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"Dual affine quantum groups"

INTRODUCTION

"Dualitas dualitatum
et omnia dualitas"
N. Barbecue, "Scholia"

Let $\hat{\mathfrak{g}}$ be an untwisted affine complex Kac-Moody algebra, with the Sklyanin-Drin-fel'd structure of Lie bialgebra; let $\hat{\mathfrak{h}}$ be its dual Lie bialgebra. Let R be the subring of complex rational functions having no poles at roots of 1. Let $U_q(\hat{\mathfrak{g}})$ be the quantum group — over the field $\mathbb{C}(q)$ — associated to $\hat{\mathfrak{g}}$: then there exists an integer form $\mathfrak{U}(\hat{\mathfrak{g}})$ of $U_q(\hat{\mathfrak{g}})$ over R which for $q \to 1$ specializes to $U(\hat{\mathfrak{g}})$ as a Poisson Hopf coalgebra (cf. [Lu2]). On the other hand, another integer form $\mathcal{U}(\hat{\mathfrak{g}})$ exists which for $q \to 1$ specializes (as a Poisson Hopf algebra) to $F\big[\widehat{H}\big]$, the function algebra of an infinite dimensional proalgebraic Poisson group $\widehat{H}$ whose tangent Lie bialgebra is $\hat{\mathfrak{h}}$ (cf. [BK]). All this can be seen as an application of (a "global version" of) the *quantum duality principle*: this claims (cf. [Dr], §7, or [CP], §6; see also [Ga3] for a proof) that the quantization of a Lie bialgebra — via a quantum universal enveloping algebra (QUEA) — provides also a quantization of the dual Lie bialgebra (through its associated formal Poisson group) — via a quantum formal series Hopf algebra (QFSHA) — and, conversely, a QFSHA which quantizes a Lie bialgebra (via its associated formal Poisson group) yields a QUEA for the dual Lie bialgebra as well.

In addition, both $\mathfrak{U}(\hat{\mathfrak{g}})$ and $\mathcal{U}(\hat{\mathfrak{g}})$ can be specialized at roots of 1, and special *quantum Frobenius morphisms* $\mathfrak{U}_\varepsilon(\hat{\mathfrak{g}}) \twoheadrightarrow \mathfrak{U}_1(\hat{\mathfrak{g}})$ and $\mathcal{U}_1(\hat{\mathfrak{g}}) \hookrightarrow \mathcal{U}_\varepsilon(\hat{\mathfrak{g}})$ exist which are quantum analogues (in characteristic zero!) of the Frobenius morphisms $U(\hat{\mathfrak{g}}_{\mathbb{Z}_p}) \twoheadrightarrow U(\hat{\mathfrak{g}}_{\mathbb{Z}_p})$ and $F\big[\widehat{H}_{\mathbb{Z}_p}\big] \hookrightarrow F\big[\widehat{H}_{\mathbb{Z}_p}\big]$ which exist in characteristic p. Such results are not predicted by the quantum duality principle: they are typical instead of the Jimbo-Lusztig's approach to quantum groups.

Our aim is to find an analogue of $U_q(\hat{\mathfrak{g}})$ for the algebra $\hat{\mathfrak{h}}$ instead of $\hat{\mathfrak{g}}$. Inspired by the quantum duality principle, and encouraged by the finite-type case (cf. [Ga1]), we choose as a reasonable candidate the linear dual $U_q(\hat{\mathfrak{g}})^*$, which has a natural structure of formal Hopf algebra. This dual can be studied by dualizing Drinfel'd's construction of the quantum double and using Tanisaki's pairings between quantum Borel (sub)algebras. So we find a description of $U_q(\hat{\mathfrak{g}})^*$, as a topological algebra with formal Hopf algebra structure, in terms of generators and relations: we call this algebra $U_q\big(\hat{\mathfrak{h}}\big)$, for in fact we prove that it is for $\hat{\mathfrak{h}}$ what $U_q(\hat{\mathfrak{g}})$ is for $\hat{\mathfrak{g}}$. In particular, $U_q\big(\hat{\mathfrak{h}}\big)$ has an integer form $\mathfrak{U}\big(\hat{\mathfrak{h}}\big)$ (over R) which is a quantization of $U\big(\hat{\mathfrak{h}}\big)$; moreover, $U_q\big(\hat{\mathfrak{h}}\big)$ has also a second integer form $\mathcal{U}\big(\hat{\mathfrak{h}}\big)$ which is a quantization of $F^\infty\big[\widehat{G}\big]$, where $\widehat{G}$ of course is a Kac-Moody Poisson group with $\hat{\mathfrak{g}}$ as tangent Lie bialgebra. More in general, both $\mathfrak{U}\big(\hat{\mathfrak{h}}\big)$ and $\mathcal{U}\big(\hat{\mathfrak{h}}\big)$ can be specialized at roots of 1, and quantum Frobenius morphisms exist (for both kind of forms), which are dual to those of $U_q(\hat{\mathfrak{g}})$ and have a similar description.

Finally, a brief sketch of the main ideas of the paper.

First, since $U_q(\hat{\mathfrak{g}})$ is a quotient of a quantum double $D_q(\hat{\mathfrak{g}}) := D\Big(U_q\big(\hat{\mathfrak{b}}_-\big), U_q\big(\hat{\mathfrak{b}}_+\big), \pi\Big)$, its linear dual $U_q(\hat{\mathfrak{g}})^*$ embeds into $D_q(\hat{\mathfrak{g}})^*$. Second, since $D_q(\hat{\mathfrak{g}}) \cong U_q\big(\hat{\mathfrak{b}}_+\big) \otimes U_q\big(\hat{\mathfrak{b}}_-\big)$ (as coalgebras) we have an isomorphism (of algebras) $D_q(\hat{\mathfrak{g}})^* \cong U_q\big(\hat{\mathfrak{b}}_+\big)^* \,\widehat{\otimes}\, U_q\big(\hat{\mathfrak{b}}_-\big)^*$, where $\widehat{\otimes}$ denotes topological tensor product. Third, since quantum Borel algebras of opposite sign are perfectly paired, their linear duals are suitable completions of quantum Borel algebras of opposite sign. Thus we find a presentation of $U_q(\hat{\mathfrak{g}})^*$ (as a topological algebra) by generators and relations which leads us to *define* $U_q\big(\hat{\mathfrak{h}}\big) := U_q(\hat{\mathfrak{g}})^*$ (actually, one has to keep track of some choice of lattices too, involved in the toral parts).

From this, all claimed results follow. In particular, the form $\mathcal{U}\big(\hat{\mathfrak{h}}\big)$ is the subset (of $U_q\big(\hat{\mathfrak{h}}\big) := U_q(\hat{\mathfrak{g}})^*$) of linear functions on $U_q(\hat{\mathfrak{g}})$ which are R–valued on $\mathfrak{U}(\hat{\mathfrak{g}})$, so $\mathcal{U}\big(\hat{\mathfrak{h}}\big) \cong Hom_R\big(\mathfrak{U}(\hat{\mathfrak{g}}), R\big)$. Therefore, all results about specialisations of $\mathcal{U}\big(\hat{\mathfrak{h}}\big)$ and its quantum Frobenius morphisms follow from those about $\mathfrak{U}(\hat{\mathfrak{g}})$. On the other hand, the form $\mathfrak{U}\big(\hat{\mathfrak{h}}\big)$ is a *proper* subset of $Hom_R\big(\mathcal{U}(\hat{\mathfrak{g}}), R\big)$, for sort of a (non-trivial) "locality condition" is required for elements of $Hom_R\big(\mathcal{U}(\hat{\mathfrak{g}}), R\big)$ to belong to $\mathfrak{U}\big(\hat{\mathfrak{h}}\big)$.

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