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"The R–matrix action of untwisted affine quantum groups at roots of 1"

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INTRODUCTION

"Oh, quant'è affine alla sua genitrice!
Osserva come anch'ella ha belle trecce
ch'ha ereditate dalla sua matrice"

N. Barbecue, "Scholia"

A Hopf algebra H is called quasitriangular (cf. [Dr], [C-P]) if there exists an invertible element $R \in H \otimes H$ (or an element of an appropriate completion of $H \otimes H$) such that

$$\mathrm{Ad}(R)(\Delta(a)) \,=\, \Delta^{\mathrm{op}}(a) \qquad \forall\, a \in H$$
$$(\Delta \otimes \mathrm{id})(R) \,=\, R_{13}R_{23}\;, \qquad (\mathrm{id} \otimes \Delta)(R) \,=\, R_{13}R_{12}$$

where $\mathrm{Ad}(R)(x) := R \cdot x \cdot R^{-1}$, Δ^{op} is the opposite comultiplication (i. e. $\Delta^{\mathrm{op}}(a) = \sigma \circ \Delta(a)$ with $\sigma\colon A^{\otimes 2} \to A^{\otimes 2}$, $a \otimes b \mapsto b \otimes a$), and $R_{12}, R_{13}, R_{23} \in H^{\otimes 3}$ (or the appropriate completion of $H^{\otimes 3}$), $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, $R_{13} = (\sigma \otimes \mathrm{id})(R_{23}) = (\mathrm{id} \otimes \sigma)(R_{12})$.

As a corollary of this definition, R satisfies the Yang-Baxter equation in $H^{\otimes 3}$, namely

$$R_{12}R_{13}R_{23} \,=\, R_{23}R_{13}R_{12}$$

so that a braid group action can be defined on tensor products of H–modules (whence applications to knot theory arise). If $\hat{\mathfrak{g}}$ is an untwisted affine Kac-Moody algebra, the quantum universal enveloping algebra $U_h(\hat{\mathfrak{g}})$, over $\mathbb{C}[[h]]$, is quasitriangular (cf. [Dr], [C-P]). On the other hand, this is not true — strictly speaking — for its "polynomial version", the $\mathbb{C}(q)$–algebra $U_q(\hat{\mathfrak{g}})$: nonetheless, it is a braided algebra, in the sense of the following

Definition. *(cf. [Re1], Definition 2) A Hopf algebra H is called braided if there exists an automorphism $\mathcal{R}$ of $H \otimes H$ (or of an appropriate completion of $H \otimes H$) distinct from $\sigma\colon a \otimes b \mapsto b \otimes a$ such that*

$$\mathcal{R} \circ \Delta \,=\, \Delta^{\mathrm{op}}$$
$$(\Delta \otimes \mathrm{id}) \circ \mathcal{R} \,=\, \mathcal{R}_{13} \circ \mathcal{R}_{23} \circ (\Delta \otimes \mathrm{id})\;, \qquad (\mathrm{id} \otimes \Delta) \circ \mathcal{R} \,=\, \mathcal{R}_{13} \circ \mathcal{R}_{12} \circ (\mathrm{id} \otimes \Delta)$$

where $\mathcal{R}_{12} := \mathcal{R} \otimes \mathrm{id}$, $\mathcal{R}_{23} = \mathrm{id} \otimes \mathcal{R}$, $\mathcal{R}_{13} = (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathcal{R}) \circ (\sigma \otimes \mathrm{id}) \in Aut(H \otimes H \otimes H)$.

It follows from this definition that $\mathcal{R}$ satisfies the Yang-Baxter equation in $End(H^{\otimes 3})$,

$$\mathcal{R}_{12} \circ \mathcal{R}_{13} \circ \mathcal{R}_{23} \,=\, \mathcal{R}_{23} \circ \mathcal{R}_{13} \circ \mathcal{R}_{12}$$

which yields a braid group action on tensor powers of H, which is still important for applications. Notice that if (H,R) is quasitriangular, then $\big(H, \mathrm{Ad}(R)\big)$ is braided.

In this paper we prove that the unrestricted specializations of $U_q(\hat{\mathfrak{g}})$ at odd roots of 1 are braided too: indeed, we show that the braiding automorphism of $U_q(\hat{\mathfrak{g}})$ — which is, roughly speaking, the conjugation by its universal R–matrix — does leave stable the integer form — of $U_q(\hat{\mathfrak{g}})$ — which is to be "specialized". This extends to the present case a result due to Reshetikhin (cf. [Re1]) for the case of the quantum group $U_q\big(\mathfrak{sl}(2)\big)$, and to Reshetikhin (cf. [Re2]) and the author (cf. [Ga1]) for $U_q(\mathfrak{g})$, with $\mathfrak{g}$ finite dimensional semisimple. The most general case is developed in [G-H]. As a consequence, we get that the action of the universal R–matrix of $U_q(\mathfrak{g})$ on tensor products of pairs of Verma modules does specialize at odd roots of 1 as well.

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