Weinstein and Xu in [WX]. We briefly sketch their results. Let G, resp. G^*, be a Poisson group with tangent Lie bialgebra $\mathfrak{g}$, resp. $\mathfrak{g}^*$: in addition, assume both G and G^* to be complete. Let D be the corresponding double Poisson group, which is given a structure of symplectic double groupoid, over G and G^* at once (further assumptions are needed, see §3 later on). Then the authors prove that there is a classical analogous of the quantum R–matrix, namely a Lagrangian submanifold $\mathcal{R}$ of $D \times D$, called *the (global) classical* $\mathcal{R}$*–matrix*, which enjoys much the same properties of a quantum R–matrix! Furthermore, for any symplectic leaf S in G^*, this $\mathcal{R}$ induces a symplectic automorphism of $S \times S$ which in turn at the level of function algebras yields a braiding for $F[S]$; then, as G^* is the union of its symplectic leaves, we get also a braiding on $F[G^*]$ and so, via completion, a braiding on $F[[\mathfrak{g}^*]]$ too.

As a first goal in this paper, we investigate more in depth the properties of the construction in [GH]. In particular, we show that the step $\big(U_\hbar(\mathfrak{g}), R\big) \mapsto \big(U_\hbar(\mathfrak{g})', \mathfrak{R}_\hbar\big)$ is functorial and preserves quantisation equivalence. Since the initial quantisation step $(\mathfrak{g}, \boldsymbol{r}) \mapsto \big(U_\hbar(\mathfrak{g}), R_\hbar\big)$ (provided by [EK], but any other would work) is functorial, and of course the final specialisation step $\big(U_\hbar(\mathfrak{g})', \mathfrak{R}_\hbar\big) \mapsto \big(F[[\mathfrak{g}^*]], \mathfrak{R}_0\big)$ is trivially functorial, we conclude that the whole construction $(\mathfrak{g}, \boldsymbol{r}) \mapsto \big(F[[\mathfrak{g}^*]], \mathfrak{R}_0\big)$ is functorial too. Moreover, whenever one has a braiding on $F[[\mathfrak{g}^*]]$ a so-called *infinitesimal braiding* $\overline{\mathfrak{R}}$ is defined on the cotangent Lie bialgebra of $F[[\mathfrak{g}^*]]^{\otimes 2}$, which is just $\mathfrak{g}^{\oplus 2}$: if the braiding is the afore mentioned $\mathfrak{R}_0$, we prove that the infinitesimal braiding $\overline{\mathfrak{R}}_0$ is trivial.

As a second goal of the paper, we compare our results with those of [WX]. First of all, a general fact is worth stressing: the purpose in [WX] is to find a geometrical counterpart of the classical $\boldsymbol{r}$–matrix, in particular an object which is of global rather than local nature: to this end, one is forced to impose some additional requirements from scratch, mainly the existence of complete Poisson groups G and G^* with tangent Lie bialgebras respectively $\mathfrak{g}$ and $\mathfrak{g}^*$). In contrast, the approach of [GH] sticks to the infinitesimal level: everything is formulated in terms of Lie bialgebras or formal Poisson groups. Therefore, the final output of [WX] is stronger but requires stronger hypotheses as well. Nevertheless, the additional requirements in [WX] are not necessary if we stick to the infinitesimal setting: indeed, a good deal of the analysis therein can be carried out as well in local terms — just on germs of Poisson groups — so that eventually one ends up with results which are perfectly comparable with those of [GH]. Thus we compare the braiding $\mathfrak{R}_{{}_{WX}}$ of [WX] with the one of [GH], call it $\mathfrak{R}_{{}_{GH}}$. Indeed, one has a theoretical reason to find strong similarities: namely, the construction in [WX] is a *geometric* quantisation of $(\mathfrak{g}, \boldsymbol{r})$, whereas the one of [GH] passes through *deformation* quantisation. As a matter of fact, first we show that the infinitesimal braiding $\overline{\mathfrak{R}}_{{}_{WX}}$ is trivial, just like $\overline{\mathfrak{R}}_{{}_{GH}}$. Second, when $\mathfrak{g} = \mathfrak{sl}_2$ with the standard $\boldsymbol{r}$–matrix we prove via explicit computation that $\overline{\mathfrak{R}}_{{}_{WX}} = \overline{\mathfrak{R}}_{{}_{GH}}$. This raises the question of whether $\overline{\mathfrak{R}}_{{}_{WX}}$ and $\overline{\mathfrak{R}}_{{}_{GH}}$ do always coincide: we give an affirmative answer in a separate paper (see [EGH]).

The paper is organized as follows. Section 1 is devoted to recall some notions and results of quantum theory. Section 2 deals with the construction of braidings via quantum groups, after [GH]: in particular we point out its "compatibility" with the equivalence relation for quantisations, we prove the triviality of the associated infinitesimal braiding, and we sketch some examples. Section 3 deals with the geometrical construction of braidings after [WX]: in particular we reformulate some results from [*loc. cit.*] to make them fit with our language, and we prove that the associated infinitesimal braiding is trivial. Finally, section 4 is devoted to explicit computation of both $\overline{\mathfrak{R}}_{{}_{WX}}$ and $\overline{\mathfrak{R}}_{{}_{GH}}$, which shows they do coincide.

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