F. Gavarini, *"A global version of the quantum duality principle"*
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INTRODUCTION

The quantum duality principle is known in literature in several formulations. One of these, due to Drinfeld ([[1]], §7, and [[2]]), states that any quantization $F_h[[G]]$ of $F[[G]]$ yields also a quantization of $U(\mathfrak{g}^*)$ and, conversely, any quantization $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$ provides a quantization of $F[[G^*]]$: here G^*, resp. $\mathfrak{g}^*$, is a Poisson group, resp. a Lie bialgebra, dual to G, resp. to $\mathfrak{g}$. Namely, Drinfeld defines two functors, inverse to each other, from the category of quantum enveloping algebras to the category of quantum formal series Hopf algebras and viceversa such that (roughly) $U_h(\mathfrak{g}) \mapsto F_h[[G^*]]$ and $F_h[[G]] \mapsto U_h(\mathfrak{g}^*)$. The *global version* of the above principle is an improvement of Drinfeld's result, which put it more in purely Hopf algebra theoretical terms, and make it much more manageable.

The general idea is the following. Quantization of groups and Lie algebras is a matter of dealing with suitable Hopf algebras. Roughly, the "classical" Hopf algebras of interest are the commutative and the connected cocommutative ones: the first are function algebras of affine algebraic groups, the second are universal enveloping algebras of Lie algebras. A "quantization" of such an object H_0 will be a Hopf algebra H depending on some parameter, say p, such that setting $p = 0$, i.e. taking the quotient $H/p\,H$, one gets back the original Hopf algebra H_0. When a quantization H is given, the classical object H_0 inherits an additional structure, that of a Poisson algebra, if $H_0 = F[G]$, or that of a co-Poisson algebra, if $H_0 = U(\mathfrak{g})$. Correspondingly, G is an affine *Poisson* group, $\mathfrak{g}$ is a *Lie bialgebra, and then also* its dual space $\mathfrak{g}^*$ is a Lie bialgebra; we'll denote by G^* any affine Poisson group with tangent Lie bialgebra $\mathfrak{g}^*$, and we say G^* is *dual* to G. In conclusion, one is lead to consider such "quantum groups", namely "p–depending" Hopf algebras which are either *commutative modulo p* or *cocommutative modulo p*.

In detail, I focus on the category $\mathcal{HA}$ of all Hopf algebras which are torsion-free modules over a PID, say R; the role of the "quantization parameter" will be played by any prime element $p \in R$. For any such p, I introduce well-defined Drinfeld's-like functors from $\mathcal{HA}$ to itself, I show that their image is contained in a category of quantum groups — quantized function algebras in one case, quantized enveloping algebras in the other. Moreover, when restricted to quantum groups these functors are inverse to each other, and they exchange the type of the quantum group — switching "function" to "enveloping" — and the underlying group — switching G to G^*. In fact, the whole picture is much richer, giving indeed (from the mathematical point of view) sort of a (inner) "Galois' correspondence" on $\mathcal{HA}$.

I wish to stress the fact that, compared with Drinfeld's result, mine is "global" in several respects. First, I deal with functors applying to general Hopf algebras, not only quantum groups (i.e. I do not require them to be "commutative up to specialization" or "cocommutative up to specialization"). Second, I work with more global objects, namely algebraic Poisson groups rather than *formal* algebraic Poisson groups. Third, I do *not require* the geometric objects — Poisson groups and Lie bialgebras — to be finite dimensional. Fourth, the ground ring R is any PID, not necessarily $k[[h]]$ as in Drinfeld's approach: therefore one may have several primes $p \in R$, and to each of them the machinery applies.

In particular we have a method to get, out of any Hopf algebra over a PID, several "quantum groups", namely two of them (of both types) for each prime $p \in R$. As an application, one can start from any Hopf algebra H over a field k and then take $H_x := k[x] \otimes_k H$, (x being an indeterminate): this is a Hopf algebra over the PID $k[x]$, to which Drinfeld's functors at any prime $p \in k[x]$ may be applied to give quantum groups.

In this note, I state the result and illustrate an application to a nice example occurring in the study of renormalization of quantum electrodynamics.

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References

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