F. Gavarini, *"On the global quantum duality principle"*
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INTRODUCTION

The quantum duality principle is known in literature with at least two formulations. One claims that quantum function algebras associated to dual Poisson groups can be taken to be (Hopf) dual to each other, and similarly for quantum enveloping algebras (cf. [FRT] and [Se]). The second one, due to Drinfeld (cf. [Dr], §7, and [Ga4]), states that any quantization $F_h[[G]]$ of $F[[G]]$ yields also a quantization of $U(\mathfrak{g}^*)$, and, conversely, any quantization $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$ provides a quantization of $F[[G^*]]$: here G^*, resp. $\mathfrak{g}^*$, is a Poisson group, resp. a Lie bialgebra, dual to G, resp. to $\mathfrak{g}$). Namely, Drinfeld defines two functors, inverse to each other, from the category of quantum enveloping algebras to the category of quantum formal series Hopf algebras and viceversa such that (roughly) $U_h(\mathfrak{g}) \mapsto F_h[[G^*]]$ and $F_h[[G]] \mapsto U_h(\mathfrak{g}^*)$.

In this paper I provide a *global* version of the above principle, improving Drinfeld's result and pushing as far as possible the treatment *in a Hopf algebra theoretical way.*

The general idea is the following. Quantization of groups and Lie algebras is a matter of dealing with suitable Hopf algebras. In short, the Hopf algebras "of geometrical interest" are (simplifying a bit) the commutative and the connected cocommutative ones: the first are function algebras of affine algebraic groups (which are their maximal spectra), the second are restricted universal enveloping algebras of Lie algebras. A "quantization" of such an object H_0 will be a Hopf algebra H depending on some parameter, say p, such that setting $p = 0$, i.e. taking the quotient of the algebra modulo p, one gets back the original Hopf algebra H_0. One must also remark that when a quantization H is given, the classical object H_0 inherits an additional structure, that of a Poisson algebra, if $H_0 = F[G]$, or that of a co-Poisson algebra, if $H_0 = U(\mathfrak{g})$. Correspondingly, G is an affine *Poisson* group, $\mathfrak{g}$ is a *Lie bialgebra, and then also* its dual space $\mathfrak{g}^*$ is a Lie bialgebra; then we'll denote by G^* any affine Poisson group with tangent Lie bialgebra $\mathfrak{g}^*$, and we say G^* is *dual* to G.

In conclusion, one is lead to consider such "quantum groups", namely "p–depending" Hopf algebras which are either *commutative modulo* p or *cocommutative modulo* p.

In detail, I focus on the category $\mathcal{HA}$ of all Hopf algebras which are torsion-free modules over a PID, say R; the role of the "quantization parameter" then will be played by any prime element $p \in R$. For any such p, I introduce well-defined Drinfeld's-like functors from $\mathcal{HA}$ to itself, I show that their image is contained in a category of quantum groups — quantized function algebras in one case, quantized enveloping algebras in the other — and that when restricted to quantum groups these functors are inverse to each other and they exchange the type of the quantum group — switching "function" to "enveloping" — and the underlying group — switching G to G^*. Other details enter the picture to show that these functors endow $\mathcal{HA}$ with sort of a (inner) "Galois' correspondence", in which quantum groups — i.e. quantized enveloping algebras and quantized function algebras —

play the role of Galois (sub)extensions, for they are exactly the objects which are fixed by the composition (in either order) of the two Galois maps.

From a purely algebraic point of view — and in characteristic zero, to make things easier — the quantum duality principle, coupled with the existence theorems for quantizations of Lie bialgebras or algebraic groups (given by [EK] and [E]), tells us (roughly speaking) that the category of *commutative* Hopf algebras and the category of *cocommutative* Hopf algebras are related in a very precise way via the "quantization + Drinfeld's functors + specialization" process. This requires passing through *general* (i.e neither commutative nor cocommutative) Hopf algebras, so we see that quantization may be a way to rule special subclasses inside the whole category of Hopf algebras.

I wish to stress the fact that, compared with Drinfeld's result, mine is "global" in several respects. First, I deal with functors applying to general Hopf algebras (not only quantum groups, i.e. I do not require them to be "commutative up to specialization" or "cocommutative up to specialization"). Second, I work with more global objects, namely (function algebras on) algebraic Poisson groups rather than (function algebras on) *formal* algebraic Poisson groups. Third, I do *not require* the geometric objects — Poisson groups and Lie bialgebras — to be finite dimensional. Fourth, the ground ring R is any PID, not necessarily $k[[h]]$ as in Drinfeld's approach: therefore one may have several points $(p) \in Spec\,(R)$, and to each of them the machinery applies: thus for any such (p) Drinfeld's functors are defined and for a given Hopf R–algebra H they yield two Hopf R–algebras, say $H'_{[p]}$ and $H^\vee_{[p]}$, such that the fibre over (p) of $H'_{[p]}$, resp. of $H^\vee_{[p]}$, is a quantum function algebra at (p), resp. a quantum enveloping algebra at (p), i.e. its reduction modulo (p) is the function algebra of a Poisson group, resp. the (restricted, if $Char\big(R/(p)\big) > 0$) universal enveloping algebra of a Lie bialgebra. In particular we have a method to get, out of any Hopf algebra over a PID, several "quantum groups", namely two of them (of the "enveloping algebra" and of the "function algebra" type) for each point of the spectrum of R. More in general, one can start from any Hopf algebra H over a field k and then take $H_x := k[x] \otimes_k H$, (x being an indeterminate): this is a Hopf algebra over the PID $k[x]$, to which Drinfeld's functors at any prime $p \in k[x]$ may be applied to get quantum groups.

In this note, I confine myself to state the result and to expound it on two examples: the case of semisimple groups and the so-called "Kostant-Kirillov structure" on any Lie algebra. All details, proofs and further examples can be found in [Ga5].

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