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INTRODUCTION

"Dualitas dualitatum
et omnia dualitas"
N. Barbecue, "Scholia"

The *quantum duality principle* is known in literature under at least two formulations. One claims that quantum function algebras associated to dual Poisson groups can be considered to be dual — in the Hopf sense — to each other; and similarly for quantum enveloping algebras (cf. [FRT] and [Se]). The second one, due to Drinfeld (cf. [Dr]), states that any quantization of the universal enveloping algebra of a Poisson group can also be understood — in some sense — as a quantization of the dual formal Poisson group, and, conversely, any quantization of a formal Poisson group also "serves" as a quantization of the universal enveloping algebra of the dual Poisson group: this is the point of view we are interested in. I am now going to describe this result more in detail.

Let $\Bbbk$ be a field of zero characteristic. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\Bbbk$, $U(\mathfrak{g})$ its universal enveloping algebra: then $U(\mathfrak{g})$ has a natural structure of Hopf algebra. Let $F[[\mathfrak{g}]]$ be the (algebra of regular functions on the) formal group associated to $\mathfrak{g}$: it is a complete topological Hopf algebra (the coproduct taking values in a suitable topological tensor product of the algebra with itself), which has two realizations. The first one is as follows: if G is an affine algebraic group with tangent Lie algebra $\mathfrak{g}$, and $F[G]$ is the algebra of regular functions on G, then $F[[\mathfrak{g}]]$ is the $\mathfrak{m}_e$–completion of $F[G]$ at the maximal ideal $\mathfrak{m}_e$ of the identity element $e \in G$, endowed with the $\mathfrak{m}_e$–adic topology. The second one is $F[[\mathfrak{g}]] := U(\mathfrak{g})^*$, the linear dual of $U(\mathfrak{g})$), endowed with the weak topology. In any case, $U(\mathfrak{g})$ identifies with the *topological dual* of $F[[\mathfrak{g}]]$, i.e. the set of all $\Bbbk$–linear continuous maps from $F[[\mathfrak{g}]]$ to $\Bbbk$, where $\Bbbk$ is given the discrete topology; similarly $F[[\mathfrak{g}]] = U(\mathfrak{g})^*$ is also the topological dual of $U(\mathfrak{g})$ if we take on the latter space the discrete topology: in particular, a (continuous) biduality theorem relates $U(\mathfrak{g})$ and $F[[\mathfrak{g}]]$, and evaluation yields a natural Hopf pairing among them. Now assume $\mathfrak{g}$ is a *Lie bialgebra*: then $U(\mathfrak{g})$ is a *co-Poisson Hopf* algebra, $F[[\mathfrak{g}]]$ is a topological *Poisson* Hopf algebra, and the above pairing is compatible with these additional co-Poisson and Poisson structures. Further, the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ is a Lie bialgebra as well, so we can consider also $U(\mathfrak{g}^*)$ and $F[[\mathfrak{g}^*]]$.

Let $\mathfrak{g}$ be a Lie bialgebra. A quantization of $U(\mathfrak{g})$ is, roughly speaking, a topological Hopf $\Bbbk[[h]]$–algebra which for $h = 0$ is isomorphic, as a co-Poisson Hopf algebra, to $U(\mathfrak{g})$: these objects form a category, called $\mathcal{QUEA}$. Similarly, a quantization of $F[[\mathfrak{g}]]$ is, in short, a topological Hopf $\Bbbk[[h]]$–algebra which for $h = 0$ is isomorphic, as a topological Poisson Hopf algebra, to $F[[\mathfrak{g}]]$: we call $\mathcal{QFSHA}$ the category formed by these objects.

The quantum duality principle (after Drinfeld) states that there exist two functors, namely $(\ )' \colon \mathcal{QUEA} \longrightarrow \mathcal{QFSHA}$ and $(\ )^\vee \colon \mathcal{QFSHA} \longrightarrow \mathcal{QUEA}$, which are inverse of each other, and if $U_h(\mathfrak{g})$ is a quantization of $U(\mathfrak{g})$ and $F_h[[\mathfrak{g}]]$ is a quantization of $F[[\mathfrak{g}]]$, then $U_h(\mathfrak{g})'$ is a quantization of $F[[\mathfrak{g}^*]]$, and $F_h[[\mathfrak{g}]]^\vee$ is a quantization of $U(\mathfrak{g}^*)$.

This paper provides an explicit thorough proof — seemingly, the first one in literature — of this result, in a slightly stronger version, too. I also point out some further details and what is true when $\Bbbk$ has positive characteristic, and sketch a plan for generalizing all this to the infinite dimensional case.

Note that several properties of the objects I consider have been discovered and exploited in the works by Etingof and Kazhdan (see [EK1], [EK2]), by Enriquez (cf. [E]) and by Kassel and Turaev (cf. [KT]), who deal with some *special cases* of quantum groups, arising from a specific construction, and also applied Drinfeld's results. The analysis in the present paper shows that that those properties are often direct consequences of more general facts.

I point out that Drinfeld's result is essentially *local* in nature, as it deals with quantizations over the ring of formal series and ends up only with infinitesimal data, i.e. objects attached to Lie bialgebras; a *global* version of the principle, dealing with quantum groups over a ring of Laurent polynomials, which give information on the global data of the underlying Poisson groups will be provided in a forthcoming paper (cf. [Ga2]): this is useful in applications, e.g. it yields a quantum duality principle for Poisson homogeneous spaces, cf. [CG].

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